Maximal pattern complexity of 2-dimensional words

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Abstract

Several works are devoted to the maximal pattern complexity of one-dimensional words ([KZ1, KZ2, KR, KX]). In this paper, we study the maximal pattern complexity $p^*_\alpha(k)$ of 2-dimensional word α. A new notion, the *strongly recurrent* is introduced. It is shown that if α is strongly recurrent, then either α is 2-periodic or $p^*_\alpha(k) \geq 2k$ $(k = 1, 2, \cdots)$. We thus define a *two dimensional pattern Sturmian word* to be a strongly recurrent word α with $p^*_\alpha(k) = 2k$. Examples of pattern Sturmian words are given.

1 Introduction

The study of complexity of words has a long history. Especially the words with low complexity have special interest. Recently, the study of complexity of words has been extended in two different directions. One direction is to study the complexity of higher dimensional words (cf. [ABI],[B],[BT], [BV],[LP], etc). Another is to consider a new complexity, the so called *maximal pattern complexity* ([KZ1],[KZ2],[KR]). In this paper we combine the above two efforts by studying the maximal pattern complexity of two dimensional words.

1.1 Maximal pattern complexity.

Let A be a finite alphabet. An element $\alpha = \alpha(0)\alpha(1)\alpha(2)\cdots \in A^{\mathbb{N}}$, where $\mathbb{N} := \{0, 1, 2, \cdots\}$, is called a *one-sided word* over A if every letter of A appears in α.

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Let k be a positive integer. By a *k-window* τ, we mean a sequence of integers of length k with

$$0 = \tau(0) < \tau(1) < \tau(2) < \cdots < \tau(k-1).$$

The k-window $\tau = \{0, 1, \cdots, k-1\}$ is called the *k-block window.* For a k-window $\tau: \ 0 = \tau(0) < \tau(1) < \cdots < \tau(k-1)$ and a word α, the word

$$\alpha[n+\tau] := \alpha(n+\tau(0))\alpha(n+\tau(1))\cdots\alpha(n+\tau(k-1))$$

is called *the pattern of α through the window τ at position n.* We denote by $F_\alpha(\tau)$ the set of all patterns of α through the window τ, i.e.,

$$F_\alpha(\tau) := \{\alpha[n+\tau]; \ n = 0, 1, 2, \cdots\}.$$

In particular, we denote $F_\alpha(k) := F_\alpha(\tau)$ for the k-block window τ.

The *maximal pattern complexity function* p^*_α for a word α is introduced by the first author together with Zamboni [KZ1] as

$$p^*_\alpha(k) := \sup_\tau \sharp F_\alpha(\tau) \quad (k = 1, 2, 3, \cdots), \tag{1.1}$$

where the supremum is taken over all k-windows τ, while the classical complexity function p_α is defined as $p_\alpha(k) = \sharp F_\alpha(k)$.

A classical result of Morse and Hedlund says that

Theorem A. ([MoHe]) *For a word α, the following statements are equivalent:*
(i) *α is eventually periodic,*
(ii) *$p_\alpha(k)$ is bounded in k,*
(iii) *$p_\alpha(k) < k+1$ for some $k = 1, 2, \cdots$.*

The following parallel result with respect to the maximal pattern complexity function is proved in [KZ1].

Theorem B. ([KZ1]) *For a word α, the following statements are equivalent:*
(i) *α is eventually periodic,*
(ii) *$p^*_\alpha(k)$ is bounded in k,*
(iii) *$p^*_\alpha(k) < 2k$ for some $k = 1, 2, \cdots$.*

A word α with block complexity $p_\alpha(k) = k+1$ $(k = 1, 2, 3, \cdots)$ is known as a *Sturmian word* and is studied extensively (see for example Berthé [B] and the references therein). Naturally, a word α with maximal pattern complexity $p^*_\alpha(k) = 2k$ $(k = 1, 2, 3, \cdots)$ is called a *pattern Sturmian word.* It is interesting that the classical Sturmian words are also pattern Sturmian words (See [KZ1]).

The maximal pattern complexity of a word α with more than two letters has been investigated in [KR]. Let 1_S be the indicator function. A word α over A is called *periodic by projection* if there exists S with $\emptyset \neq S \subsetneq A$ such that the word

$$1_S \circ \alpha := 1_S(\alpha_0)1_S(\alpha_1)1_S(\alpha_2)\cdots \in \{0,1\}^{\mathbb{N}}$$

is eventually periodic. Note that if $\ell = 2$, then α is periodic by projection if and only if α is eventually periodic. It is shown that

Theorem C. ([KR]) *Let α be a word over ℓ letters with $\ell \geq 2$. If $p^*_\alpha(k) < \ell k$ holds for some $k = 1, 2, \cdots$, then α is periodic by projection.*

Accordingly, a word over ℓ letters is said to be a *pattern Sturmian word* if $p^*_\alpha(k) = \ell k$ and it is not periodic by projection.

Example 1.1. Let θ be an irrational number and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let $\ell \geq 2$ and $\mathcal{P} = \{I_0, I_1, \cdots, I_{\ell-1}\}$ be a partition of $\mathbb{T}$ into ℓ intervals with nonempty interiors. Define $\alpha \in \{0, 1, \cdots, \ell-1\}^{\mathbb{N}}$ by

$$\alpha(n) = i \ \text{ if } \ n\theta \in I_i.$$

Then α is a pattern Sturmian word ([KR]).

[KR] found another classes of pattern Sturmian words over ℓ letters which are not recurrent. When $\ell = 2$, a third class which is called *Topelitz words* are found by [KZ1]. It is interesting to know a new type of examples of pattern Sturmian words.

1.2 Maximal pattern complexity of two-sided word.

One advantage of the notion of the maximal pattern complexity is that it can be generalized easily to higher dimensions. For a 2-dimensional word α on A, we consider a mapping from $\mathbb{Z}^2$ to A instead of from $\mathbb{N}^2$ to A since the former is more natural and simpler than the latter.

