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Width Deviation of Convex Polygons

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Abstract

We consider the width $X_T(\omega)$ of a convex n-gon T in the plane along the random direction $\omega \in \mathbb{R}/2\pi\mathbb{Z}$ and study its deviation rate:

$$\delta(X_T) = \frac{\sqrt{\mathbb{E}(X_T^2) - \mathbb{E}(X_T)^2}}{\mathbb{E}(X_T)}.$$

We prove that the maximum is attained if and only if T degenerates to a 2-gon. Let $n \geq 2$ be an integer which is not a power of 2. We show that

$$\sqrt{\frac{\pi}{4n \tan(\pi/(2n))} + \frac{\pi^2}{8n^2 \sin^2(\pi/(2n))} - 1}$$

is the minimum of $\delta(X_T)$ among all n-gons and determine completely the shapes of T's which attain this minimum. They are characterized as polygonal approximations of equi-Reuleaux bodies, found and studied by Reinhardt (Jahresber. Deutsch. Math. Verein. **31**, 251–270 (1922)). In particular, if n is odd, then the regular n-gon is one of the minimum shapes. When n is even, we see that regular n-gon is far from optimal. We also observe an unexpected property of the deviation rate on the truncation of the regular triangle.

Keywords Width distribution · Convex polygon · Minimum deviation rate · Reinhalt polygon

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1 Introduction

The width of an image along various directions is basic information in the Image Processing Technique. We are interested in the deviation of the widths of compact convex sets in the plane along a random direction. Although it is an important and useful quantity both practically and theoretically, as far as we know, there is no serious study on it.

Let $\Omega = \mathbb{R}/2\pi\mathbb{Z} = (-\pi, \pi]$ and $\mathbb{P}$ be the normalized Lebesgue measure on Ω. Consider the probability space $(\Omega, \mathbb{P})$; we write Ω for short. Let a compact convex set T in the complex plane $\mathbb{C}$ be given. We identify T with its boundary. We discuss the length of the orthogonally projected shadow of T by the light from a random direction ω, say $X_T(\omega)$. It is also interpreted as the width of T along the orthogonal direction of ω, that is, $\omega \pm \pi/2$.

We are interested in the uniformity (or its contrary) of the deviation of $X_T(\omega)$ with respect to $\omega \in \Omega$. For this purpose, we consider the *deviation rate* of the random variable X_T, say $\delta(X_T)$, defined by

$$\delta(X_T) = \frac{\sqrt{\mathbb{E}(X_T^2) - \mathbb{E}(X_T)^2}}{\mathbb{E}(X_T)},$$

where $\mathbb{E}(\ )$ is the expectation of the random variables. Clearly, $\delta(X_T)$ is invariant among the similarity images of T. It is also clear that $\delta(X_T) = 0$ if and only if T is a closed curve of constant width. The family of closed curves of constant width is rich, and attracted many researchers. In Sect. 2, we review Reuleaux bodies briefly. For additional background, the reader may consult [5, 6, 8].

Hence, $\delta(X_T)$ measures how far T is from convex bodies of constant width. For the ellipse $T = \{(x, y) : x^2/a^2 + y^2/b^2 = 1\}$ with the perimeter L, it is not difficult to see that

$$\delta(X_T) = \frac{\sqrt{2\pi^2(a^2 + b^2) - L^2}}{L}.$$

In this paper, we study the deviation rate of the convex n-gons T with $n \geq 2$. We decompose X_T as the sum of random variables coming from its edges, say $X_T = \sum_{j=1}^{n} X_{\alpha_j}$. Throughout this paper, the branch of the argument of a complex number is chosen to be $(-\pi, \pi]$. For a complex number $\alpha \in \mathbb{C}$ with $\arg(\alpha) = \theta$, define a random variable X_α on Ω by

$$X_\alpha(\omega) = |\alpha| \sin(\theta - \omega)_+$$

(this means $(\sin(\theta - \omega))_+$), where $x_+ = \max\{x, 0\}$. In other words, $X_\alpha(\omega)$ is the length of the orthogonally projected shadow of the right side of the vector $\overrightarrow{\mathbf{0}\alpha}$ by

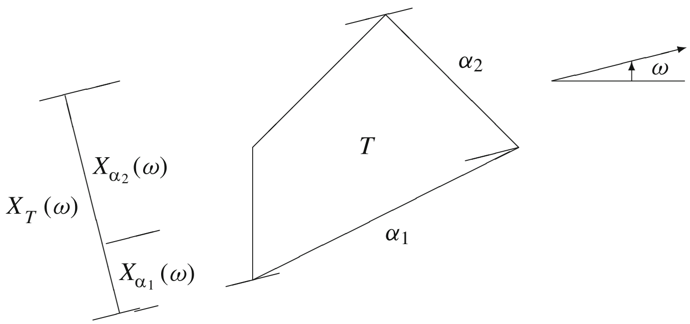

Fig. 1 $X_T(\omega)$

the light from the ω direction (the left side of the vector is permeable and makes no shadow).

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$, $n \geq 2$, be a sequence of distinct complex numbers. They are arranged in counter-clockwise order forming a convex n-gon if and only if

$$\begin{aligned} \arg(\alpha_2 - \alpha_1) &\leq \arg(\alpha_3 - \alpha_2) \leq \ldots \\ &\leq \arg(\alpha_{n+1} - \alpha_n) \leq \arg(\alpha_2 - \alpha_1) + 2\pi \pmod{2\pi}, \end{aligned} \tag{1}$$

where *we always consider the suffix j related to the n-gon T in modulo n*, so that $\alpha_{n+1} = \alpha_1, \alpha_{n+2} = \alpha_2$, etc. In this case, the convex n-gon with vertices $\alpha_1, \ldots, \alpha_n$ is denoted by $T = T(\alpha_1, \ldots, \alpha_n)$.

The above T is non-degenerate (i.e., all the vertices are extremal points) if and only if "$<$" holds everywhere in the above inequalities. If this is not the case, then we identify $T = T(\alpha_1, \ldots, \alpha_n)$ with $T(\alpha'_1, \ldots, \alpha'_m)$, where $\{\alpha'_1, \ldots, \alpha'_m\}$ are the set of extremal points among $\{\alpha_1, \ldots, \alpha_n\}$ arranged in counter clockwise order. Define a random variable X_T by

$$X_T(\omega) = \sum_{j=1}^{n} X_{\alpha_{j+1}-\alpha_j}(\omega),$$

which agrees with the former explanation as to the length of the shadow of T (see Fig. 1).

The set of convex n-gons $T = T(\alpha_1, \ldots, \alpha_n)$ can be identified with the sequence of n distinct complex numbers $(\alpha_1, \ldots, \alpha_n)$ satisfying (1). We denote the space consisting of these $(\alpha_1, \ldots, \alpha_n)$ by Θ_n. We consider the usual structures coming from $\mathbb{C}^n$ on Θ_n. Then, $\bigcup_{k=2}^{n-1} \Theta_k$ is considered as the boundary of Θ_n in the above sense.

Denote the factor space of Θ_n divided by the similarity equivalence by $\tilde{\Theta}_n$. Then it is a compact space. Most of our notions like the regular n-gon are notions in $\tilde{\Theta}_n$ rather than Θ_n. The deviation rate $\delta(X_T)$ can be considered as a continuous and piecewise smooth functional on $\tilde{\Theta}_n$. Hence, it has the minimum and the maximum in Θ_n.

In this paper, we prove the following results.

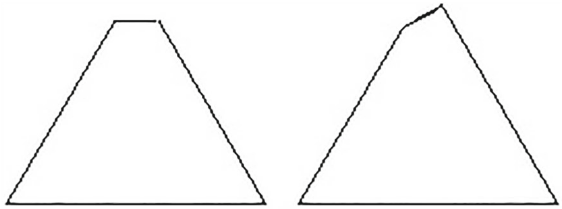

Fig. 2 Parallel truncation (left) and non-parallel truncation (right)

(i) The deviation rate for the compact convex sets T attains the maximum if and only if T is the 2-gon (Theorem 1).

(ii) For the integer $n \geq 2$ which is not a power of 2, the minimum of $\delta(X_T)$ for $T \in \Theta_n$ is

$$\nu_n := \sqrt{\frac{\pi}{4n \tan(\pi/(2n))} + \frac{\pi^2}{8n^2 \sin^2(\pi/(2n))} - 1}. \tag{2}$$

Theorem 5 gives a complete characterization of the shapes that attain the minimum value ν_n. As a consequence, we show that the minimum shape is nothing but a Reinhardt n-gon (Theorem 6). The minimum value for odd $n \geq 3$ is strictly decreasing in n.

(iii) For even $m \geq 4$, the regular m-gon is far from the minimum shape. Let $n(< m)$ be the odd number such that either $n = m/2$ or $n = m/2 + 1$. If $n = m/2$, then $\delta(X_{T_m}) = \delta(X_{T_n})$ holds, and if $n = m/2 + 1$, then $\delta(X_{T_m}) > \delta(X_{T_n})$ holds, where T_m, T_n are the regular m-gon and n-gon, respectively (Theorem 3).

(iv) Let T be the regular triangle. We call a *truncation* of it a quadrangle obtained by cutting off a vertex by a line near it. It is called a *parallel* truncation if the line is parallel to the opposite side, otherwise a *non-parallel* truncation (Fig. 2). We prove that a small parallel truncation of the regular triangle increases the deviation rate, while a small non-parallel truncation decreases it (Theorem 7).

2 Review on Ruleaux Bodies and Reinhardt Polygons

It turned out that the solution of the minimization problem of $\delta(X_T)$ is related to the convex bodies of constant width. A *Reuleaux body*[1] is a convex body of constant width whose boundary consists of a finite number of circular arcs with the center in it and the radius equal to the width. For the self-containedness, we review the Reuleaux body here briefly.

