Dynamical system at infinity, Super-stationarity, Uniform complexity and Pattern recognition

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1 Motivation

I studied the maximal pattern complexity with Luca Zamboni [5, 6] arround 2000 which is developed to the study of pattern recognition problems with Yu-Mei Xue [12, 13] arround 2010 to find an optimal set of observation points to distinguish the maximal possible patterns in a given set of patterns among the sets of observations of the same finite size. An optimal position is by definition a set of points whose every finite subset attains this optimal. The restriction of the set of observation points to such an optimal position defines a uniform set and uniform complexity attaining the maximal pattern complexity always. At the same time, I was interested in the symbolic dynamics at time infinities in the sense of Stone-Cech compactification and arrived at super-stationary sets which are subclass of uniform sets. They are determined by finite sets of prohibited super-subwords and the structures are easy to analyze. Moreover, every uniform complexities are realized by them (partly with Hui Rao, Bo Tan and Yu-Mei Xue [7, 8, 9, 10, 11, 14, 15]). This makes the structure of uniform complexity clear. Different from the usual complexity, the entropy of the uniform complexity takes values only at the logarithm of positive integers (see Remark 4). It comes from the exponentially increasing main term, and dividing by this term, we get polynomial order term and not in between. We also discussed the meaning of the symbolic dynamics at time infinities coming from geometrical dynamics and found out that it corresponds to the infinitesimal quantities the infinite time (that is, non-princicipal ultra-filter) has. Finally, we discuss conditions that the super-stationary sets coming from a symbolic dynamics are always the full set. For example, Thue-Morse system has this property.

2 Remote moves and super-stationary sets

In the study of symbolic dynamics with discrete time, we concern specially the system at time of a fixed infinity in the sense of Stone-Cech compactification.

That is, let Ω be a nonempty closed subset of $\mathbb{A}^{\mathbb{N}}$ (not necessarily be shift-invariant), where $\mathbb{A}$ is a finite set with $\#\mathbb{A} \geq 2$ and $\mathbb{N} = \{0, 1, 2, \cdots\}$. An element $\omega \in \mathbb{A}^{\mathbb{N}}$ is often written as an infinite sequence $\omega(0)\omega(1)\omega(2)\cdots$. Let $\beta\mathbb{N}$ be the Stone-Cech compactification of $\mathbb{N}$ and $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$. Note that χ is a non-principal ultra-filter on $\mathbb{N}$, which can be considered as an *infinite time* while $\mathbb{N}$ is interpreted as the set of finite times. Another interpretation is that $\beta\mathbb{N} \setminus \mathbb{N}$ is the set of finitely additive, non-atomic probability measures on $\mathbb{N}$ taking the values either 1 or 0 at S according to $S \in \chi$ or not, whereas $n \in \mathbb{N}$ is identified with the unit measure at n (hence, σ-additive).

For $\omega \in \Omega$ and $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$, $\omega(\chi) \in \mathbb{A}$ is defined so that $\omega(\chi) = a$ if and only if $\{i \in \mathbb{N};\ \omega(i) = a\} \in \chi$. That is, $\omega(i) = a$ almost surely with respect to χ. For $i \in \mathbb{N}$, define $\Omega[i] = \{\omega(i);\ \omega \in \Omega\} \subset \mathbb{A}$. Then, it is extended to the above χ so that $\Omega[\chi] = S \subset \mathbb{A}$ if and only if $\{i \in \mathbb{N};\ \Omega[i] = S\} \in \chi$. We remark that $\Omega[\chi] = \{\omega(\chi);\ \omega \in \Omega\}$ does not hold in general for $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ (Example 1).