For the comparison, let us first have a look at one-dimensional two-sided words, where a *two-sided word*

$$\alpha = \cdots\alpha(-2)\alpha(-1);\alpha(0)\alpha(1)\alpha(2)\cdots \in A^{\mathbb{Z}}$$

over A is a mapping from $\mathbb{Z}$ to A. A word α is said to be *recurrent* if for any positive integer N, there exists a nonzero integer L such that $\alpha(i) = \alpha(L+i)$ for any $i \in \mathbb{Z}$ with $|i| \leq N$. Moreover, for a two-sided word α, $p^*_\alpha(k)$ is defined by (1.1) with

$$F_\alpha(\tau) := \{\alpha[n+\tau];\ n \in \mathbb{Z}\}.$$

It is not hard to generalize Theorem B to the case of two-sided words.

Proposition 1.2. *For a two-sided word α, the following conditions are equivalent to each other:*

(i) *α is periodic.*

(ii) *$p^*_\alpha(k)$ is bounded in k.*

(iii) *α is recurrent and $p^*_\alpha(k) < 2k$ for some $k = 1, 2, 3, \cdots$.*

Remark 1.3. The recurrent condition cannot be dropped, for it is easy to check the word $\alpha = \cdots 000; 1000 \cdots$ where $\alpha(0) = 1$ is not periodic but $p^*_\alpha(k) = k + 1$ for any $k = 1, 2, 3, \ldots$. If we do not want to assume recurrent property for α, then we can only have: *A two-sided word α is eventually periodic if and only if $p^*_\alpha(k) < 2k$ for some $k = 1, 2, 3, \cdots$.* A two-sided word α is said to be *eventually periodic* if both of the one-sided words $\alpha(0)\alpha(1)\alpha(2)\cdots$ and $\alpha(-1)\alpha(-2)\alpha(-3)\cdots$ are eventually periodic. As the above example shows, an eventually periodic two-sided word α does not necessarily satisfy that $p^*_\alpha(k)$ is bounded in k.

1.3 Maximal pattern complexity and strongly recurrent property of two-dimensional words

The main purpose of this paper is to generalize Proposition 1.2 to two dimension words. First, let us give some notations and definitions.

Definitions. Let $\alpha \in A^{\mathbb{Z}^2}$ be a 2-dimensional word. By a *k-window*, we mean a subset τ of $\mathbb{Z}^2$ with $\sharp\tau = k$ and $O \in \tau$, where $O = (0, 0)$ is the origin. Let $\xi \in \mathbb{Z}^2$ and τ be a k-window. We denote

$$\alpha[\xi + \tau] := (\alpha(\xi + x))_{x\in\tau} \in A^\tau,$$

which is called a *τ-factor* of α. Sometimes we also call it a *pattern of α through the window τ*. Let $F_\alpha(\tau)$ be the set of τ-factors of α. We define the *maximal pattern complexity* by

$$p^*_\alpha(k) := \sup_{\tau;\ \sharp\tau=k} \sharp F_\alpha(\tau) \quad (k = 1, 2, 3, \cdots).$$

For any positive integer N, we denote $\Lambda_N := ([-N, N] \times [-N, N]) \cap \mathbb{Z}^2$, which is a $(2N + 1) \times (2N + 1)$ square.

For $u = (u_1, u_2) \in \mathbb{Z}^2 \setminus \{O\}$, let $\|u\| := \sqrt{u_1^2 + u_2^2}$ be the Euclidean norm and $\arg(u) := \arg(u_1 + u_2\sqrt{-1})$ be the argument of the complex number $u_1 + u_2\sqrt{-1}$. We call $\arg(u)$ the *angle* of u. Angles are always considered in modulo 2π.

A nonzero vector $\xi \in \mathbb{Z}^2$ is called a *period* of a word α if $\alpha(x) = \alpha(x + \xi)$ for any $x \in \mathbb{Z}^2$. A word α is called 2-*periodic* (or *doubly periodic*) if there exist two linearly independent vector ξ and η which are periods of α.

A word α is called 2-*recurrent* if for any positive integer N, there eixst two linearly independent vectors ξ and η in $\mathbb{Z}^2$ such that $\alpha[\Lambda_N] = \alpha[\xi + \Lambda_N] = \alpha[\eta + \Lambda_N]$.

A word α is called *strongly recurrent* if there exists $\delta > 0$ such that for any positive integer N, there exist two nonzero vectors ξ and η in $\mathbb{Z}^2$ such that

(i) $\delta < |\arg(\xi) - \arg(\eta)| < \pi - \delta$,

(ii) $\alpha[\Lambda_N] = \alpha[\xi + \Lambda_N] = \alpha[\eta + \Lambda_N]$.

Now we can state our main result.

Theorem 1.4. *For a two-dimensional word α, the following statements are equivalent to each other.*

(i) *α is 2-periodic.*

(ii) *$p^*_\alpha(k)$ is bounded in k.*

(iii) *α is strongly recurrent and $p^*_\alpha(k) < 2k$ holds for some $k = 1, 2, 3, \cdots$.*

We thus define a *two dimensional pattern Sturmian word* to be a strongly recurrent word α with $p^*_\alpha(k) = 2k$ for $k = 1, 2, \cdots$. Examples of these words are given in Section 6. The following example shows that the strongly recurrent property cannot be replaced by the 2-recurrent property.

Example 1.5. Let $\alpha \in \{0,1\}^{\mathbb{Z}^2}$ be the word over two letters $\{0,1\}$ such that $\alpha(m,n) = 1$ if and only if $n \le am$, where $a \ne 0$ is a real number.

First, the maximal pattern complexity $p^*_\alpha(k) = k + 1$. For when we move a k-window τ in the plane, the pattern in τ is completely determined by the number of 0 in it, which can only takes values $0, 1, \ldots, k$.

Secondly, it is seen that α is not two-recurrent when a is rational, and we will see α is 2-recurrent but not strongly recurrent when a is irrational.

Recurrent angle or direction. Motivated by this example, we define a recurrent angle or direction of a word α. An angle θ is said to be a *recurrent angle* of a word α, if for any $\delta > 0$ and any $N > 0$, there exists a nonzero vector $\xi = \xi(\delta, N)$ such that $\alpha[\Lambda_N] = \alpha[\xi + \Lambda_N]$ and $|\arg(\xi) - \theta| < \delta$. A nonzero vector ξ or a half line l starting at O is called a *recurrent direction* if $\arg \xi$ or $\arg l$ is a recurrent angle.