Let D be a Reuleaux body with width r. Then, $C = \partial D$ consists of a finite number of circular arcs, say $C_1, \ldots, C_p$, of radius r. The endpoints of the circular arcs are

[1] In literature, it is often referred to as a "Reuleaux polygon", though its boundary is not linear. In this paper, we use the word "body" to distinguish it from a genuine polygon.

called *essential* if the two neighboring circular arcs have different centers, and hence cannot be joined into a single circular arc.

We assume that the circular arcs $C_1, \dots, C_p$ are arranged in counter clockwise order and all the endpoints are essential. Let A_1, A_2 be the endpoints of C_1; A_2, A_3 be the endpoints of $C_2, \dots$; A_p, A_{p+1} be the endpoints of C_p (the suffixes of C or A are considered modulo p so that $A_{p+1} = A_1$). Note that a center of any one of the circular arcs is an essential endpoint of some other circular arc on C, since otherwise, we have two points in C with a distance larger than r. Conversely, any endpoint is a center of some circular arc since there must be the same number of circular arcs and centers.

Hence, the center of C_1 is one of $A_3, \dots, A_p$. Let it be A_k. Since D is strictly convex, the tangent vector of C at $c \in C$ to the counter clockwise direction rotates counter clockwise as c moves. The center of the circular arc containing $c \in C$ is the other intersection of the normal line to the tangent vector at c with C. This intersection moves at the endpoints of the circular arcs in the counter clockwise direction since the normal lines from the same point rotate in the counter clockwise direction. Hence, the center of this circular arc is the next end point to that before. This means that if the center of C_1 is A_k, then the center of C_2 is $A_{k+1}, \dots$

On the other hand, the center of the circular arc C_k must be A_2 since otherwise, either it is A_1 or there is $l \neq 1, 2$ such that $\overline{A_k A_l} = r$. The latter is impossible since if so, by the convexity and the assumption on C, the circular arc C_1 can be extended to A_l contradicting the assumption that both A_1 and A_2 are essential end points. The former is impossible since if so, then we have a contradiction that $\overline{A_2 A_{k+1}} > r$. Thus, we have

$$A_2 = \text{ the center of } C_k = A_{2k-1},$$

and hence, $2 \equiv 2k - 1 \pmod{p}$. This implies that p is odd.

Consider the diagonals connecting two points in C, one of the center and an interior point of the circular arc centered by it. Any of two diagonals with different centers always intersect just at one point. Let Λ_j be the set of angles from A_j to the points in the circular arc centered by A_j. We always consider the angles modulo 2π. Then, by the above argument, Λ_j and Λ_l with $j \neq l$ are essentially disjoint. The same thing holds for Λ_j and $\Lambda_l + \pi$. Let $\Lambda = \bigcup_{j=1}^{p} \Lambda_j$. Then, the union is essentially disjoint. Moreover, Λ and $\Lambda + \pi$ are also essentially disjoint. It also holds that $\Lambda \cup (\Lambda + \pi) = (-\pi, \pi]$. This is because the Lebesgue measure of Λ is π, since the integration of the exterior angle of C is 2π, which counts the angles in Λ_j, $j = 1, \dots, p$, twice, once at C_j, once at its center, where the exterior angle jumps just the amount of angles in Λ_j.

Thus, it holds that the circular arcs $C_1, \dots, C_p$ can be embedded by parallel translations by $\mathbf{v}_1, \dots, \mathbf{v}_p \in \mathbb{C}$ into a circle $\mathbb{S}$ of radius r so that

$$\tilde{\Lambda} := \bigcup_{j=1}^{p} (C_j + \mathbf{v}_j) \subset \mathbb{S}$$

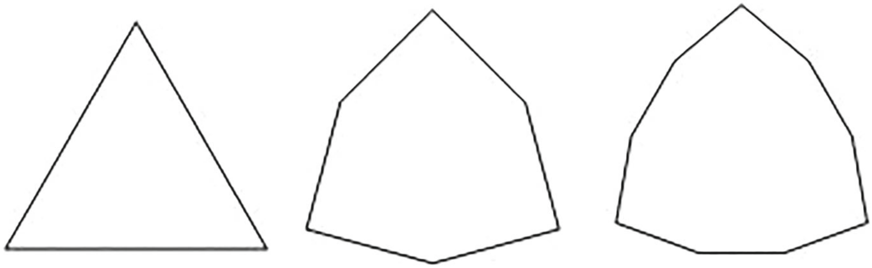

Fig. 3 Reinhardt i^3-gon for $i = 1, 2, 3$

satisfies that

$$\begin{aligned} &\#\left((C_j + \mathbf{v}_j) \cap (C_k + \mathbf{v}_k)\right) < \infty \quad \text{for any } j \neq k, \\ &\#\,(\tilde{\Lambda} \cap R_\pi \tilde{\Lambda}) < \infty \quad \text{and} \quad \tilde{\Lambda} \cup R_\pi \tilde{\Lambda} = \mathbb{S}, \end{aligned} \tag{3}$$

where R_π is the rotation on $\mathbb{S}$ by angle π, and "#" denotes the number of elements in a set.

We write a *Reuleaux p-body* to indicate the number p of circular arcs with different centers. In particular, a Reuleaux p-body is called *regular* if all the circular arcs have the same length. A regular Reuleaux 3-body is well known as "Reuleaux triangle".

A *Reinhardt n-gon* [3, 9] is an equilateral convex n-gon that can be inscribed in a Reuleaux body, containing all the essential endpoints of the circular arcs. It is referred to as a *Reinhaldt polygon* if n is not necessarily specified. If the lengths of the circular arcs of a Reuleaux p-body have ratio $n_1 : n_2 : \ldots : n_p$ with integer n_i's, we can divide the circular arcs into n_i, $i = 1, \ldots, p$, parts of equal length and take the convex-hull of all the division points to get a Reinhardt n-gon, where $n = n_1 + \cdots + n_p$, which is called a Reinhardt $(n_1, \ldots, n_p)$-gon.

In particular, let $p \geq 3$ be an odd integer and take a regular Reuleaux p-body D. We divide all the circular arcs of D into q parts of equal length. Then, the convex hull of all the division points forms a Reinhardt pq-gon, which is a Reinhardt q^p-gon, where

$$q^p = (\underbrace{q, \ldots, q}_{p \text{ times}})$$

for short; see Fig. 3. Note that a Reinhardt 1^p-gon is the regular p-gon.

A Reinhardt polygon gives the solution of three optimization problems on convex n-gons when n is not a power of 2:

Problem 1 Maximize the perimeter for a fixed diameter [9].
Problem 2 Maximize the width for a fixed diameter [2].
Problem 3 Maximize the width for a fixed perimeter [1].

Given a cyclic integer vector $(n_1, n_2, \ldots, n_p)$, there exists a unique way to construct a Reinhardt $(n_1, n_2, \ldots, n_p)$-gon when this is possible. We reproduce a necessary and

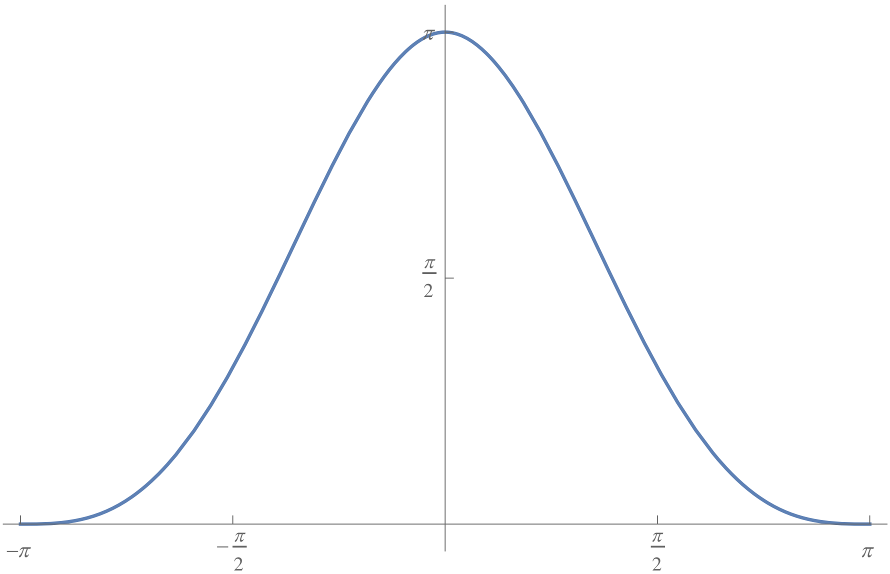

Fig. 4 Function V

sufficient condition of Reinhardt [9] that a given integer cyclic vector $(n_1, n_2, \ldots, n_p)$ forms a Reinhardt $(n_1, n_2, \ldots, n_p)$-gon as a byproduct of our result (Theorems 5 and 6). In particular, when n is not a power of 2, there exists a Reinhardt $(n/p)^p$-gon for any odd divisor $p > 1$. Many interesting properties on Reinhardt polygons are discussed in [3, 4, 6, 7]. When n is odd, the regular n-gon is one of the Reinhardt polygons but it is not unique when n is in addition a composite. The Reinhardt n-gon is unique if and only if $n = p$ or $2p$ with p an odd prime. A Reinhardt polygon may have no symmetry at all (see the figures in [3]).