For $\chi_1, \chi_2 \in \beta\mathbb{N}$, we define $\chi_1 \otimes \chi_2 \in \beta(\mathbb{N} \times \mathbb{N})$ so that for any $S \subset \mathbb{N} \times \mathbb{N}$, $S \in \chi_1 \otimes \chi_2$ if and only if

$$\{i \in \mathbb{N};\ \{j \in \mathbb{N};\ (i,j) \in S\} \in \chi_2\} \in \chi_1 \quad ([1,\ 15,\ 16]).$$

This is just defining the product measure by the successive integrations with the 2nd coordinate first. Since the measure $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ is not σ-additive, Fubini's Theorem does not hold. In fact, we have

$$\{(i,j) \in \mathbb{N}^2;\ i < j\} \in \chi \otimes \chi \text{ but } \{(j,i) \in \mathbb{N}^2;\ i < j\} \notin \chi \otimes \chi.$$

Of course for $i, j \in \mathbb{N}$, $i \otimes j$ is just the couple $(i,j) \in \mathbb{N} \times \mathbb{N}$ and $\Omega[i \otimes j]$ is defined as $\{\omega(i)\omega(j);\ \omega \in \Omega\} \subset \mathbb{A}^2$. This operation is extended to $\chi_1 \otimes \chi_2$ so that $U = \Omega[\chi_1 \otimes \chi_2] \subset \mathbb{A}^2$ if and only if $S := \{(i,j);\ \Omega[i \otimes j] = U\} \in \chi_1 \otimes \chi_2$, that is

$$\begin{aligned} &\{i \in \mathbb{N};\ \{j \in \mathbb{N};\ (i,j) \in S\} \in \chi_2\} \\ &\qquad = \{i \in \mathbb{N};\ \{j \in \mathbb{N};\ \{\omega(i)\omega(j);\ \omega \in \Omega\} = U\} \in \chi_2\} \in \chi_1. \end{aligned}$$

Different from a finite time $i \in \mathbb{N}$, where $\Omega[i \otimes i] = \{aa;\ a \in \Omega[i]\}$, $\Omega[\chi \otimes \chi]$ may contain elements ab with $a \neq b$ as is shown below.

Example 1. Let $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ and

$$\Omega = \{0^i 1^\infty;\ i = 0, 1, 2, \cdots\} \subset \{0, 1\}^{\mathbb{N}}.$$

Then, we have $\Omega[\chi] = \{0, 1\}$ while $\{\omega[\chi];\ \omega \in \Omega\} = \{1\}$. Moreover, we have $\Omega[\chi \otimes \chi] = \{00, 01, 11\}$ for any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ since

$$\{(i,j) \in \mathbb{N} \times \mathbb{N};\ \{\omega(i)\omega(j);\ \omega \in \Omega\} = \{00, 01, 11\}\} = \{(i,j);\ i < j\} \in \chi \otimes \chi.$$

To make clear what happens at infinity, we define $\Omega[\chi^\infty] := \Omega[\chi \otimes \chi \otimes \cdots]$. Let

$$\chi_1 \otimes \cdots \otimes \chi_n \otimes \chi_{n+1} = (\chi_1 \otimes \cdots \otimes \chi_n) \otimes \chi_{n+1}$$

inductively for $n = 1, 2, \cdots$. Let $\Omega[\chi_1 \otimes \chi_2 \otimes \cdots] \subset \mathbb{A}^{\mathbb{N}}$ be the project limit of $\Omega[\chi_1 \otimes \cdots \otimes \chi_n]$ as $n \to \infty$. Thus we define $\Omega[\chi^\infty]$, which is called the *remote move* of Ω at χ. Note that for a closed set $\Omega \subset \mathbb{A}^N$ and an increasing sequence of nonnegative integers $n_0 < n_1 < n_2 < \cdots$,

$$\Omega[n_0 \otimes n_1 \otimes n_2 \otimes \cdots] = \{\omega(n_0)\omega(n_1)\omega(n_2)\cdots\ ;\ \omega \in \Omega\} \subset \mathbb{A}^{\mathbb{N}}.$$