Lemma 1.6. *A two-dimensional word is strongly recurrent if and only if it has two linearly independent recurrent directions.*

Proof. If a word has two linearly independent recurrent directions, then clearly it is strongly recurrent.

Now suppose a word α is strongly recurrent. Then there exists $\delta > 0$ such that for any positive integer N, there exist vectors ξ_N and η_N such that (i) and (ii) in the definition of "strongly recurrent" holds with ξ_N and η_N for ξ and η. Take a limiting pair (ξ, η) of $(\xi_N / \parallel \xi_N \parallel, \eta_N / \parallel \eta_N \parallel)$ as $N \to \infty$. Then, ξ and η are linearly independent recurrent directions of α, which proves the lemma. □

For the word in Example 1.5, the only recurrent angles are $\pm \arg(1 + a\sqrt{-1})$ and hence it is not strongly recurrent.

1.4 Outline of the paper

The equivalence of (i) and (ii) in Theorem 1.4 is an easy matter and it is proved in Section 2 as Proposition 2.1. That (i) implies (iii) is trivial, hence to prove Theorem 1.4, it is sufficient to prove that (iii) implies (i). This can be reduced to the case $A = \{0, 1\}$ by the following lemma.

Lemma 1.7. *Let α be a two-dimensional word over A.*

(i) *If α is strongly recurrent, then the word $1_{\{a\}} \circ \alpha$ is strongly recurrent for any $a \in A$.*

(ii) *If the word $1_{\{a\}} \circ \alpha$ is 2-periodic for any $a \in A$, then α is 2-periodic.*

The proof is strait forward and we omit it. Therefore, we need only consider a word α over $\{0, 1\}$ satisfying

(#) There is an positive integer k such that $p^*_\alpha(k) < 2k$.

In Section 3 we show that our main theorem is valid for a class of very special words, the monotone words. In fact, Theorem 3.5 claims that if α is a monotone word satisfying the condition (iii) of Theorem 1.4, then α is a constant word.

In Section 4, we show that a 2-recurrent word satisfying (#) has a monotone structure.

In Section 5, using this monotone structure, we decompose α which satisfies the condition (iii) of Theorem 1.4 into several lattice subwords which are monotone. Since these subwords also satisfy the condition (iii) of Theorem 1.4, they are constant words by Theorem 3.5. This implies that α is 2-periodic, the condition (i) of Theorem 1.4.

In Section 6, some examples of words with $p^*_\alpha(k) = 2k$ $(k = 1, 2, \cdots)$ are given.

2 Preliminaries

In this section we prove the equivalence of (i) and (ii) in Theorem 1.4. Let $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ be the canonical base of $\mathbb{Z}^2$.

Proposition 2.1. *A two dimensional word α is 2-periodic if and only if $p^*_\alpha(k)$ is bounded in k.*

Proof. Let ξ and η be linear independent periods of α. Let

$$\Omega := \{s\xi + t\eta \in \mathbb{Z}^2;\ 0 \le s < 1 \text{ and } 0 \le t < 1\} = \{x_0, \ldots, x_{q-1}\}.$$

For any $x \in \mathbb{Z}^2$ and any window τ, the periodicity of α yields

$$\alpha[x + \tau] = \alpha[x_i + \tau],$$

where $x_i \in \Omega$ is the element such that $x = x_i + m\xi + n\eta$ for some integers m, n. Hence $p^*_\alpha(k) \le \sharp\Omega$ holds for any $k = 1, 2, 3, \cdots$. Namely, $p^*_\alpha(k)$ is bounded in k.

Now suppose that $\sup_k p^*_\alpha(k) = M < \infty$. Recall that $\Lambda_k = \{(m, n) \in \mathbb{Z}^2;\ -k \le m, n \le k\}$ is a square window. Then at least one pattern appears two times among $\alpha[\Lambda_k], \alpha[\mathbf{e}_1 + \Lambda_k], \ldots, \alpha[M\mathbf{e}_1 + \Lambda_k]$. Hence $\alpha[i_k\mathbf{e}_1 + \Lambda_k] = \alpha[j_k\mathbf{e}_1 + \Lambda_k]$ for some $0 \le i_k < j_k \le M$. It follows that $\alpha[\Lambda_{k-M}] = \alpha[(j_k - i_k) + \Lambda_{k-M}]$. Now there exists a subsequence of k such that $j_k - i_k = i$ is a constant. Since $\lim_{k\to\infty} \Lambda_k = \mathbb{Z}^2$, it follows that ie_1 is a period of α.

Applying the same argument to the $\mathbf{e}_2$ direction, we obtain $i'\mathbf{e}_2$ which is a period of α. Hence α is 2-periodic. □

Proof of Proposition 1.2. Let $\alpha = \cdots \alpha(-2)\alpha(-1); \alpha(0)\alpha(1)\alpha(2) \cdots \in A^{\mathbb{Z}}$.

The proof of (i) $\Leftrightarrow$ (ii) is analogous to the proof of Proposition 2.1 but simpler. (i) $\Rightarrow$ (iii) is trivial. In the following, we prove (iii)$\Rightarrow$ (i).

We may and do assume without loss of generality that α is recurrent in the positive direction. Notice that the one-sided word $\alpha^+ := \alpha(0)\alpha(1)\alpha(2) \cdots$ satisfies that

$$p^*_{\alpha^+}(k) \le p^*_\alpha(k) < 2k$$

for some $k = 1, 2, 3, \cdots$. Hence α^+ is eventually periodic by Property 1.

Let $p > 0$ be a period of α^+. We prove that p is a period of α. Suppose that this is not the case. Then, there exists i such that $\alpha(i) \ne \alpha(i + p)$. Hence

$$i_0 := \max\{i \in \mathbb{Z};\ \alpha(i) \ne \alpha(i + p)\}.$$

is a finite number. Let $N := |i_0| + p$, then $\tau = \{-N, \ldots, 0, \ldots, N\}$ is a window containing i_0 and $i_0 + p$. Since α is recurrent in positive direction, we have $\alpha[\tau] = \alpha[i + \tau]$ for some $i \ge 1$. Now $\alpha(i_0) \ne \alpha(i_0 + p)$ implies $\alpha(i_0 + i) \ne (i_0 + i + p)$, which contradicts the the maximality of i_0. Hence p must be a period of α. □

3 Monotone word

In this section, we show that our main theorem (Theorem 1.4) is valid for a class of special words, namely the monotone words. A two-dimensional word α over $\{0, 1\}$

is called a *monotone word* if

$$\alpha(x+\mathbf{e}_1) \geq \alpha(x), \quad \alpha(x+\mathbf{e}_2) \geq \alpha(x) \tag{3.1}$$

holds for all $x \in \mathbb{Z}^2$. We will show that a monotone word satisfying (#) has a structure very similar to the words in Example 1.5.