3 Maximum of the Deviation Rate

Lemma 1 *For any* $\alpha, \beta \in \mathbb{C}$ *with* $\eta = \arg(\beta/\alpha) \in (-\pi, \pi]$, *the following statements hold:*

(i) $\mathbb{E}(X_\alpha) = |\alpha|/\pi$,
(ii) $\mathbb{E}(X_\alpha X_\beta) = |\alpha| \cdot |\beta| \cdot V(\eta)/(4\pi)$, *where* V *is a periodic function of period* 2π *such that*

$$V(x) = (\pi - |x|)\cos x + \sin|x|, \qquad -\pi < x \leq \pi \text{ (See Fig. 4).}$$

Proof (i) Let $\theta = \arg(\alpha)$. Then, we have

$$\mathbb{E}(X_\alpha) = \frac{|\alpha|}{2\pi}\int_0^{2\pi} \sin(\theta - \xi)_+ \, d\xi = \frac{|\alpha|}{2\pi}\int_0^{\pi} \sin\xi \, d\xi = \frac{|\alpha|}{\pi}.$$

(ii) We have

$$\begin{aligned}\mathbb{E}(X_\alpha X_\beta) &= \frac{|\alpha|\cdot|\beta|}{2\pi}\int_0^{2\pi}\sin(\theta-\xi)_+\sin(\theta+\eta-\xi)_+\,d\xi\\ &= \frac{|\alpha|\cdot|\beta|}{2\pi}\int_{[0,\pi]\cap[-\eta,\pi-\eta]}\sin\xi\,\sin(\xi+\eta)\,d\xi\\ &= \frac{|\alpha|\cdot|\beta|}{4\pi}\int_{[0,\pi]\cap[-\eta,\pi-\eta]}(\cos\eta-\cos(2\xi+\eta))\,d\xi.\end{aligned}$$

Therefore, if $\eta<0$, then

$$\begin{aligned}\mathbb{E}(X_\alpha X_\beta) &= \frac{|\alpha|\cdot|\beta|}{4\pi}\int_{-\eta}^{\pi}(\cos\eta-\cos(2\xi+\eta))\,d\xi\\ &= \frac{|\alpha|\cdot|\beta|}{4\pi}((\pi+\eta)\cos\eta-\sin\eta) = \frac{|\alpha|\cdot|\beta|}{4\pi}V(\eta).\end{aligned}$$

If $\eta\geq 0$, then

$$\begin{aligned}\mathbb{E}(X_\alpha X_\beta) &= \frac{|\alpha|\cdot|\beta|}{4\pi}\int_0^{\pi-\eta}(\cos\eta-\cos(2\xi+\eta))\,d\xi\\ &= \frac{|\alpha|\cdot|\beta|}{4\pi}((\pi-\eta)\cos\eta+\sin\eta) = \frac{|\alpha|\cdot|\beta|}{4\pi}V(\eta).\end{aligned}$$ □

Theorem 1 *The maximum of $\delta(X_T)$ for the compact convex sets T is attained by a 2-gon.*

Proof Since all 2-gons are similar, they have the same δ-value. Let $U=T(0,1)$ be a 2-gon. Then by Lemma 1, we have

$$\mathbb{E}(X_U)=\frac{1}{\pi}+\frac{1}{\pi}=\frac{2}{\pi},\qquad \mathbb{E}(X_U^2)=\frac{2V(0)+2V(\pi)}{4\pi}=\frac{1}{2}.$$

Hence, $\delta(X_U)=\sqrt{(\pi^2/8)-1}$.

Take any convex polygon $T=T(\alpha_1,\dots,\alpha_n)$. We prove $\delta(X_T)\leq\sqrt{(\pi^2/8)-1}$, and the equality holds only when T is a 2-gon. Let

$$\beta_j=\alpha_{j+1}-\alpha_j=r_je^{\mathbf{i}\theta_j},\qquad j=1,\dots,n,$$

and let $\theta_{jk}\in(-\pi,\pi]$ satisfy that

$$\theta_{jk}\equiv\theta_j-\theta_k \pmod{2\pi},\qquad j,k=1,\dots,n.$$

It is sufficient to prove that

$$\frac{\mathbb{E}(X_T^2)}{\mathbb{E}(X_T)^2}\leq\frac{\pi^2}{8}.$$

Hence, it is sufficient to prove

$$I := \frac{\pi^3}{2}\mathbb{E}(X_T)^2 - 4\pi\mathbb{E}(X_T^2) \geq 0.$$

Then by Lemma 1, we have

$$\begin{aligned} I &= \frac{\pi^3}{2}\mathbb{E}(X_T)^2 - 4\pi\,\mathbb{E}(X_T^2) = \frac{\pi^3}{2}\left(\sum_{j=1}^{n}\mathbb{E}(X_j)\right)^2 - 4\pi\sum_{j,k=1}^{n}\mathbb{E}(X_jX_k) \\ &= \frac{\pi^3}{2}\left(\sum_{j=1}^{n}\frac{r_j}{\pi}\right)^2 - \sum_{j,k=1}^{n} r_jr_kV(\theta_j-\theta_k) = \sum_{j,k=1}^{n} r_jr_k\left(\frac{\pi}{2} - V(\theta_j-\theta_k)\right) \\ &= \sum_{j,k=1}^{n} r_jr_k\left(\frac{\pi}{2} - (\pi - |\theta_{jk}|)\cos\theta_{jk} - |\sin\theta_{jk}|\right). \end{aligned} \tag{4}$$

Since $r_jr_k\cos\theta_{jk}$ is the inner product between $\overrightarrow{\mathbf{0}\beta_j}$ and $\overrightarrow{\mathbf{0}\beta_k}$, we have

$$\sum_{j,k=1}^{n} r_jr_k\cos\theta_{jk} = \left\langle \sum_{j=1}^{k}\overrightarrow{\mathbf{0}\beta_j},\ \sum_{j=1}^{k}\overrightarrow{\mathbf{0}\beta_j}\right\rangle = \langle\,\vec{\mathbf{0}}\,,\,\vec{\mathbf{0}}\,\rangle = 0.$$

This implies that in the above equality (4), the constant in the coefficient of $\cos\theta_{jk}$ can be changed anyway keeping the equality. Hence, we have

$$I = \sum_{j,k=1}^{n} r_jr_k\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - |\theta_{jk}|\right)\cos\theta_{jk} - |\sin\theta_{jk}|\right).$$

Thus, to prove $I \geq 0$, it is sufficient to prove that

$$\left(\frac{\pi}{2} - |x|\right)\cos x + |\sin x| \leq \frac{\pi}{2}$$

for any $x \in (-\pi, \pi]$, which can be verified easily. Moreover, the equality holds if and only if $x = 0$ or π, which implies that T is a 2-gon.

For a general compact convex set S with $C = \partial S$, replacing the above I by

$$I = \iint_{C\times C}\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta(u,v)\right)\cos\theta(u,v) - |\sin\theta(u,v)|\right)du\,dv,$$

where $\theta(u,v)$ is the angle between the tangent lines of C at u and v, we have the same statement, which completes the proof. ◻

4 Symmetrization and Asymmetrization of *n*-Gons

Let

$$\mathcal{H} = \{\alpha \in \mathbb{C} : \Im(\alpha) > 0, \text{ or } \Im(\alpha) = 0 \text{ and } \Re(\alpha) \geq 0\}$$

be the upper half plane endowed with the quotient topology of $\mathbb{C}$ by identifying z and $-z$. For $\alpha \in \mathbb{C}$, we define

$$\iota(\alpha) = \begin{cases} \alpha & \alpha \in \mathcal{H}, \\ -\alpha & \alpha \notin \mathcal{H}. \end{cases}$$

For a finite set of nonzero complex numbers $S = \{\alpha_1, \ldots, \alpha_n\}$, define $\iota(S) \subset \mathcal{H}$ by

$$\iota(S) = \text{Abbreviation}\,\{\iota(\alpha_1), \ldots, \iota(\alpha_n)\},$$

where Abbreviation $\{\alpha'_1, \ldots, \alpha'_n\}$ is the set of complex numbers obtained by replacing any pair α'_j, α'_k having $\arg(\alpha'_j) = \arg(\alpha'_k)$ by $\alpha'_j + \alpha'_k$.

A nonzero element in $\mathcal{H}$ is sometimes called a *pre-edge*. For a sequence of pre-edges $\beta_1, \ldots, \beta_m \in \mathcal{H}$, we call $\mathcal{B} = \mathcal{B}(\beta_1, \ldots, \beta_m)$ a *pre-edge bundle* (of size m) if

$$0 \leq \arg(\beta_1) < \arg(\beta_2) < \ldots < \arg(\beta_m) < \pi.$$

Denote by Ξ_m the set of pre-edge bundles of size m. For a convex n-gon $T = T(\alpha_1, \ldots, \alpha_n)$, we define its *asymmetrization* $\iota(T)$ as the pre-edge bundle $\mathcal{B} = \mathcal{B}(\beta_1, \ldots, \beta_m)$ such that

$$\iota(\{\alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \ldots, \alpha_{n+1} - \alpha_n\}) = \{\beta_1, \beta_2, \ldots, \beta_m\}.$$

In this case, T is called a *realization* of $\mathcal{B}$. Let $T = T(\alpha_1, \ldots, \alpha_n)$ be a convex n-gon and $\mathcal{B} = \mathcal{B}(\beta_1, \ldots, \beta_m)$ be its asymmetrization. Then,

$$U = T\left(\gamma, \gamma + \frac{\beta_1}{2}, \ldots, \gamma + \frac{\beta_1 + \cdots + \beta_m}{2}, \gamma + \frac{\beta_2 + \cdots + \beta_m}{2}, \ldots, \gamma + \frac{\beta_m}{2}\right)$$

is another realization of $\mathcal{B}$, where $\gamma = -(\beta_1 + \cdots + \beta_m)/4$. We call U the *symmetrization* of T (or $\mathcal{B}$). In this case, U is *symmetric*, that is, $U = T(\gamma_1, \ldots, \gamma_{2k})$ with even size $2k$ and

$$\gamma_{k+1} = -\gamma_1, \quad \gamma_{k+2} = -\gamma_2, \quad \ldots, \quad \gamma_{2k} = -\gamma_k$$

holds. For a symmetric $U = T(\gamma_1, \ldots, \gamma_{2k})$, it holds that

$$X_U(\omega) = 2|\gamma_j| \sin\left(\arg(\gamma_j) - \omega\right) \tag{5}$$

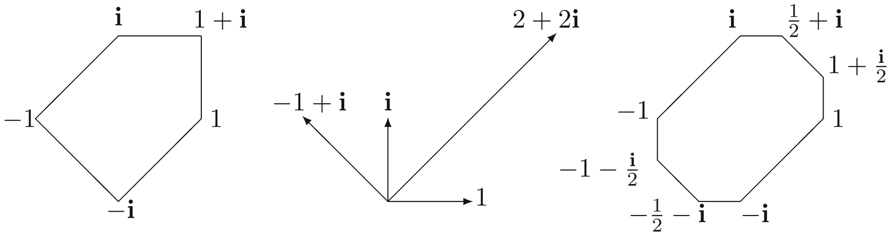

Fig. 5 T (left), $\mathcal{B}$ (center), and U (right)

for any ω with $\arg(\gamma_{j-1}-\gamma_j) \le \omega \le \arg(\gamma_j-\gamma_{j+1})$. This representation of X_U is called the *diagonal representation*.