A closed subset $\Omega \subset \mathbb{A}^{\mathbb{N}}$ is called *super-stationary* if for any increasing sequence of nonnegative integers $n_0 < n_1 < n_2 < \cdots$,

$$\Omega[n_0 \otimes n_1 \otimes n_2 \otimes \cdots] = \Omega$$

holds. It is known (Theorem 1) that $\Omega[\chi^\infty]$ is super-stationary for any $\chi \in \beta\mathbb{N}$. This means in special by taking $n_i = i + l$ ($i = 0, 1, 2, \cdots; l \in \mathbb{N}$), $\Omega[\chi^\infty]$ is shift invariant, that is, invariant under where to start the observation. Furthermore, it is invariant under any choice of the observation times as long as keeping the order. Thus in the remote moves, the quantities of time, hence *the clocks are of no use, only asking which is before or after makes sense.*

A remote move is called *trivial* if it is contained in $\{a^\infty;\ a \in \mathbb{A}\}$.

For a subset S of a compact metric set K, the set of accumulate points of S is denoted by S' and called the *derived* set of S. The derived set of S' is denoted by S'' or $S^{(2)}$ and is called the second derived set of S. In the same way, we define k-th derived set $S^{(k)}$. The *accmulation degree* of S is defined to be the smallest k such that $S^{(k+1)} = \emptyset$ if such k exists, otherwise ∞.

Theorem 1. [15]
Let $\Omega \subset \mathbb{A}^{\mathbb{N}}$ be an arbitrary nonempty closed set. Then for any $\chi \in \beta\mathbb{N}$, we have
(1) $\Omega[\chi^\infty]$ *is super-stationary,*
(2) $\Omega[\chi^\infty] = \Omega$ *holds if Ω is super-stationary and $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$,*
(3) *if there exists an increasing sequence $n_0 < n_1 < n_2 < \cdots$ of nonnegative integers such that $\Omega[n_0 \otimes n_1 \otimes n_2 \otimes \cdots]$ is super-stationary, then $\Omega[\chi^\infty] = \Omega[n_0 \otimes n_1 \otimes n_2 \otimes \cdots]$ holds for any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ such that $\{n_0, n_1, n_2, \cdots\} \in \chi$,*
(4) *the increasing sequence as* (3) *exists if Ω has a finite accumulation degree,*
(5) $\Omega[\chi^\infty]$ *is trivial if $\chi \in \mathbb{N}$ or $\#\Omega < \infty$.*
If (3) *holds, we say that the remote move of Ω at χ is attainable.*

Example 2. Let $X = \mathbb{R}/\mathbb{Z}$ and $f : X \to X$ be $f(x) = x + \alpha$ (mod 1), where α is an irrational number. Let $0 < u < v < 1$ and define $\kappa : X \to \mathbb{A}$ with $\mathbb{A} = \{0, 1, 2\}$ by

$$\kappa(x) = \begin{cases} 0 & x \in [0, u) \\ 1 & x \in [u, v) \\ 2 & x \in [v, 1) \end{cases} \quad (mod\ 1).$$

Define $\omega_0 \in \mathbb{A}^{\mathbb{N}}$ by $\omega_0(n) = \kappa(f^n(0))$ ($\forall n \in \mathbb{N}$). Let Ω be the closure of $\{T^n\omega_0;\ n \in \mathbb{N}\}$, where T is the shift on $\mathbb{A}^{\mathbb{N}}$.

Let $f_0 : \mathbb{N} \to \mathbb{R}/\mathbb{Z}$ be $f_0(n) = n\alpha$ (mod 1). Then, this mapping is extended to the mapping $\beta\mathbb{N} \to \beta(\mathbb{R}/\mathbb{Z})$ so that for $\chi \in \beta\mathbb{N}$, $f_0(\chi)$ is defined by $f_0(\chi) = \{S \subset \mathbb{R}/\mathbb{Z};\ \{n \in \mathbb{N};\ f_0(n) \in S\} \in \chi\}$. Then, $p \in \mathbb{R}/\mathbb{Z}$ is determined by

$f_0(\chi)$ so that $\{x \in \mathbb{R}/\mathbb{Z};\ |x-p| < \epsilon\} \in f_0(\chi)$ for any $\epsilon > 0$ or equivalently, $\{n \in \mathbb{N};\ |n\alpha - p| < \epsilon\ (mod\ 1)\} \in \chi$. This p is denoted as $\lim f_0(\chi)$. Furthermore, either

$$\{n \in \mathbb{N};\ 0 < n\alpha - p < \epsilon\ (mod\ 1)\} \in \chi$$

holds for any $\epsilon > 0$, or

$$\{n \in \mathbb{N};\ -\epsilon < n\alpha - p < 0\ (mod\ 1)\} \in \chi$$

holds for any $\epsilon > 0$. The former case, we say $f_0(\chi)$ (or χ) is of $(-)$type, while the latter case, of $(+)$type.