For $x \in \mathbb{Z}^2$, we denote by $C_x^- := x - \mathbb{N}\mathbf{e}_1 - \mathbb{N}\mathbf{e}_2$ the negative cone with vertex x, and $C_x^+ := x + \mathbb{N}\mathbf{e}_1 + \mathbb{N}\mathbf{e}_2$ the positive cone. The following lemma is trivial.

Lemma 3.1. *Let α be a monotone word. If $\alpha(x) = 1$, then α is constantly 1 in the cone C_x^+; if $\alpha(x) = 0$, then α is constantly 0 in the cone C_x^-.*

Let θ be a rational angle in the sense that $\tan\theta$ is a rational number. We denote by l_θ the half line starting at O with $\arg(l_\theta) = \theta$. Let L_θ be the set of integer points on the half line, which we write as $\{O = P_0, P_1, P_2, \ldots,\}$ so that $\|P_n - P_0\| = n\|P_1 - P_0\|$. Restricting α on the set L_θ, we obtain a 1-dimensional one-sided word φ_θ by setting $\varphi_\theta(n) = \alpha(P_n)$ $(n = 0, 1, 2, \cdots)$.

If we assume (#) for α, then $p^*_{\varphi_\theta}(k) \leq p^*_\alpha(k) < 2k$ for some $k = 1, 2, 3, \cdots$, and hence φ_θ is eventually periodic by Theorem B.

Type of φ_θ. We say that φ_θ is of *type*-0 if it contains only finitely many 1, and is of *type*-1 if it contains only finitely many 0. We say that φ_θ is *mixed* if it is neither type-0 nor type-1.

From now on, we assume that α is a non-constant monotone word. Then by Lemma 3.1, it is clear that φ_θ is of type-1 if $0 < \theta < \pi/2$; φ_θ is of type-0 if $\pi < \theta < 3\pi/2$. When θ belongs to $[\pi/2, \pi] \cup [3\pi/2, 2\pi]$, the type of φ_θ is not clear. But we have the following lemma.

Lemma 3.2. *Let α be a non-constant monotone word satisfying (#). If $\varphi_{\theta'}$ is either of type-1 or mixed for some $\theta' \in (\pi/2, \pi]$, then φ_θ is of type-1 for all $\theta \in [\pi/2, \theta')$.*

Proof. Take $\theta \in [\pi/2, \theta')$. Write $L_{\theta'} = (P_0, P_1, P_2, \ldots)$. Since $\varphi_{\theta'}$ is eventually periodic and contains infinitely many 1, there exist positive integers N, M such that $\varphi_{\theta'}(P_{N+iM}) = 1$ for any $i = 0, 1, 2, \cdots$. Hence α is constantly 1 in the cones $C^+_{P_{N+iM}}$. Note that since $\theta \in [\pi/2, \theta')$, the half line l_θ is contained in the union of those cones except for a finite part of the ray near O (see Figure 1). Thus, φ_θ is of type-1.

□

Similarly, under the assumption of Lemma 3.2, we have the following fact:

If $\varphi_{\theta'}$ is of type-1 or mixed for some $\theta' \in [\frac{3\pi}{2}, 2\pi)$, then φ_θ is of type-1 for all $\theta \in (\theta', 2\pi]$.

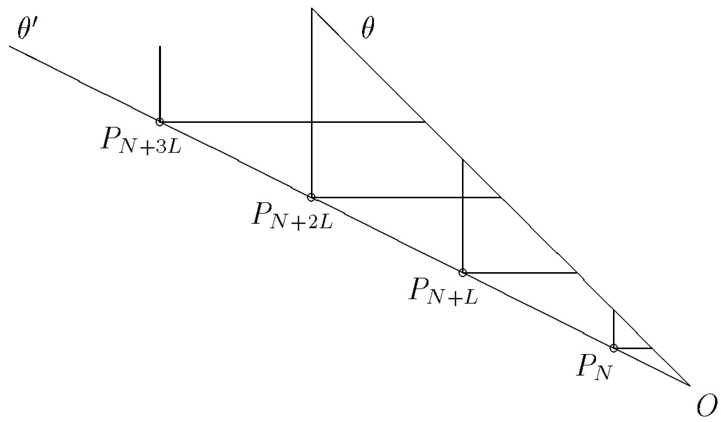

Figure 1:

If $\varphi_{\theta'}$ is of type-0 or mixed for some $\theta' \in [\frac{\pi}{2}, \pi)$, then φ_θ is of type-0 for all $\theta \in (\theta', \pi]$.

If $\varphi_{\theta'}$ is of type-0 or mixed for some $\theta' \in (\frac{3\pi}{2}, 2\pi]$, then φ_θ is of type-0 for all $\theta \in [\frac{3\pi}{2}, \theta')$.

Accordingly, there exist two angles $\theta_1 \in [\frac{\pi}{2}, \pi]$ and $\theta_2 \in [\frac{3\pi}{2}, 2\pi]$ such that φ_θ is of type-1 when $\theta_2 - 2\pi < \theta < \theta_1$; φ_θ is of type-0 when $\theta_1 < \theta < \theta_2$. We fix θ_1 and θ_2 as this.

Lemma 3.3. *Let α be a non-constant monotone word satisfying (#). Then for any $\epsilon > 0$ there exists $C > 0$ such that for any $\|x\| \geq C$,*

$$\alpha(x) = \begin{cases} 0 & \text{if } \theta_1 + \epsilon < \arg x < \theta_2 - \epsilon \\ 1 & \text{if } \theta_2 - 2\pi + \epsilon < \arg x < \theta_1 - \epsilon. \end{cases} \tag{3.2}$$

Proof. Choose two angles θ_3 and θ_4 with $\theta_1 - \epsilon < \theta_3 < \theta_1$ and $\theta_2 < \theta_4 < \theta_2 + \epsilon$. Then both of φ_{θ_3} and φ_{θ_4} are of 1-type.