Example 1 Let $T = T(-\mathbf{i}, 1, 1+\mathbf{i}, \mathbf{i}, -1)$. Then, its asymmetrization is $\mathcal{B} = \mathcal{B}(1, 2+2\mathbf{i}, \mathbf{i}, -1+\mathbf{i})$ and its symmetrization is

$$U = T\left(-\frac{1}{2}-\mathbf{i}, -\mathbf{i}, 1, 1+\frac{\mathbf{i}}{2}, \frac{1}{2}+\mathbf{i}, \mathbf{i}, -1, -1-\frac{\mathbf{i}}{2}\right).$$

See Fig. 5.

Let β be a pre-edge with $\arg(\beta) = \theta$. Define a random variable $\tilde{X}_\beta$ as

$$\tilde{X}_\beta(\omega) = \frac{|\beta|}{2}|\sin(\theta-\omega)|,$$

and for a pre-edge bundle $\mathcal{B} = \mathcal{B}(\beta_1, \ldots, \beta_m)$, let $\tilde{X}_\mathcal{B} = \sum_{j=1}^m \tilde{X}_{\beta_j}$.

Theorem 2 *If T is a realization of $\mathcal{B}$, then we have $X_T = \tilde{X}_\mathcal{B}$.*

Proof Since $T = T(\alpha_1, \ldots, \alpha_n)$ is a convex polygon, we have $X_T(\omega) = X_T(\omega+\pi)$ for any $\omega \in \Omega$. Hence,

$$\begin{aligned}
X_T(\omega) &= \frac{X_T(\omega)+X_T(\omega+\pi)}{2} = \sum_{j=1}^n \frac{X_{\alpha_j}(\omega)+X_{\alpha_j}(\omega+\pi)}{2} \\
&= \sum_{j=1}^n \frac{|\alpha_j|}{2}(\sin(\theta_j-\omega)_+ + \sin(\theta_j-\omega-\pi)_+) \\
&= \sum_{j=1}^n \frac{|\alpha_j|}{2}(\sin(\theta_j-\omega)_+ + (-\sin(\theta_j-\omega))_+) \\
&= \sum_{j=1}^n \frac{|\alpha_j|}{2}|\sin(\theta_j-\omega)| = \sum_{j=1}^n \tilde{X}_{\iota(\alpha_j)} = \sum_{j=1}^m \tilde{X}_{\beta_j} = \tilde{X}_\mathcal{B}.
\end{aligned}$$ ◻

5 Minimum of Deviation Rate Among the Pre-Edge Bundles of Fixed Size

A regular n-gon is denoted by T_n, $n = 2, 3, \ldots$ A pre-edge bundle $\mathcal{B}(\beta_1, \ldots, \beta_m)$ is said to be *regular* if

$$|\beta_1| = \ldots = |\beta_m|, \qquad \arg(\beta_{j+1}) - \arg(\beta_j) \equiv \frac{\pi}{m} \ (\mathrm{mod}\ \pi), \qquad j = 1, \ldots, m.$$

Let R_m be the regular pre-edge bundle of size m.

Theorem 3 (i) *With ν_m defined in (2), it holds that*

$$\delta(\tilde{X}_{R_m}) = \nu_m$$

for $m = 1, 2, \ldots$, and hence, $\delta(\tilde{X}_{R_1}) > \delta(\tilde{X}_{R_2}) > \delta(\tilde{X}_{R_3}) > \ldots$

(ii) *If $m \geq 2$ is even, then let n $(< m)$ be the odd number such that either $n = m/2$ or $n = m/2 + 1$. If $n = m/2$, then $\delta(X_{T_m}) = \delta(X_{T_n}) = \delta(\tilde{X}_{R_n})$ holds, and if $n = m/2 + 1$, then $\delta(X_{T_m}) = \delta(\tilde{X}_{R_{m/2}}) > \delta(\tilde{X}_{R_n}) = \delta(X_{T_n})$ holds.*

Proof Statement (ii) holds since both of T_n and T_{2n} are realizations of R_n if $n = m/2$ is odd, and $\delta(X_{T_n}) = \delta(X_{T_{2n}}) = \delta(X_{T_m}) = \delta(\tilde{X}_{R_n})$ by Theorem 2. If $n = m/2 + 1$, then by the monotonicity in (i), $\delta(X_{T_n}) = \delta(\tilde{X}_{R_n}) < \delta(\tilde{X}_{R_{m/2}})$.

By Theorem 2, to prove (i), it is sufficient to prove that

$$\delta(X_{T_{2m}}) = \nu_m, \qquad m = 1, 2, \ldots,$$

for $T_{2m} = T\left(e^{\mathbf{i}\pi/2m}, e^{\mathbf{i}3\pi/2m}, \ldots, e^{\mathbf{i}\pi(1+2(2m-1))/2m}\right)$. Using the diagonal representation (5), we have

$$\begin{aligned}\mathbb{E}(X_{T_{2m}}) &= \frac{2m}{2\pi} \int_{-\pi/2}^{-\pi/2+\pi/m} 2\sin\left(\frac{\pi}{2m} - \omega\right) d\omega \\ &= \frac{2m}{\pi} \int_{-\pi/(2m)}^{\pi/(2m)} \sin\left(\frac{\pi}{2} - \omega\right) d\omega = \frac{2m}{\pi} \int_{-\pi/(2m)}^{\pi/(2m)} \cos\omega \, d\omega,\end{aligned}$$

hence

$$\mathbb{E}(X_{T_{2m}}) = \frac{4m}{\pi} \sin\frac{\pi}{2m}.$$

Also,

$$\begin{aligned}\mathbb{E}(X_{T_{2m}}^2) &= \frac{2m}{2\pi} \int_{-\pi/2}^{-\pi/2+\pi/m} 4\sin^2\left(\frac{\pi}{2m} - \omega\right) d\omega \\ &= \frac{4m}{\pi} \int_{-\pi/(2m)}^{\pi/(2m)} \sin^2\left(\frac{\pi}{2} - \omega\right) d\omega\end{aligned}$$

$$= \frac{4m}{\pi} \int_{-\pi/(2m)}^{\pi/(2m)} \cos^2 \omega \, d\omega = \frac{2m}{\pi} \sin \frac{\pi}{m} + 2.$$

Thus,

$$\delta(X_{T_{2m}}) = \sqrt{\frac{\mathbb{E}(X_{T_{2m}}^2)}{\mathbb{E}(X_{T_{2m}})^2} - 1} = \sqrt{\frac{\pi \sin(\pi/m)}{8m \sin^2(\pi/(2m))} + \frac{\pi^2}{8m^2 \sin^2(\pi/(2m))} - 1}$$
$$= \sqrt{\frac{\pi}{4m \tan(\pi/(2m))} + \frac{\pi^2}{8m^2 \sin^2(\pi/(2m))} - 1} = \nu_m.$$

Let $x = \pi/(2m)$ and I be the term inside the root in the above formula. Then, we have

$$I = \frac{x}{2 \tan x} + \frac{x^2}{2 \sin^2 x} - 1.$$

We show that I is an increasing function of $x \in (0, \pi/2]$. We have

$$I'(x) = \frac{\cos x \sin x - x}{2 \sin^2 x} + \frac{x \sin x - x^2 \cos x}{\sin^3 x}$$
$$= \frac{\cos x \sin^2 x + x \sin x - 2x^2 \cos x}{2 \sin^3 x}.$$

Since

$$\cos x \le 1 - \frac{x^2}{2} + \frac{x^4}{24} \quad \text{and} \quad \sin x \ge x - \frac{x^3}{6},$$

we have

$$\begin{aligned}
&(2 \sin^3 x) I'(x) \\
&= \cos x \sin^2 x + x \sin x - 2x^2 \cos x \\
&\ge x\left(x - \frac{x^3}{6}\right) - (2x^2 - \sin^2 x)\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) \\
&\ge x\left(x - \frac{x^3}{6}\right) - 2x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) + \left(x - \frac{x^3}{6}\right)^2\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) \\
&= \frac{11x^6}{72} - \frac{x^8}{36} + \frac{x^{10}}{864} = \frac{x^6}{864}(x^4 - 24x^2 + 132)
\end{aligned}$$

which is positive on $x \in (0, \pi/2]$. Thus, $I(x)$ is strictly increasing in x, and hence, $\delta(X_{T_m})$ is strictly decreasing in $m = 1, 2, \ldots$ □

By a numerical calculation, we have $\delta(\tilde{X}_{R_m})$ as follows.

m	1	2	3	4	5	6	7
δ	0.48342	0.09772	0.04196	0.02333	0.01485	0.01028	0.00754

Theorem 4 *The deviation rate $\delta(\tilde{X}_{\mathcal{B}})$ attains the minimum among $\mathcal{B} \in \Xi_m$ if and only if $\mathcal{B} = R_m$.*

Proof We use induction on m. If $m = 1$, the statement is clear since R_1 is essentially the only element in Ξ_1. Let $m \geq 2$ and assume that the statement holds for Ξ_j, $j = 1, \dots, m-1$. The boundary of the closure of Ξ_m consists of $\bigcup_{j=1}^{m-1} \Xi_j$, and the δ-values there are larger than $\delta(\tilde{X}_{R_m})$ by the assumption of the induction and Theorem 3. Hence, there is $\mathcal{B}_0 \in \Xi_m$ attaining the minimum of $\delta(\tilde{X}_{\mathcal{B}})$ in Ξ_m. We prove that $\mathcal{B}_0 = R_m$.