Define $\Pi_+,\ \Pi_- \subset \{0,1,2\}^{\mathbb{N}}$ by

$$\Pi_+ = \{0^n 1^\infty;\ n \in \mathbb{N}\} \cup \{1^n 2^\infty;\ n \in \mathbb{N}\} \cup \{2^n 0^\infty;\ n \in \mathbb{N}\}$$
$$\Pi_- = \{0^n 2^\infty;\ n \in \mathbb{N}\} \cup \{2^n 1^\infty;\ n \in \mathbb{N}\} \cup \{1^n 0^\infty;\ n \in \mathbb{N}\}.$$

Then, the following statement holds.

Lemma 1. *For any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$, $\Omega[\chi^\infty] = \Pi_+$ if $f_0(\chi)$ is of $(+)$type, while $\Omega[\chi^\infty] = \Pi_-$ if $f_0(\chi)$ is of $(-)$type. Moreover, some of them are attainable, but some of them are not.*

Proof Let $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$. Note that for any $k = 1, 2, \cdots$, $M \in \mathbb{N}$ and increasing functions $\psi : \mathbb{N} \to \mathbb{N}$,

$$\{(i_0, i_1, \cdots, i_k) \in \mathbb{N}^{k+1};\ M < i_0,\ \psi(i_{j-1}) < i_j\ (j = 1, 2 \cdots, k)\} \in \chi^{k+1}.$$

Without loss of generality, assume that $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ is of $(+)$type. Let $p = \lim f_0(\chi)$. Then for any $\epsilon > 0$, using the above statement, we have

$$\{(i_0, j_1, \cdots, i_k) \in \mathbb{N}^{k+1};\ p - \epsilon < j_0\alpha < j_1\alpha < \cdots < j_k\alpha < p\ (mod\ 1)\} \in \chi^{k+1}.$$

Hence,

$$\Omega[\chi^{k+1}] = \{0^n 1^{k+1-n};\ 0 \le n \le k\} \cup \{1^n 2^{k+1-n};\ 0 \le n \le k\}$$
$$\cup \{2^n 0^{k+1-n};\ 0 \le n \le k\}.$$

Taking the project limit, we have $\Omega[\chi^\infty] = \Pi_+$, proving the one half of our Lemma.

Now, let us prove the existences of both attainable and non-attainable χ. The existence of attainable χ is clear by taking any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ such that there exists an increasing sequence $n_0 < n_1 < n_2 < \cdots$ of nonnegative integers satisfying that $\{n_i\alpha\}$ converges monotonously to 1 and $\{n_0, n_1, n_2, \cdots\} \in \chi$, where $\{x\} \in [0,1)$ denotes the fractional part of $x \in \mathbb{R}$. Then,

$$\Omega[\chi^\infty] = \Omega[n_0 \otimes n_1 \otimes n_2 \otimes \cdots] = \Pi_+.$$

To prove the existence of non-attainable $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$, take any χ which is contained in

$$\{S \subset \mathbb{N};\ \{\{n\alpha\};\ n \in S\} \subset (1-\epsilon, 1) \text{ and } 1 \in \{\{n\alpha\};\ n \in S\}''\}$$

for any $\epsilon > 0$. Then, it holds that $\Omega[\chi^\infty] = \Pi_+$, but for any $S = \{n_0 < n_1 < n_2 < \cdots\} \in \chi$, $\Omega[n_0 \otimes n_1 \otimes n_2 \otimes \cdots]$ contains ω having $10, 21, 02$ as its subwords. Hence, $\Omega[n_0 \otimes n_1 \otimes n_2 \otimes \cdots] \neq \Pi_+$, □