Let us denote the set L_{θ_3} by $\{P_0, P_1, P_2, \dots,\}$ and denote the set L_{θ_4} by $\{Q_0, Q_1, Q_2, \dots,\}$. There exists a $N > 0$ such that $\alpha(P_n) = \alpha(Q_n) = 1$ for $n \geq N$. It is easy to see that the area $\{x;\ \theta_2 - 2\pi + \epsilon < \arg x < \theta_1 - \epsilon\}$ is covered by the cones $C^+_{P_n}$ and $C^+_{Q_n}$ $(n \geq N)$ except a finite part near O (see Figure 2). Therefore we have a half part of (3.2). The other half part can be proved similarly. □

Lemma 3.4. *It holds that $\theta_2 = \theta_1 + \pi$.*

Proof. Suppose that $\theta_2 \neq \theta_1 + \pi$. Then, either $\theta_2 < \theta_1 + \pi$ or $\theta_2 > \theta_1 + \pi$. Without loss of generality, let us assume $\theta_2 > \theta_1 + \pi$. Take $\epsilon > 0$ small. By Lemma 3.3, there

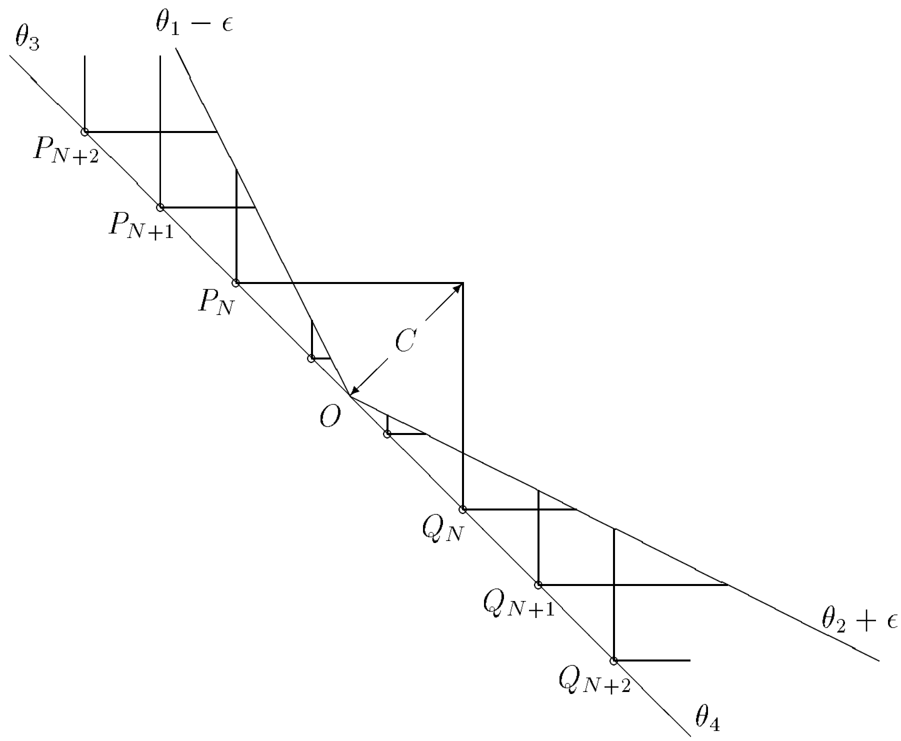

Figure 2:

exists $C > 0$ such that for any $\|x\| \geq C$, it holds

$$\alpha(x) = \begin{cases} 0 & \text{if } \theta_1 + \epsilon < \arg x < \theta_2 - \epsilon \text{ and } \|x\| \geq C \\ 1 & \text{if } \theta_2 + \epsilon - 2\pi < \arg x < \theta_1 - \epsilon \text{ and } \|x\| \geq C. \end{cases}$$

Let us call these two regions Region 0 and Region 1 respectively.

Take $N > 0$ and take a line ℓ which passes the point $N\mathbf{e}_1$ and has slope -1. Clearly ℓ will intersect both Region 0 and Region 1. We choose N sufficiently large so that ℓ does not intersect the circle with center O and radius C (see Figure 3). Hence the configuration of α on the line ℓ is (from left to right)

$$\cdots 0\ 0\ 0\ \ast \cdots \ast 1 \cdots 1 \ast \cdots \ast 0\ 0\ 0 \cdots,$$

where $\ast$ represents an undetermined letter 0 or 1 (Figure 3). The consecutive 1 in the middle can be arbitrarily long if we choose N large.

Take any $k > 0$, let τ be the k-window

$$\tau = \{i(\mathbf{e}_1 - \mathbf{e}_2) \in \mathbb{Z}^2;\ i = 0, 1, \cdots, k-1\}.$$

Moving this window along ℓ, we obtain at least the following patterns

$$\{\underbrace{0\cdots 0}_{i}\underbrace{1\cdots 1}_{k-i};\ i = 1, 2, \cdots, k\} \cup \{\underbrace{1\cdots 1}_{i}\underbrace{0\cdots 0}_{k-i};\ i = 1, 2, \cdots, k\} \subset F_\alpha(\tau).$$