For this purpose, we take the symmetrization $T_0 \in \Theta_{2m}$ of $\mathcal{B}_0$. Then, $\delta(X_{T_0})$ is minimal among $\delta(X_T)$ for symmetric $T \in \Theta_{2m}$. This is equivalent to saying that $\kappa(X_{T_0})$ is minimum among $\kappa(X_T)$ for symmetric $T \in \Theta_{2m}$, where $\kappa(X_T) = \mathbb{E}(X_T^2)/\mathbb{E}(X_T)^2$. We'll conclude from this that T_0 is the regular $2m$-gon.

Let $T_0 = T(\alpha_1, \dots, \alpha_{2m})$. Consider the diagonal representation (5) of X_{T_0}. Then, for any $j = 1, \dots, 2m$, we have

$$X_{T_0}(\omega) = 2|\alpha_j| \sin(\arg(\alpha_j) - \omega) \quad \text{ if } \omega \in \Omega_j,$$

where

$$\Omega_j = \{\omega \in \Omega : \arg(\alpha_{j-1} - \alpha_j) < \omega \leq \arg(\alpha_j - \alpha_{j+1})\}.$$

For a fixed $j = 1, \dots, m$ and a real number λ near 0, let

$$T_0^{\mathbf{i}\lambda} = T\big(\alpha_1, \dots, (1 + \mathbf{i}\lambda)\alpha_j, \alpha_{j+1}, \dots, (1 + \mathbf{i}\lambda)\alpha_{j+m}, \dots, \alpha_{2m}\big).$$

By the minimality, we must have

$$\frac{d\kappa(X_{T_0^\lambda})}{d\lambda}\bigg|_{\lambda=0} = 0.$$

Let

$$A = \mathbb{E}(X_{T_0}), \quad B = \mathbb{E}(X_{T_0}^2), \quad a_j = \mathbb{E}(X_{T_0} 1_{\Omega_j}), \quad b_j = \mathbb{E}(X_{T_0}^2 1_{\Omega_j}).$$

Then, we have

$$\frac{\mathbb{E}({X_{T_0^\lambda}}^2)}{\mathbb{E}(X_{T_0^\lambda})^2} = \frac{B - 2b_j + 2(1+\lambda)^2 b_j}{(A - 2a_j + 2(1+\lambda)a_j)^2} + o(\lambda).$$

Therefore, we have

$$0 = \frac{d\kappa(X_{T_0^\lambda})}{d\lambda}\bigg|_{\lambda=0} = \frac{4b_j A - 4a_j B}{A^3},$$

and hence,

$$\frac{b_j}{a_j} = \frac{B}{A} \qquad \text{for any } j = 1, \dots, 2m.$$

Denoting

$$u_j = \arg(\alpha_j) - \arg(\alpha_{j-1} - \alpha_j) - \frac{\pi}{2}, \quad v_j = \arg(\alpha_j) - \arg(\alpha_j - \alpha_{j+1}) - \frac{\pi}{2},$$

it holds that

$$\begin{aligned}
a_j &= \mathbb{E}(X_{T_0} 1_{\Omega_j}) = \int_{\arg(\alpha_{j-1}-\alpha_j)}^{\arg(\alpha_j - \alpha_{j+1})} 2|\alpha_j| \sin(\arg(\alpha_j) - \omega) \frac{d\omega}{2\pi} \\
&= \frac{|\alpha_j|}{\pi} \int_{u_j+\pi/2}^{v_j+\pi/2} (-1) \sin\omega \, d\omega \\
&= \frac{|\alpha_j|}{\pi} \left(\cos\left(v_j + \frac{\pi}{2} \right) - \cos\left(u_j + \frac{\pi}{2} \right) \right) \\
&= \frac{|\alpha_j|}{\pi} (\sin u_j - \sin v_j) \\
b_j &= \mathbb{E}(X_{T_0}^2 1_{\Omega_j}) = \int_{\arg(\alpha_{j-1}-\alpha_j)}^{\arg(\alpha_j - \alpha_{j+1})} 2|\alpha_j|^2 \sin^2(\arg(\alpha_j) - \omega) \frac{d\omega}{2\pi} \\
&= \frac{|\alpha_j|^2}{\pi} \int_{u_j+\pi/2}^{v_j+\pi/2} (-1) \sin^2\omega \, d\omega \\
&= \frac{|\alpha_j|^2}{\pi} \left(\frac{u_j - v_j}{2} + \frac{\sin(2v_j + \pi) - \sin(2u_j + \pi)}{4} \right) \\
&= \frac{|\alpha_j|^2}{\pi} \left(\frac{u_j - v_j}{2} + \frac{\sin 2u_j - \sin 2v_j}{4} \right).
\end{aligned}$$

For $j = 1, \dots, 2m$, let

$$n_j = \text{the perpendicular leg from 0 to the line } \alpha_{j-1}\alpha_j.$$

Then, the above u_j and v_j have another representation (see Fig. 6) that

$$\begin{aligned}
u_j = \arg(\alpha_j) - \arg(n_j), \qquad v_j &= \arg(\alpha_j) - \arg(n_{j+1}) \in (-\pi, \pi], \\
u_j - v_j &= \text{ the exterior angle at } \alpha_j > 0,
\end{aligned}$$

and we have

$$\frac{B}{A} = \frac{|\alpha_j|(2u_j - 2v_j + \sin 2u_j - \sin 2v_j)}{4(\sin u_j - \sin v_j)} \qquad \text{for } j = 1, \dots, 2m. \tag{6}$$

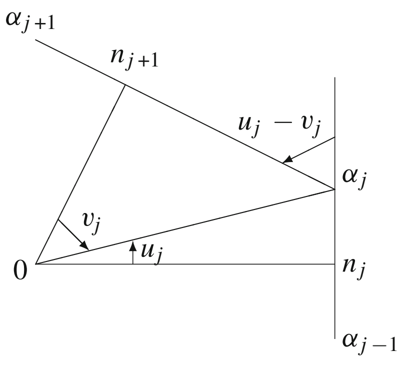

Fig. 6 $\alpha_{j-1}, \alpha_j, \alpha_{j+1}, n_j, n_{j+1}, u_j, v_j$

For a fixed $j = 1, \ldots, m$ and a real number λ near 0, let

$$T_0^{\mathbf{i}\lambda} = T\big(\alpha_1, \ldots, (1+\mathbf{i}\lambda)\alpha_j, \alpha_{j+1}, \ldots, (1+\mathbf{i}\lambda)\alpha_{j+m}, \ldots, \alpha_{2m}\big).$$

Then, we have

$$\frac{\mathbb{E}(X_{T_0^\lambda}{}^2)}{\mathbb{E}(X_{T_0^\lambda})^2} = \frac{B + 2d_j^2\lambda/(2\pi) - 2d_{j+1}^2\lambda/(2\pi)}{(A + 2d_j\lambda/(2\pi) - 2d_{j+1}\lambda/(2\pi))^2} + o(\lambda),$$

where

$$\begin{aligned} d_j &= X_{T_0}(\arg(\alpha_{j-1} - \alpha_j)) \\ &= 2|\alpha_j| \sin(\arg(\alpha_j) - \arg(\alpha_{j-1} - \alpha_j)) = 2|\alpha_j| \cos u_j, \\ d_{j+1} &= X_{T_0}(\arg(\alpha_j - \alpha_{j+1})) \\ &= 2|\alpha_j| \sin(\arg(\alpha_j) - \arg(\alpha_j - \alpha_{j+1})) = 2|\alpha_j| \cos v_j. \end{aligned}$$

Therefore, we have

$$0 = \frac{d\kappa(X_{T_0^{\mathbf{i}\lambda}})}{d\lambda}\bigg|_{\lambda=0} = \frac{(d_j^2 - d_{j+1}^2)A/\pi - 2(d_j - d_{j+1})B/\pi}{A^3},$$

and hence, either $d_j = d_{j+1}$ or $(d_j + d_{j+1})/2 = B/A$.

If $(d_j + d_{j+1})/2 = B/A$ holds, then by (6), we have

$$\frac{|\alpha_j|(2u_j - 2v_j + \sin 2u_j - \sin 2v_j)}{4(\sin u_j - \sin v_j)} = |\alpha_j|(\cos u_j + \cos v_j).$$

Hence,

$$2(u_j - v_j) = \sin 2u_j - \sin 2v_j + 4\sin(u_j - v_j).$$

Since $u_j - v_j > 0$ and if $u_j - v_j \le \pi/2$, then $\sin(u_j - v_j) \ge 2(u_j - v_j)/\pi$, we have a contradiction that $\sin 2u_j - \sin 2v_j + 4\sin(u_j - v_j) > 2(u_j - v_j)$. Therefore, $u_j - v_j > \pi/2$ and the exterior angle of α_j is larger than $\pi/2$. By the symmetry, the exterior angle of α_k for $k \ne j, j+\pi$ is smaller than $\pi/2$. Therefore, for these k, $(d_j + d_{j+1})/2 = B/A$ is impossible, and hence, $d_k = d_{k+1}$. Thus, $u_k = -v_k > 0$. This implies $|n_k| = |n_{k+1}|$ for any $k \ne j, j+m$.