Remark 1. For any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ and $n \in \mathbb{N}$, we can define $\chi + n \in \beta\mathbb{N}$ so that $S \in \chi + n$ if and only if $S - n = \{s - n;\ s \in S,\ s \geq n\} \in \chi$. Then, $\Omega[\chi \otimes (\chi+1) \otimes (\chi+2) \otimes \cdots]$ is a T-invariant closed set including $\Omega_\infty := \bigcap_{n=0}^{\infty} T^n \Omega$, where T is the shift on $\mathbb{A}^{\mathbb{N}}$. Because of the remote move, it is usually strictly bigger than Ω_∞.

3 Full remote moves

The remote move of Ω seems to be small if Ω is small, but sometimes not. In this section, we obtain a sufficient condition for it to be the full set in the case $\mathbb{A} = \{0, 1\}$. This condition is satisfied, for example, by the Thue-Morse system.

Definition 1. For $\omega \in \{0,1\}^{\mathbb{N}}$ and $S = \{s_1 < s_2 < \ldots\} \subset \mathbb{N}$ with $\#S < \infty$ or $\#S = \infty$, define

$$\lambda(S, \omega) = \sup_{m \in \mathbb{N}} \sum_i (\omega(s_i + m) - \omega(s_{i+1} + m))^2,$$

where $\displaystyle\sum_i$ denotes $\displaystyle\sum_{i=1}^{k-1}$ if $\#S = k < \infty$ and $\displaystyle\sum_{i=1}^{\infty}$ if $\#S = \infty$.

Theorem 2. [15] *Let $\omega_0 \in \{0,1\}^{\mathbb{N}}$ and Ω be the closure of $\{T^n \omega_0;\ n \in \mathbb{N}\} \subset \{0,1\}^{\mathbb{N}}$, where $T : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is the shift. For any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$, the following statements hold.*
(1) $\Omega[\chi^\infty] = \{0,1\}^{\mathbb{N}}$ *holds if there exists a sequence $(U_k \in \chi^k;\ k = 1, 2, \ldots)$ such that*

$$\lim_{k \to \infty} \inf_{S \in U_k \cap \Delta_k} \lambda(S, \omega_0) = \infty,$$

where

$$\Delta_k := \{(s_1, \ldots, s_k) \in \mathbb{N}^k;\ s_1 < \ldots < s_k\}$$

and $(s_1, \ldots, s_k) \in U_k \cap \Delta_k$ is identified with $\{s_1 < \ldots < s_k\} \subset \mathbb{N}$.

(2) $\Omega[\chi^\infty] \neq \{0,1\}^{\mathbb{N}}$ *holds if there exists $U \in \chi$ such that $\lambda(U, \omega_0) < \infty$.*

Example 3. Let $\omega_0 = 0110100110010110\ldots \in \{0,1\}^{\mathbb{N}}$ be the Thue-Morse word. That is, $\omega_0(n) = 0$ if and only if the number of 1 in the 2-adic representation of n is even. Let $T : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the shift and $\Omega \subset \{0,1\}^{\mathbb{N}}$ be the closure of $\{T^n \omega_0;\ n \in \mathbb{N}\}$. Then, we have $\Omega[\chi^\infty] = \{0,1\}^{\mathbb{N}}$ for any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$.

In fact, let $\varphi(n) = 2^{2n}$ ($\forall n \in \mathbb{N}$). For any $k = 1, 2, \ldots$ and $\xi = \xi_1 \xi_2 \ldots \xi_k \in \{0,1\}^k$, let $N_\xi = l 2^{2k} + \sum_{i=1}^{k} (1 - \xi_i) 2^{2(i-1)}$, where $l \in \{0, 1\}$ is determined depending on ξ so that $\omega_0(N_\xi) = 0$. Then, we have $\omega_0(N_\xi + \varphi(i-1)) = \xi_i$ for any $i = 1, 2, \ldots, k$. Hence,

$$\xi \in \Omega[\varphi(0) \otimes \varphi(1) \otimes \cdots \otimes \varphi(k-1)]$$

for any $\xi \in \{0,1\}^k$. That is,

$$\Omega[\varphi(0) \otimes \varphi(1) \otimes \cdots \otimes \varphi(k-1)] = \{0,1\}^k \quad (k = 1, 2, \ldots),$$

which implies that

$$\Omega[\varphi(0) \otimes \varphi(1) \otimes \varphi(2) \otimes \cdots] = \{0,1\}^{\mathbb{N}}.$$