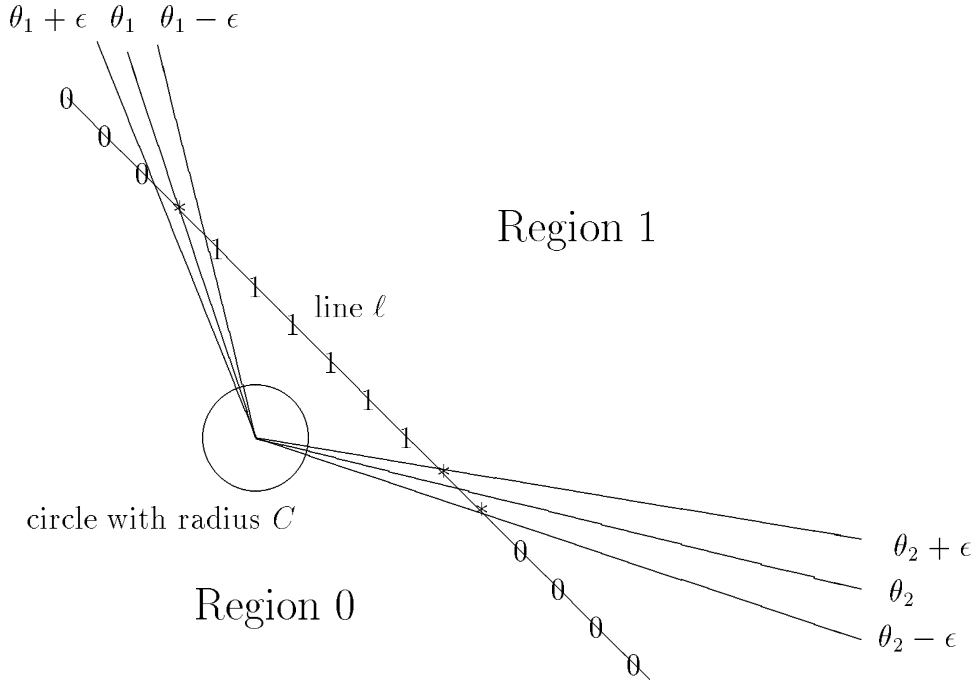

Figure 3:

Hence, $\sharp F_\alpha(\tau) \geq 2k$, which contradicts with (#). The lemma is proved. $\square$

Theorem 3.5. *If* $\alpha \in \{0,1\}^{\mathbb{Z}^2}$ *is a non-constant monotone word satisfying (#), then there exists an angle* θ^* *such that any recurrent angle* θ *of* α *satisfies that* $\theta \in \{\theta^*, \theta^* + \pi\}$*. Therefore, if* α *is a monotone word satisfying (#) which is strongly recurrent as well, then* α *is a constant word.*

Proof. Let us denote by θ^* the common value of θ_1 and $\theta_2 - \pi$ in Lemma 3.4. Suppose that θ is a recurrent angle of α such that $\theta \notin \{\theta^*, \theta^* + \pi\}$. Without out loss of generality, let us assume that $\theta^* < \theta < \theta^* + \pi$. Let $\delta = \min\{\theta - \theta^*, \theta^* + \pi - \theta\}/4$. We choose N_0 large so that $\alpha[\Lambda_{N_0}]$ contains at least one 1. For any $N > N_0$, there is a nonzero vector λ_N such that

$$\alpha[\Lambda_N] = \alpha[\lambda_N + \Lambda_N] \text{ and } |\arg\lambda_N - \arg\theta| < \delta.$$

Particularly, $\theta^* + 3\delta < \arg\lambda_N < \theta^* + \pi - 3\delta$.

Case 1: Suppose the sequence $\{\lambda_N\}_{N=1}^{\infty}$ is bounded. Then there is a vector λ which appears infinitely many times in the sequence. Since $\Lambda_N \to \mathbb{Z}^2$ as $N \to \infty$, we have $\alpha[\mathbb{Z}^2] = \alpha[\lambda + \mathbb{Z}^2]$ and hence, λ is a period of α.

Take any point x with $\alpha(x) = 1$. Let us denote by $l_{\arg\lambda,x}$ the half line starting at x which is parallel to $l_{\arg\lambda}$. Since $\theta^* < \arg\lambda < \theta^* + \pi$, sooner or later the half line $l_{\arg\lambda,x}$ will fall into the Region 0. Hence the configuration of α on $l_{\arg\lambda,x}$ is eventually 0. On the other hand, α takes 1 infinitely often on $l_{\arg\lambda,x}$ by the periodicity, which is a contraction.

Case 2: Suppose that $\{\lambda_N\}_{N=1}^{\infty}$ is unbounded. It it easy to see that when $\|\lambda_N\|$ is very large, the square $\lambda_N + \Lambda_{N_0}$ is contained in the cone $\{x;\ \theta + 2\delta < \arg x < \theta - 2\delta\}$. Let C be the constant in Lemma 3.3 with $\epsilon = 2\delta$. Then if $\|\lambda_N\|$ is larger than $C + 3N_0$, $\alpha[\lambda_N + \Lambda_{N_0}]$ is purely 0. Since $\alpha[\lambda_N + \Lambda_{N_0}] = \alpha[\Lambda_{N_0}]$ implies $\alpha[\lambda_N + \Lambda_{N_0}]$ contains at least one 1. This is a contradiction. □

4 Critical window and monotone lemma

The main purpose of this section is to prove the following theorem.

Theorem 4.1. *If a word $\alpha \in \{0,1\}^{\mathbb{Z}^2}$ is 2-recurrent satisfying (#), then there exist linearly independent vectors ξ and η in $\mathbb{Z}^2$ such that*

$$\alpha(x+\xi) \geq \alpha(x), \quad \alpha(x+\eta) \geq \alpha(x) \tag{4.1}$$

for all $x \in \mathbb{Z}^2$.

Critical window. Let τ be a k-window. We call τ' an *immediate extension* of τ if τ' is a $k+1$-window containing τ as a subset.

Lemma 4.2. *If α is a word satisfying (#), then there exists a window τ such that*

$$\sharp F_\alpha(\tau') \leq \sharp F_\alpha(\tau) + 1 \tag{4.2}$$

holds for any immediate extension τ'.

We call such a window τ a *critical window*. This idea comes from [KZ2].

Proof. If $p_\alpha^*(k) < 2k$ for $k = 1$, then the word α consists of just one letter and the lemma is true. So we assume that $p_\alpha^*(1) \geq 2$. Let k be the smallest integer such that $p_\alpha^*(k+1) < 2(k+1)$. Then, $p_\alpha^*(k) \geq 2k$. Let τ be a k-window which attains $p_\alpha^*(k)$. That is, $\sharp F_\alpha(\tau) = p_\alpha^*(k)$. Then, for any immediate extension $\tau'$of τ, we have

$$\sharp F_\alpha(\tau') \leq p_\alpha^*(k+1) \leq p_\alpha^*(k) + 1 = \sharp F_\alpha(\tau) + 1,$$

which proves (4.2). □

For a window τ, we say that a square Λ_N is a *sample square* if

$$\{\alpha[x+\tau];\ x \in \Lambda_N\} = F_\alpha(\tau).$$

It amounts to say that all the patterns of τ appear at some position in Λ_N.