In any case, we have $|n_k| = |n_{k+1}|$ except for $k = j, j+m$. This implies that there are two classes

$$|n_{j+1}| = |n_{j+2}| = \ldots = |n_{j+m}|,$$
$$|n_{j+m+1}| = |n_{j+m+2}| = \ldots = |n_{j+2m}|$$

(suffixes are considered modulo $2m$). By symmetry, the values of these two classes coincide. Thus, $|n_1| = |n_2| = \ldots = |n_{2m}|$, which implies that T_0 has an inscribed circle with radius $r := |n_1|$. Hence, $u_j = -v_j$, $j = 1, \ldots, 2m$. Then by (6),

$$\begin{aligned}\frac{B}{A} &= \frac{|\alpha_j|(2u_j + \sin 2u_j)}{4\sin u_j} = \frac{|\alpha_j|(2u_j + \sin 2u_j)\cos u_j}{4\sin u_j \cos u_j}\\ &= \frac{r(2u_j + \sin 2u_j)}{2\sin 2u_j} = \frac{r}{2}\cdot\frac{2u_j}{\sin 2u_j} + \frac{r}{2}, \qquad j = 1, \ldots, 2m.\end{aligned}$$

It follows from this that $2u_j/(\sin 2u_j)$ is the same for $j = 1, \ldots, 2m$. Since the correspondence $x \mapsto x/(\sin x)$ for $x > 0$ is one-to-one, we have $u_1 = \ldots = u_{2m}$. Thus, T_0 has the same exterior angle $2u_j$ at vertex α_j for $j = 1, \ldots, 2m$. Together with the fact that T_0 has an inscribed circle, T_0 is a regular $2m$-gon, which completes the proof. □

6 Minimum of Deviation Rate Among *n*-Gons

Lemma 2 *If $n \ge 2$, then the set*

$$P(n) := \left\{ (c_0, \ldots, c_{n-1}) \in \{-1, 1\}^n : \sum_{j=0}^{n-1} c_j \exp\frac{j\pi\mathbf{i}}{n} = 0 \right\}$$

is empty if and only if n is a power of 2.

Proof If n is not a power of 2, then take an odd factor p of n. Denote $j = 0, 1, \ldots, n-1$ as

$$j = \frac{kn}{p} + \ell, \qquad \ell = 0, 1, \ldots, \frac{n}{p} - 1, \quad k = 0, 1, \ldots, p-1.$$

Given $c_\ell \in \{-1, 1\}$ and $\ell = 0, 1, \ldots, n/p - 1$ arbitrarily, define

$$c_j = c_\ell(-1)^k, \qquad j = 0, 1, \ldots, n-1.$$

Then, we have

$$
\begin{aligned}
\sum_{j=0}^{n-1} c_j e^{j\pi \mathbf{i}/n} &= \sum_{j=0}^{n-1} c_\ell (-1)^k \exp \frac{(kn/p+\ell)\pi \mathbf{i}}{n} \\
&= \sum_{\ell=0}^{n/p-1} c_\ell \exp \frac{\ell\pi \mathbf{i}}{n} \sum_{k=0}^{p-1} (-1)^k \exp \frac{k\pi \mathbf{i}}{p} \\
&= \sum_{\ell=0}^{n/p-1} c_\ell \exp \frac{\ell\pi \mathbf{i}}{n} \left(\sum_{k=0}^{(p-1)/2} \exp \frac{2k\pi \mathbf{i}}{p} - \sum_{k=1}^{(p-1)/2} \exp \frac{(2k-1)\pi \mathbf{i}}{p} \right) \\
&= \sum_{\ell=0}^{n/p-1} c_\ell \exp \frac{2\pi \ell \mathbf{i}}{n} \left(\sum_{k=0}^{(p-1)/2} \exp \frac{2k\pi \mathbf{i}}{p} + \sum_{k=1}^{(p-1)/2} \exp \frac{(2k-1+p)\pi \mathbf{i}}{p} \right) \\
&= \sum_{\ell=0}^{n/p-1} c_\ell \exp \frac{2\pi \ell \mathbf{i}}{n} \left(\sum_{k=0}^{(p-1)/2} \exp \frac{2k\pi \mathbf{i}}{p} + \sum_{k=(p+1)/2}^{p-1} \exp \frac{2k\pi \mathbf{i}}{p} \right) \\
&= \sum_{\ell=0}^{n/p-1} c_\ell \exp \frac{2\pi \ell \mathbf{i}}{n} \sum_{k=0}^{p-1} \exp \frac{2k\pi \mathbf{i}}{p} = 0.
\end{aligned}
$$

Therefore $P(n)$ contains a non-empty subset

$$
Q(p) := \left\{ (c_0, \dots, c_{n-1}) \in \{-1, 1\}^n : c_{kn/p+\ell} = c_\ell (-1)^k \right\}
$$

with $\#(Q(p)) = 2^{n/p}$. Thus we see that $P(n)$ is non-empty. Next assume that $n = 2^s$ with $s \geq 1$. Since $x^n + 1 \in \mathbb{Z}[x]$ is the minimum polynomial of $w := \exp(\pi \mathbf{i}/n)$,

$$
\sum_{j=0}^{n-1} c_j \exp \frac{j\pi \mathbf{i}}{n} = c_0 + c_1 w + \cdots + c_{n-1} w^{n-1}
$$

cannot be 0 for any $(c_0, c_1, \dots, c_{n-1}) \in \{-1, 1\}^n$. Hence, $P(n) = \emptyset$. □

Theorem 5 *Let $n \geq 2$ be an integer which is not a power of 2. Then we have*

$$
\min_{T \in \Theta_n} \delta(X_T) = \delta(\tilde{X}_{R_n}) = \nu_n
$$

(*see* (2)). *The minimum is attained if and only if the asymmetrization of T is the regular pre-edge bundle R_n, and hence, if and only if T is similar to the polygon $T(\alpha_1, \dots, \alpha_n)$ with*

$$
\{\alpha_{j+1} - \alpha_j : j = 1, \dots, n\} = \left\{ c_j \exp \frac{j\pi \mathbf{i}}{n} : (c_0, \dots, c_{n-1}) \in P(n) \right\}.
$$

Here $P(n)$ is defined in Lemma 2.

Proof In light of Theorem 4, $T^* \in \Theta_n$ attains the minimum of $\delta(X_T)$ among $T \in \Theta_n$ if T^* is a realization of the regular pre-edge bundle R_n. A realization of R_n, say T, may have more than n edges but this number is reduced to n if and only if there exists a one-to-one correspondence between the set of pre-edges $\{\beta_1, \ldots, \beta_n\}$ of R_n and the set of edges of T, i.e., there exists $(c_0, c_1, \ldots, c_{n-1}) \in \{-1, 1\}^n$ such that $\{c_0\beta_0, c_1\beta_1, \ldots, c_{n-1}\beta_{n-1}\}$ is the set of edges of T. Since we may assume $\beta_j = \exp(j\pi\mathbf{i}/n)$ for $i = 0, \ldots, n-1$, this condition is satisfied if and only if

$$\sum_{j=0}^{n-1} c_j \exp\frac{j\pi\mathbf{i}}{n} = 0 \tag{7}$$

is solvable in $c_j \in \{-1, 1\}$, i.e., $P(n)$ is non-empty. □

Hereafter $n > 1$ is always assumed to be an integer which is not a power of 2. There is a natural map σ and τ from $P(n)$ to itself defined by

$$\begin{aligned}\sigma((c_0, c_1, \ldots, c_{n-1})) &= (c_1, \ldots c_{n-1}, -c_0) \quad \text{and}\\ \tau((c_0, c_1, \ldots, c_{n-1})) &= (c_{n-1}, c_{n-2}, \ldots, c_0),\end{aligned}$$

which corresponds to the symmetry of dihedral group D_{2n}: the rotation of angle π/n and the reflection. Two elements $(c_0, \ldots, c_{n-1})$ and $(c'_0, \ldots, c'_{n-1})$ of $P(n)$ give congruent realizations if and only if

$$(c_0, \ldots, c_{n-1}) = \sigma^j(c'_0, \ldots, c'_{n-1}) \quad \text{or} \quad (c_0, \ldots, c_{n-1}) = \sigma^j\tau(c'_0, \ldots, c'_{n-1})$$

for some $j \in \{0, \ldots, 2n-1\}$. Lemma 2 and Theorem 5 can be restated in a geometric form.

Theorem 6 *If* $1 < n \in \mathbb{N}$ *is not a power of* 2*, then the minimum polygon in* Θ_n *is a Reinhardt* n*-gon and vice versa. A Reinhardt* n*-gon exists if and only if* n *is not a power of* 2.

Though it is not stated in this manner, the latter statement follows from the characterization of Reinhardt [9], see the discussion after the proof.

Proof Assume that $1 < n \in \mathbb{N}$ is not a power of 2. By Theorem 5, the minimum n-polygon T^* can be considered, without loss of generality, to have the asymmetrization

$$\mathcal{B}\left(c\exp\frac{\pi\mathbf{i}}{2n}, c\exp\frac{3\pi\mathbf{i}}{2n}, \ldots, c\exp\frac{(2n-1)\pi\mathbf{i}}{2n}\right)$$

for some appropriate $c > 0$, so that its symmetrization is

$$T_{2n} = T\left(e^{-n\pi\mathbf{i}/(2n)}, e^{(-n+2)\pi\mathbf{i}/(2n)}, \ldots, e^{(3n-2)\pi\mathbf{i}/(2n)}\right).$$

Let $T^* = T(P_1, P_2, \ldots, P_n)$. Since $X_{T^*} = X_{T_{2n}}$ and, by (5), $X_{T^*}(\omega)$ is a periodic function of period π/n such that if $\omega \equiv \eta \pmod{\pi/n}$ with $\eta \in (-\pi/(2n), \pi/(2n)]$,

$$X_{T^*}(\omega) = 2\cos\eta.$$

Therefore, $X_{T^*}(\omega)$ for $\omega \in (0, 2\pi]$ repeats its maximum 2 and its minimum $2\cos(\pi/(2n))$, $2n$ times.

Since T^* has the asymmetrization of the same size n, there are no parallel edges in T^*. Hence, the endpoints of the minimum shadow of T^*, say at $\omega = j\pi/n + \pi/(2n)$ come from a vertex, say P_k, and an edge, say $P_l P_{l+1}$. Those of the two neighboring maximum shadows come from the vertices P_k, P_l and from the vertices P_k, P_{l+1}, respectively. Hence, $\overline{P_k P_l} = \overline{P_k P_{l+1}}$ holds. Replace the edge $P_l P_{l+1}$ by the circular arc centered at P_k having the endpoints at P_l and P_{l+1}. Repeating this replacement for $j = 1, 2, \ldots, n$, we get a convex body of constant width. It is easy to see that this convex body is a Reuleaux p-body for some p. Hence, T^* is a Reinhardt n-gon.