Therefore for any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$, if $\{2^0, 2^2, 2^4, \cdots\} \in \chi$, then $\Omega[\chi^\infty] = \{0,1\}^{\mathbb{N}}$ and χ is attainable (Theorem 1).

Moreover, we can prove $\Omega[\chi^\infty] = \{0,1\}^{\mathbb{N}}$ for any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ using Theorem 2 ([15]).

4 Super-subwords

The set $\cup_{n=0}^{\infty} \mathbb{A}^n$ of finite words over $\mathbb{A}$ is denoted by $\mathbb{A}^*$. The empty word is denoted by ε, which is the unique element in $\mathbb{A}^0$. We also denote $\mathbb{A}^+ = \mathbb{A}^* \setminus \{\varepsilon\}$. For $\xi \in \mathbb{A}^k$, k is called the *length* of ξ which is denoted by $|\xi|$. We denote $\xi \in \mathbb{A}^*$ with $|\xi| = k$ as $\xi_1\xi_2 \cdots \xi_k$, where $\xi_i \in \mathbb{A}$ $(i = 1, 2, \cdots .k)$. For $\xi, \eta \in \mathbb{A}^*$, $\xi\eta = \xi_1 \cdots \xi_k \eta_1 \cdots \eta_l \in \mathbb{A}^{k+l}$ is the *concatenation*, where $\xi = \xi_1 \cdots \xi_k \in \mathbb{A}^k$ and $\eta = \eta_1 \cdots \eta_l \in \mathbb{A}^l$. Furthermore, $\xi^{-1}\eta = \xi_k^{-1} \cdots \xi_1^{-1}\eta \in \mathbb{A}^*$ and $\xi\eta^{-1} = \xi\eta_l^{-1} \cdots \eta_1^{-1} \in \mathbb{A}^*$ are defined step by step by

$$a^{-1}\eta = \begin{cases} \eta_2 \cdots \eta_l & (\text{if } \eta_1 = a) \\ \eta_1\eta_2 \cdots \eta_l & (\text{otherwise}) \end{cases}, \quad \xi a^{-1} = \begin{cases} \xi_1 \cdots \xi_{k-1} & (\text{if } \xi_k = a) \\ \xi_1 \cdots \xi_{k-1}\xi_k & (\text{otherwise}) \end{cases}$$

for all $a \in \mathbb{A}$. Thus for any $\xi, \zeta, \eta \in \mathbb{A}^*$, $\xi^{-1}\zeta\eta^{-1} \in \mathbb{A}^*$ is defined, and for $\Xi \subset \mathbb{A}^*$ and $\xi, \eta \in \mathbb{A}^*$, $\xi^{-1}\Xi\eta^{-1} \subset \mathbb{A}^*$ is defined as $\xi^{-1}\Xi\eta^{-1} = \{\xi^{-1}\zeta\eta^{-1};\ \zeta \in \Xi\}$.