Lemma 4.3. (Monotone Lemma) *Let τ be a critical window of a word α, and Λ_N a sample square of τ. If a nonzero vector $\xi \in \mathbb{Z}^2$ satisfies $\alpha[\Lambda_N] = \alpha[\xi + \Lambda_N]$, then we have the dichotomy either*

$$\alpha(x+\xi) \geq \alpha(x) \text{ for all } x \in \mathbb{Z}^2 \quad \text{or} \tag{4.3}$$

$$\alpha(x+\xi) \leq \alpha(x) \text{ for all } x \in \mathbb{Z}^2. \tag{4.4}$$

Proof. First, let us consider the case that $\xi \in \tau$. Take any $\omega \in F_\alpha(\tau)$, then there exists $x \in \Lambda_N$ such that $\omega = \alpha[x+\tau]$. The assumption $\alpha[\xi+\Lambda_N] = \alpha[\Lambda_N]$ yields $\omega(O) = \alpha(x) = \alpha(x+\xi) = \omega(\xi)$. Since ω is an arbitrary pattern of α through τ, we conclude that $\alpha(x) = \alpha(x+\xi)$ holds for any $x \in \mathbb{Z}^2$. The lemma is proved in this case.

Next, we consider the case that $\xi \notin \tau$. Let $\tau' = \tau \cup \{\xi\}$. Take any $\omega \in F_\alpha(\tau)$. We define $\omega' \in A^{\tau'}$ by $\omega'(z) = \omega(z)$ for any $z \in \tau$ and $\omega'(\xi) = \omega(O)$. Let $\omega = \alpha[x+\tau]$ with $x \in \Lambda_N$. Then, we have $\omega'(z) = \omega(z) = \alpha(x+z)$ for any $z \in \tau$. Moreover, since $\alpha(x+\xi) = \alpha(x)$, we have $\omega'(\xi) = \omega(O) = \alpha(x) = \alpha(x+\xi)$. Hence, $\omega' = \alpha[x+\tau']$.

Thus, we proved that ω' is an element in $F_\alpha(\tau')$ which is an extension of $\tau \in F_\alpha(\tau)$. Since $\sharp F_\alpha(\tau') \leq \sharp F_\alpha(\tau)+1$, there exists at most one $\omega \in F_\alpha(\tau)$ which has an extension ω'' to $F_\alpha(\tau')$ other than ω'.

Assume that $\omega''(\xi) > \omega'(\xi) = \omega(O)$. Take any $x \in \mathbb{Z}^2$. If $\eta := \alpha[x+\tau] \neq \omega$, then we have $\alpha(x+\xi) = \alpha(x)$ since η has the unique extension η' to $F_\alpha(\tau')$ satisfying that $\eta'(\xi) = \eta(O)$. If $\alpha[x+\tau] = \omega$, then we have $\alpha(x+\xi) \geq \alpha(x)$ since $\omega'(\xi) = \omega(O)$ and $\omega''(\xi) > \omega(O)$. In any case, we have $\alpha(x+\xi) \geq \alpha(x)$ for any $x \in \mathbb{Z}^2$.

Assume that $\omega''(\xi) < \omega'(\xi) = \omega(O)$. Then, by the same argument as above, we have $\alpha(x+\xi) \leq \alpha(x)$ for any $x \in \mathbb{Z}^2$.

The two formulas imply (4.3) and (4.4) respectively. □

Proof of Theorem 4.1. Since $p^*_\alpha(k) < 2k$ for some $k = 1, 2, 3, \cdots$, there exists a critical window τ with a sample square Λ_N. Since α is 2-recurrent, there exist linearly independent integer vectors ξ and η such that $\alpha[\Lambda_N] = \alpha[\xi + \Lambda_N] = \alpha[\eta + \Lambda_N]$. If (4.3) holds for ξ, we use ξ for ξ in (4.1). If (4.4) holds for ξ, we use $-\xi$ for ξ in (4.1). Do the same for η. Thus we have (4.1). □

5 Lattice decomposition

Let α be a strongly recurrent word and satisfying (#). Then there exist ξ and η such that (4.1) hold. Let

$$\Omega := \{s\xi + t\eta \in \mathbb{Z}^2;\ 0 \le s < 1 \text{ and } 0 \le t < 1\} \tag{5.1}$$

be the set of integers in the polygon with vertices O, ξ, η and $\xi + \eta$. Let us write $\Omega = \{x_0, x_2, \ldots, x_{q-1}\}$. Take $x_i \in \Omega$, we define a *lattice subword* of α by restricting α on the lattice $x_i + \xi\mathbb{Z} + \eta\mathbb{Z}$. We denote this subword by $\alpha^{(i)}$. Clearly α is the union of the subwords $\alpha^{(i)}$ for $i = 1, \ldots, q$.

Corresponding to $\alpha^{(i)}$, we define a word β_i as

$$\beta_i(m, n) = \alpha(x_i + m\xi + n\eta) \quad (i = 0, 1 \cdots, q-1). \tag{5.2}$$

Then it is easy to see that β_i's are monotone words satisfying (#). We are going to show that β_i's are strongly recurrent, and hence constant words by Theorem 3.5. The next lemma implies that if θ is a recurrent direction of α, then it is also a recurrent direction of $\alpha^{(i)}$'s.

Lemma 5.1. *Assume that θ is a recurrent angle of α. Then for any $N > 0$ and $\delta > 0$, there is a nonzero vector $\lambda \in \xi\mathbb{Z} + \eta\mathbb{Z}$ such that*

$$\alpha[\lambda + \Lambda_N] = \alpha[\Lambda_N] \quad \textit{and} \quad |\arg\lambda - \theta| < \delta.$$

Proof. Since θ is a recurrent angle, there exists a vector λ_0 such that

$$\alpha[\lambda + \Lambda_{N_0}] = \alpha[\Lambda_N], \quad |\arg\lambda_0 - \theta| < \delta/q.$$

We choose N_1 large enough so that $\lambda + \Lambda_{N_0}$ is contained in the square Λ_{N_1}. Then we choose λ_1 such that

$$\alpha[\lambda_1 + \Lambda_{N_1}] = \alpha[\Lambda_{N_1}], \quad |\arg\lambda_1 - \theta| < \delta/q.$$