We prove the converse. Let T be a Reinhardt n-gon coming from a Reuleaux p-body of width r. By (3), we can rearrange the circular arc of S by parallel translations into a circle $\mathbb{S}$ of radius r so that the circular arcs are essentially disjoint and cover just a half part of $\mathbb{S}$, and by the rotation of angle π, they moved to the other half part of $\mathbb{S}$. The edges of T correspond to the chord of the circular arcs. By the above property, it is easy to see that the asymmetrization of T is a regular pre-edge bundle of size n. Hence, $\delta(X_T) = \nu_n$ and T is the minimum polygon.

The "if" part of the last statement follows from the first part. To prove the "only if" part, suppose that a Reinhardt n-polygon T exists for $n = 2^k$. Then by the above argument, T has the asymmetrization R_n which has a realization T of the same size. This contradicts Lemma 2. □

Let us describe the correspondence between $(c_0, \ldots, c_{n-1}) \in P(n)$ and cyclic integer vectors. Choose $i \in \{0, 1, \ldots, 2n-1\}$ so that $\sigma^i(c_0, \ldots, c_{n-1}) = (d_0, \ldots, d_{n-1})$ with $d_0 = d_{n-1}$. Count the number of runs of 1 and -1 in $(d_0, \ldots, d_{n-1})$, i.e., we write $(d_0, \ldots, d_{n-1})$ like $1^{n_1}(-1)^{n_2}\ldots$ or $(-1)^{n_1}1^{n_2}\ldots$ Then $(n_1, n_2, \ldots, n_p)$ is the desired cyclic vector. For the converse direction, we choose either $1^{n_1}(-1)^{n_2}\ldots$ or $(-1)^{n_1}1^{n_2}\ldots$

Reinhardt [9] gave an alternative characterization of the cyclic vector $(n_1, n_2, \ldots, n_p)$ with $n = \sum_{i=1}^{p} n_i$: it is a cyclic vector if and only if p is odd and the polynomial

$$1 - z^{n_1} + z^{n_1+n_2} - \cdots + z^{n_1+n_2+\cdots+n_{p-1}}$$

is divisible by $\Phi_{2n}(z)$, the $2n$-th cyclotomic polynomial. For completeness, we show that this characterization is equivalent to ours. As above, we assume that $d_0 = d_{n-1}$. From (7) we have

$$\sum_{j=0}^{n-1} d_j z^j = 0 \quad \text{with} \quad z = \exp\frac{\pi\mathbf{i}}{n}.$$

We consider that z is a variable and multiply by $z-1$. Then we see

$$
\begin{aligned}
&(z-1)\sum_{j=0}^{n-1} d_j z^j \\
&= 1 - 2z^{n_1} + 2z^{n_1+n_2} - \cdots + 2z^{n_1+\cdots+n_{p-1}} - z^n \\
&\equiv 2 - 2z^{n_1} + 2z^{n_1+n_2} - \cdots + 2z^{n_1+\cdots+n_{p-1}} \pmod{\Phi_{2n}(z)}.
\end{aligned} \tag{8}
$$

Dividing by 2, we see the condition of Reinhardt. To get the converse, we just go backwards. Note that the polynomial in the last line of (8) is divisible by $z-1$ as p is odd. By Lemma 2, the second statement of Theorem 6 is derived from the Reinhardt criterion.

The subset $Q(p)$ in the proof of Lemma 2 corresponds to p-fold rotational symmetry. We can find Reinhardt polygons without any symmetry [3, 4].

7 Truncation of the Regular Triangle

Let X, Y be general $\mathbb{R}$-valued, square integrable random variables on the probability space Ω. Assume further that $X \geq 0$ everywhere and $\mathbb{E}(X) > 0$. Recall that

$$\kappa(X) = \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2} = \delta(X)^2 + 1.$$

It holds for any $t \in \mathbb{R}$ with sufficiently small modulus that

$$
\begin{aligned}
\kappa(X+tY) &= \frac{\mathbb{E}((X+tY)^2)}{\mathbb{E}(X+tY)^2} = \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2}\cdot\frac{1+2t\mathbb{E}(XY)/\mathbb{E}(X^2)+t^2\mathbb{E}(Y^2)/\mathbb{E}(X^2)}{1+2t\mathbb{E}(Y)/\mathbb{E}(X)+t^2\mathbb{E}(Y)^2/\mathbb{E}(X)^2} \\
&= \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2}\left(1+2t\left(\frac{\mathbb{E}(XY)}{\mathbb{E}(X^2)}-\frac{\mathbb{E}(Y)}{\mathbb{E}(X)}\right)+O(t^2)\right).
\end{aligned}
$$

If $\mathbb{E}(XY)/\mathbb{E}(X^2) - \mathbb{E}(Y)/\mathbb{E}(X) = 0$, then we have

$$\kappa(X+tY) = \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2}\left(1+t^2\left(\frac{\mathbb{E}(Y^2)}{\mathbb{E}(X^2)}-\frac{\mathbb{E}(Y)^2}{\mathbb{E}(X)^2}\right)+O(t^3)\right).$$

Hence, the following lemma holds.

Lemma 3 (i) *Respectively,*

$$\left.\frac{d\delta(X+tY)}{dt}\right|_{t=0} >,=,< 0 \quad\Longleftrightarrow\quad \frac{\mathbb{E}(XY)}{\mathbb{E}(X^2)} >,=,< \frac{\mathbb{E}(Y)}{\mathbb{E}(X)}.$$

(ii) *Assume that "=" holds in* (i). *Then, there exists* $\varepsilon > 0$ *such that*

$$\delta(X+tY) >,< \delta(X)$$

for any $t \in (-\varepsilon, \varepsilon) \setminus \{0\}$ *if*

$$\frac{\mathbb{E}(Y^2)}{\mathbb{E}(X^2)} >, < \frac{\mathbb{E}(Y)^2}{\mathbb{E}(X)^2},$$

respectively.

Proof Statement (i) follows since

$$\frac{d\delta(X+tY)}{dt} = \frac{(\kappa(X+tY)-1)^{-1/2}}{2} \cdot \frac{d\kappa(X+tY)}{dt},$$

and (ii) follows since $\delta(X+tY)$ is a monotone increasing function of $\kappa(X+tY)$. □

Lemma 4 *Let T and S be triangles in $\mathbb{C}$. Then, $X_T = X_S$ (a.s.) holds if and only if there exists $z \in \mathbb{C}$ such that either $S = T + z$ or $S = -T + z$.*

Proof $X_{-T} = X_T$ holds since they have the same asymmetrization. Hence, the "if" part holds. Let us prove the "only if" part. Let $T = T(\alpha, \beta, \gamma)$. Consider $X_T(\omega)$ as a function of $\omega \in \mathbb{R}/\mathbb{Z}$. Then, it is locally minimal if and only if ω is equal to either of

$$\pm(\arg(\beta)-\arg(\alpha)), \quad \pm(\arg(\gamma)-\arg(\beta)), \quad \pm(\arg(\alpha)-\arg(\gamma)) \tag{9}$$

modulo 2π. If $X_T = X_S$ (a.s.), then they should have the same set of ω as this. Also, at any of these ω, they should have the same height. This implies that either $S = T + z$ or $S = -T + z$ for some $z \in \mathbb{C}$. □

Lemma 5 *Let T and S be triangles in $\mathbb{C}$. Then,*

$$\mathbb{E}(X_T X_S) \le \mathbb{E}(X_T^2)^{1/2}\mathbb{E}(X_S^2)^{1/2}.$$

The equality holds if and only if there exist $\lambda \ge 0$ and $z \in \mathbb{C}$ such that either $S = \lambda T + z$ or $S = -\lambda T + z$.

Proof This is the Cauchy–Schwarz inequality for the inner product $\langle X, Y\rangle = E(XY)$. The equality holds if and only if there exists $\lambda > 0$ such that $X_S = \lambda X_T = X_{\lambda T}$ (a.s.). Hence, by Lemma 4, if and only if $S = \lambda T + z$ or $S = -\lambda T + z$ for some $z \in \mathbb{C}$. □

Lemma 6 *Let $T = T(\alpha, \beta, \gamma)$ be a regular triangle. Let $\sigma \in \mathbb{C} \setminus \{0\}$. Then,*

$$\frac{\mathbb{E}(X_T X_\sigma)}{\mathbb{E}(X_T^2)} \le \frac{\mathbb{E}(X_\sigma)}{\mathbb{E}(X_T)}.$$

The equality holds if and only if $\overrightarrow{0\sigma}$ is parallel to one of the edges of T, that is, $\arg(\sigma)$ *is equal to one of* (9) *modulo 2π.*

Proof Without loss of generality, we assume that the length of the edges of T is 1. Recall Lemma 1. It holds that $\mathbb{E}(X_{\mu z_1} X_{\mu z_2}) = \mathbb{E}(X_{z_1} X_{z_2})$ for any $\mu, z_1, z_2 \in \mathbb{C}$ with $|\mu| = 1$. Moreover, since $e^{2\pi \mathbf{i}/3} T = T$, we have

$$\mathbb{E}(X_T X_\sigma) = \mathbb{E}(X_T X_{e^{2\pi \mathbf{i}/3}\sigma}) = \mathbb{E}(X_T X_{e^{4\pi \mathbf{i}/3}\sigma}).$$

Hence, $\mathbb{E}(X_T X_\sigma) = \mathbb{E}(X_T X_S)/3$ with the regular triangle $S = T(0, \sigma, e^{\pi \mathbf{i}/3}\sigma)$. Therefore by Lemma 5,

$$\begin{aligned}\mathbb{E}(X_T X_\sigma) &= \frac{\mathbb{E}(X_T X_S)}{3} \leq \frac{\mathbb{E}(X_T^2)^{1/2}\,\mathbb{E}(X_S^2)^{1/2}}{3} \\ &= \frac{\mathbb{E}(X_T^2)^{1/2}\,\mathbb{E}((|\sigma| X_T)^2)^{1/2}}{3} = \frac{|\sigma|}{3}\mathbb{E}(X_T^2).\end{aligned}$$