For any $\omega \in \Omega$ and $\xi, \eta \in \mathbb{A}^*$, with $|\xi| = k$, ξ is called a *super-subword* of ω or η if there exists $1 \le n_1 < n_2 < \cdots < n_k$ such that $\omega(n_1-1)\omega(n_2-1) \cdots \omega(n_k-1) = \xi$, or $n_k \le |\eta|$ and $\eta_{n_1}\eta_{n_2} \cdots \eta_{n_k} = \xi$. In these cases, we denote $\xi \ll \omega$ or $\xi \ll \eta$. For $\Xi \subset \mathbb{A}^*$, let Ξ_{min} be the set of *minimal* elements in Ξ with respect to the relation $\ll$, that is

$$\Xi_{min} = \{\xi \in \Xi;\ \text{there does not exists } \eta \in \Xi \text{ such that } \eta \ll \xi \text{ and } \eta \ne \xi\}.$$

Let $\Xi \subset \mathbb{A}^*$. We denote $\mathcal{P}(\Xi)$ the set of $\omega \in \mathbb{A}^{\mathbb{N}}$ such that $\xi \ll \omega$ does not hold for any $\xi \in \Xi$. If $\Xi = \emptyset$ (empty set), then $\mathcal{P}(\Xi) = \mathbb{A}^{\mathbb{N}}$. To the contrary if $\varepsilon \in \Xi$, then $\mathcal{P}(\Xi) = \emptyset$. In the same way, for any nonempty set Ξ of $\mathbb{A}^*$, we denote $\mathcal{Q}(\Xi)$ the set of $\omega \in \mathbb{A}^{\mathbb{N}}$ such that $\xi \ll \omega$ does not hold for some $\xi \in \Xi$.

A set $\Xi \subset \mathbb{A}^*$ is said to satisfy (#) if

Condition (#): there does not exist $\xi, \eta \in \mathbb{A}^*$ such that

$$(\xi^{-1}\Xi\eta^{-1})_{min} = \{a;\ a \in \mathbb{A}\}.$$

Theorem 3. [10] *For a nonempty closed subset $\Omega \subset \mathbb{A}^{\mathbb{N}}$, Ω is super-stationary if and only if there exists a finite set $\Xi \subset \mathbb{A}^+$ satisfying (#) such that $\Omega = \mathcal{P}(\Xi)$.*

Example 4. Let $\mathbb{A} = \{0, 1\}$ and $\Xi = \{00, 10\}$. Then, $\mathcal{P}(\Xi) = \{01^\infty, 1^\infty\}$ and is not super-stationary since $\mathcal{P}(\Xi)[1 \otimes 2 \otimes 3 \cdots] = \{1^\infty\} \neq \mathcal{P}(\Xi)$.

Another example is $\Xi = \{000, 010\}$. Then,

$$\mathcal{P}(\Xi) = \{1^n \xi 1^\infty;\ n = 0, 1, 2 \cdots,\ \xi \in \{\varepsilon, 0, 00\}\}$$

and is not super-stationary since

$$\mathcal{P}(\Xi)[0 \otimes 2 \otimes 4 \cdots] = \{1^n \xi 1^\infty;\ n = 0, 1, 2 \cdots,\ \xi \in \{\varepsilon, 0\}\} \neq \mathcal{P}(\Xi).$$

Both cases, the condition (#) does not hold.

Let $\Xi \subset \mathbb{A}^*$. It is called *non-comparable*, if for any $\xi, \eta \in \Xi$ with $\xi \neq \eta$, neither $\xi \ll \eta$ nor $\eta \ll \xi$ holds. It is known [10] that any non-comparable set is finite. We call $\zeta \in \mathbb{A}^*$ a *cover* of Ξ if $\xi \ll \zeta$ holds for any $\xi \in \Xi$. A cover ζ of Ξ is called *minimal* if there does not exists a cover $\zeta' \neq \zeta$ of Ξ such that $\zeta' \ll \zeta$. The set of minimal covers of Ξ is called the *least common multiple* of Ξ and is denoted by $lcm(\Xi)$. We call $\zeta \in \mathbb{A}^*$ a *core* of Ξ if $\zeta \ll \xi$ holds for any $\xi \in \Xi$. A core ζ of Ξ is called *maximal* if there does not exists a core $\zeta' \neq \zeta$ of Ξ such that $\zeta \ll \zeta'$. The set of maximal cores of Ξ is called the *greatest common divisor* of Ξ and is denoted by $gcd(\Xi)$.