In this way, if N_k and λ_k are chosen, we can choose N_{k+1} large enough so that $\Lambda_{N_k} + \lambda_k \subset \Lambda_{N_{k+1}}$, and then choose λ_{k+1} to be a vector satisfying that

$$\alpha[\lambda_{k+1} + \Lambda_{N_{k+1}}] = \alpha[\Lambda_{N_{k+1}}], \quad |\arg\lambda_{k+1} - \theta| < \delta/q.$$

Set $\lambda'_k = \lambda_0 + \lambda_1 + \cdots \lambda_k$, then among $\lambda'_0, \lambda'_1, \ldots, \lambda'_q$, at least two of them are in the same residue class modulo the lattice $\xi\mathbb{Z} + \eta\mathbb{Z}$. Say, λ'_i and λ'_j $(i < j)$ are in the same residue class, then $\lambda := \lambda'_j - \lambda'_i = \lambda_{i+1} + \cdots \lambda_j$ belongs to the lattice $\xi\mathbb{Z} + \eta\mathbb{Z}$. It is easy to check that

$$\alpha[\lambda + \Lambda_N] = \alpha[\Lambda_N] \text{ and } |\arg\lambda - \theta| < (j-i)\delta/q \le \delta. \quad \square$$

Let T be the linear transformation determined by $T\xi = \mathbf{e}_1,\ T\eta = \mathbf{e}_2$.

Lemma 5.2. *If a half line l starting at O is a recurrent direction of α, then Tl is a recurrent direction of β_i for any $i = 0, 1, \cdots, q-1$.*

Proof. Let l be a half line starting at O which is a recurrent direction of α. Let $x_i \in \Omega$. Take any $N > 0$ and $\delta > 0$. Since T is nonsingular, there exist $M > 0$, $\epsilon > 0$ and $C > 0$ such that $T^{-1}\Lambda_N + x_i \subset \Lambda_M$ and that $|\arg(T\lambda) - \arg(Tl)| < \delta$ holds for any vector λ with $|\arg\lambda - \arg l| < \epsilon$. By Lemma 5.1, there exists a nonzero vector $\lambda \in \xi\mathbb{Z} + \eta\mathbb{Z}$ such that

$$\alpha[\lambda + \Lambda_M] = \alpha[\Lambda_M] \quad \text{and} \quad |\arg\lambda - \arg l| < \epsilon.$$

Then, $T\lambda$ is a nonzero vector blonging to $\mathbb{Z}^2$ such that $\beta_i[T\lambda + \Lambda_N] = \beta_i[\Lambda_N]$. Moreover, $|\arg(T\lambda) - \arg(Tl)| < \delta$. Thus, Tl is a recurrent direction of β_i for $i = 0, 1, \cdots, q-1$. □

Proof of Theorem 1.4: It remains to prove that (iii) implies (i). As we pointed out before, this can be reduced to the case that $A = \{0, 1\}$.

Assume that $\alpha \in \{0,1\}^{\mathbb{Z}^2}$ satisfies the condition (iii) of Theorem 1.4. By Theorem 4.1, there exist linearly independent vectors ξ and η in $\mathbb{Z}^2$ such that

$$\alpha(x + \xi) \geq \alpha(x), \quad \alpha(x + \eta) \geq \alpha(x)$$

for all $x \in \mathbb{Z}^2$. Let Ω be as (5.1) wyth these ξ and η. Let $q = \#\Omega$. Then, α is decomposed into q lattice subword $\alpha^{(i)}$ with $i = 0, \ldots, q-1$, such that the corresponding words β_i's defined in (5.2) are monotone.

The word β_i's still satisfy (#). Since α has two linearly independent recurrent directions, so do β_i's (Lemma 5.2). Hence, β_i's are constant words by Theorem 3.5. Hence the lattice subwords $\alpha^{(i)}$'s are constant which implies that α is two-periodic. Actually, ξ and η are two linear independent periods of α. □

6 Examples of 2-dimensional pattern Sturmian word

Example 6.1. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the torus. Let $0 < \theta < 1$ be a real number. Let r, s be two real numbers. Let $\mathcal{P} = \{I_0, I_1\}$ be a partition of $\mathbb{T}$, where $I_0 = [0, \theta), I_1 = [\theta, 1)$. Define a word α by

$$\alpha(m, n) = i \text{ if } mr + ns \in I_i.$$

This word has been studied in [ABI]. Here we show that α is a Sturmian word provided at least one of r, s is irrational.

Let $\tau = \{(0,0), (m_1, n_1), \ldots, (m_{k-1}, n_{k-1})\}$ be a k-window. Denote $S - x = \{s - x;\ s \in S\}$. A factor of α, $a_0 a_1 \cdots a_{k-1}$ is a pattern through τ if and only if

$$I_{a_0} \ \cap\ (I_{a_1} - m_1 r - n_1 s) \ \cap \cdots \cap\ (I_{a_{k-1}} - m_{k-1} r - n_{k-1} s) \neq \emptyset.$$

Therefore,

$$\sharp F_\alpha(\tau) \leq \sharp\,(\mathcal{P}\ \vee\ (\mathcal{P}-m_1r-n_1s)\ \vee\cdots\vee\ (\mathcal{P}-m_{k-1}r-n_{k-1}s)), \tag{6.1}$$

where"$\vee$" is the common refinement of partitions. Since the right side of (6.1) is no greater than the number of the end points of the intervals

$$I_i - m_j r - n_j s \quad (i=0,1;\ j=0,1,\cdots,k-1),$$

we have $p^*_\alpha(k) \leq 2k$ $(k=1,2,\cdots)$. Suppose r is irrational, then the factor $\alpha(m,0), m\in\mathbb{Z}$ is an one-dimensional pattern Sturmian word, and hence $p^*_\alpha(k) = 2k$ $(k=1,2,\cdots)$. Moreover, one can show that α is strongly recurrent.

Example 6.2. For an integer n, we define $\deg_2 n$ to be the largest integer such that $2^p|n$. Define α as

$$\alpha(m,n) = \begin{cases} 1 & \text{if } \deg_2 m = \deg_2 n \\ 0 & \text{otherwise.} \end{cases}$$

It is shown in [KX] that $p^*_\alpha(k) = 2k$. This word is also strongly recurrent.

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