The equality holds if and only if $\overrightarrow{0\sigma}$ is parallel to one of the edges of T. Thus, we have

$$\frac{\mathbb{E}(X_T X_\sigma)}{\mathbb{E}(X_T^2)} \leq \frac{|\sigma|}{3} = \frac{\mathbb{E}(X_\sigma)}{\mathbb{E}(X_T)}$$

with the equality if and only if $\overrightarrow{0\sigma}$ is parallel to one of the edges of T. □

Theorem 7 *A sufficiently small parallel truncation of the regular triangle increases the deviation rate, while a sufficiently small non-parallel truncation decreases it.*

Parallel Truncation: Let $T = T(\alpha, \beta, \gamma)$ be a regular triangle of the edge length 1. We also assume that it is of counter clockwise order. Let $t > 0$ be sufficiently small. Let

$$\beta_t = (1-t)\alpha + t\beta, \qquad \gamma_t = (1-t)\alpha + t\gamma.$$

Let

$$V_t = T(\beta_t, \beta, \gamma, \gamma_t)$$

be a parallel truncation of T at α. Then, we have

$$\begin{aligned}X_{V_t} &= X_{\beta-\beta_t} + X_{\gamma-\beta} + X_{\gamma_t-\gamma} + X_{\beta_t-\gamma_t} \\ &= X_T - tX_{\beta-\alpha} - tX_{\alpha-\gamma} + tX_{\beta-\gamma} = X_T - tX_Y\end{aligned}$$

with $Y = X_{\mathbf{c}} + X_{\mathbf{b}} - X_{-\mathbf{a}}$, where we denote $\mathbf{a} = \gamma - \beta$, $\mathbf{b} = \alpha - \gamma$, $\mathbf{c} = \beta - \alpha$. Since $\mathbb{E}(X_{\mathbf{c}}) = \mathbb{E}(X_{\mathbf{b}}) = \mathbb{E}(X_{-\mathbf{a}}) = 1/\pi$ by Lemma 1, we have,

$$\frac{\mathbb{E}(X_T X_{\mathbf{c}})}{\mathbb{E}(X_T^2)} = \frac{\mathbb{E}(X_T X_{\mathbf{b}})}{\mathbb{E}(X_T^2)} = \frac{\mathbb{E}(X_T X_{-\mathbf{a}})}{\mathbb{E}(X_T^2)} = \frac{1/\pi}{\mathbb{E}(X_T)}$$

by Lemma 6 and hence,

$$\frac{\mathbb{E}(X_T Y)}{\mathbb{E}(X_T^2)} = \frac{\mathbb{E}(X_T X_{\mathbf{c}}) + \mathbb{E}(X_T X_{\mathbf{b}}) - \mathbb{E}(X_T X_{-\mathbf{a}})}{\mathbb{E}(X_T^2) + \mathbb{E}(X_T^2) - \mathbb{E}(X_T^2)} = \frac{1/\pi}{\mathbb{E}(X_T)} = \frac{\mathbb{E}(Y)}{\mathbb{E}(X_T)}.$$

Now, we prove that

$$\frac{\mathbb{E}(Y^2)}{\mathbb{E}(X_T^2)} > \frac{\mathbb{E}(Y)^2}{\mathbb{E}(X_T)^2}$$

so that $\delta(V_t) > \delta(T)$ for sufficiently small $|t| \neq 0$ by Lemma 3. We have

$$\begin{aligned}
&\mathbb{E}(Y) = \mathbb{E}(X_{\mathbf{b}} + X_{\mathbf{c}} - X_{-\mathbf{a}}) = \frac{1}{\pi}, \qquad \mathbb{E}(X_T) = \frac{3}{\pi},\\
&\mathbb{E}(Y^2) = \mathbb{E}((X_{\mathbf{b}} + X_{\mathbf{c}} - X_{-\mathbf{a}})^2) = \frac{3}{4} + \frac{2(V(2\pi/3) - 2V(\pi/3))}{4\pi} = \frac{1}{3} - \frac{\sqrt{3}}{4\pi},\\
&\mathbb{E}(X_T^2) = \frac{3}{4} + \frac{6V(2\pi/3)}{4\pi} = \frac{1}{2} + \frac{3\sqrt{3}}{4\pi},\\
&\frac{\mathbb{E}(Y^2)}{\mathbb{E}(X_T^2)} = \frac{1/3 - \sqrt{3}/(4\pi)}{1/2 + 3\sqrt{3}/(4\pi)} = 0.214\ldots > \frac{1}{9} = \frac{\mathbb{E}(Y)^2}{\mathbb{E}(X_T)^2},
\end{aligned}$$

and complete the proof that $\delta(V_t) > \delta(T)$ for sufficiently small $|t| \neq 0$ in the case of parallel truncation.

Non-Parallel Truncation: Let $T = T(\alpha, \beta, \gamma)$ be a regular triangle of the edge length 1. We also assume that it is of counter clockwise order. Let $0 < \lambda \neq 1$ and $t > 0$ be sufficiently small. Let

$$\beta_t = (1-t)\alpha + t\beta, \qquad \gamma_t = (1-\lambda t)\alpha + \lambda t\gamma.$$

Let $V_t = T(\beta_t, \beta, \gamma, \gamma_t)$ be a non-parallel truncation of T at α. Then, we have

$$\begin{aligned}
X_{V_t} &= X_{\beta-\beta_t} + X_{\gamma-\beta} + X_{\gamma_t-\gamma} + X_{\beta_t-\gamma_t}\\
&= X_T - tX_{\mathbf{c}} - tX_{\lambda\mathbf{b}} + tX_{\mathbf{e}} = X_T - tY,
\end{aligned}$$

where we denote $\mathbf{b} = \alpha - \gamma$, $\mathbf{c} = \beta - \alpha$, $\mathbf{e} = \mathbf{c} + \lambda\mathbf{b}$, and $Y = X_{\mathbf{c}} + X_{\lambda\mathbf{b}} - X_{\mathbf{e}}$. By Lemma 6, we have

$$\frac{\mathbb{E}(X_T X_{\mathbf{c}})}{\mathbb{E}(X_T^2)} = \frac{\mathbb{E}(X_T X_{\mathbf{b}})}{\mathbb{E}(X_T^2)} = \frac{1/\pi}{\mathbb{E}(X_T)} \quad \text{and} \quad \frac{\mathbb{E}(X_T X_{\mathbf{e}})}{\mathbb{E}(X_T^2)} < \frac{|\mathbf{e}|/\pi}{\mathbb{E}(X_T)}.$$

Hence,

$$\frac{\mathbb{E}(X_T X_{\mathbf{c}}) + \lambda\mathbb{E}(X_T X_{\mathbf{b}}) - \mathbb{E}(X_T X_{\mathbf{e}})}{\mathbb{E}(X_T^2) + \lambda\mathbb{E}(X_T^2) - |\mathbf{e}|\mathbb{E}(X_T^2)} > \frac{1/\pi}{\mathbb{E}(X_T)}$$

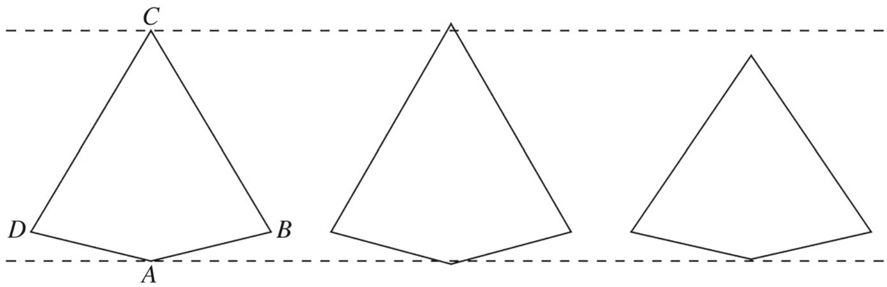

Fig. 7 Left: δ-minimum, Center: Problems 1 and 2, Right: Problem 3

and we have

$$\frac{\mathbb{E}(X_T Y)}{\mathbb{E}(X_T^2)} > \frac{(1+\lambda-|\mathbf{e}|)/\pi}{\mathbb{E}(X_T)}.$$

Thus, by Lemma 3,

$$\left.\frac{d\delta(X_T+tY)}{dt}\right|_{t=0} > 0,$$

which implies that

$$\delta(X_{V_t}) = \delta(X_T - tY) < \delta(X_T)$$

if $t > 0$ is small, which completes the proof.

8 Remaining Problems

When n is a power of 2, the method in this paper does not apply because by Lemma 2, there is no n-gon realization of the regular pre-edge bundle R_n. The case $n = 4$ may be of special interest.

By numerical calculation, a possible minimum of δ in Θ_4 is attained by the kite-shape with vertices

$$(A, B, C, D) \approx ((0, -0.24213332485), (1, 0), (0, 1.67502597318), (-1, 0))$$

having the deviation rate 0.035306425. This is not a solution to any of three optimization problems for $n = 4$ in the introduction. See Fig. 7 for a comparison of solutions having the same horizontal width. Indeed from $AB < AC < BC < BD$, we see that this shape is not optimal for Problems 1 and 2. The optimizers for these problems are the same, which is based on the regular triangle with an additional vertex similar to the construction of Reinhardt polygon, see [2, 6]. The kite $ABCD$ is not optimal for

Problem 3 either. Indeed, [1] showed that the maximum width of quadrangles with the unit perimeter is close to $\sqrt{-9+6\sqrt{3}}/4 \approx 0.295$, possibly attained by

$$\left(\left(0, -\frac{\sqrt{-3+2\sqrt{3}}}{3}\right), (1,0), \left(0, \sqrt{1+\frac{2}{\sqrt{3}}}\right), (-1,0)\right),$$

while the kite shape gives the value 0.288. For Problems 1–3, the solutions have algebraic expressions. We do not know if our kite has such an algebraic expression.

In a subsequent paper, we will discuss the minimality of δ-values and the minimal shapes under all infinitesimal deformations including the cases when n is a power of 2.

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