Theorem 4. [9]
Let $\Xi \subset \mathbb{A}^{\mathbb{N}}$ be an arbitrary non-comparable (hence, finite), nonempty set. Then,
(1) *$gcd(lcm(\Xi)) = \Xi$ holds, hence $\mathcal{P}(lcm(\Xi)) = \mathcal{Q}(\Xi)$,*
(2) *$lcm(gcd(\Xi)) = \Xi$ holds if and only if $\Xi = lcm(\Theta)$ holds for some nonempty finite set $\Theta \subset \mathbb{A}^{\mathbb{N}}$.*

5 Pattern recognitions and optimal positions

Let $\Sigma = \mathbb{R}^2$ and take $\omega \in \mathbb{A}^\Sigma$. It is considered as a *picture* drawn on $\mathbb{R}^2$, where $\omega(x) \in \mathbb{A}$ is the color put on the point $x \in \mathbb{R}^2$ and $\mathbb{A}$ is considered as the set of colors.

Here, we restrict to the monochromatic case that $\mathbb{A} = \{0, 1\}$. Then, the picture $\omega \in \Omega$ is identified with the subset $\{x \in \Sigma;\ \omega(x) = 1\} \subset \Sigma$ and the restriction $\omega|_S$ the subset $\omega \cap S$. Take a finite observation points $S \subset \Sigma$. We are interested in maximizing $\#\{\omega \cap S;\ \omega \in \Omega\}$ among S of the same size. That is, to maximize the number of distinguished pictures by the observation of a fixed number of points. This maximum value for $\#S = k$ is denoted by $p^*_\Omega(k)$ and the function $p^*_\Omega : \mathbb{N} \to \mathbb{N}$ is called the *maximal pattern complexity* of Ω.

An infinite subset $\Theta \subset \Sigma$ is called an *optimal position* if for any $k \in \mathbb{N}$ and $S \subset \Theta$ with $\#S = k$, $\#\{\omega \cap S;\ \omega \in \Omega\} = p^*_\Omega(k)$ holds. In this case, we can restrict our observation points in Θ, and the maximal pattern complexity becomes a *uniform complexity* (Section 5).

As Ω we take the following closed sets and discuss the maximal pattern complexity together with the existence of an optimal position.

$\mathcal{L}$ = the class of straight lines in $\mathbb{R}^2$

$\mathcal{H}$ = the class of half planes bounded by straight lines in $\mathbb{R}^2$

$\mathcal{D}_1 =$ the class of unit discs in $\mathbb{R}^2$

$\mathcal{D} =$ the class of discs in $\mathbb{R}^2$

$\mathcal{Q}_1 =$ the class of unit squares in $\mathbb{R}^2$ with edges parallel to fixed orthogonal directions

$\mathcal{Q} =$ the class of squares in $\mathbb{R}^2$ with edges parallel to fixed orthogonal directions

$\mathcal{R} =$ the class of rectangles in $\mathbb{R}^2$ with edges parallel to fixed orthogonal directions

$\mathcal{C}_n =$ the class of convex n-polygons in $\mathbb{R}^2$, $(n = 3, 4, \cdots)$

$\mathcal{C}_\infty =$ the class of convex n-polygons with arbitrary $n = 3, 4, \cdots$ in $\mathbb{R}^2$

Theorem 5. [13]
(1) *We have* $p^*_{\mathcal{L}}(k) = \frac{1}{2}k^2 + \frac{1}{2}k + 1$ $(k = 1, 2, \cdots)$. *Moreover,* $\Theta \subset \mathbb{R}^2$ *with* $\#\Theta = \infty$ *is an optimal position for* $\mathcal{L}$ *if and only if any* 3 *points in* Θ *are not on a line.*
(2) *We have* $p^*_{\mathcal{D}_1}(k) = k^2 - k + 2$ $(k = 1, 2, \cdots)$. *Moreover,* $\Theta \subset \mathbb{R}^2$ *with* $\#\Theta = \infty$ *is an optimal position for* $\mathcal{D}_1$ *if* Θ *is a subset of a circle with radius* δ *such that* $0 < \delta < 1$.
(3) *We have* $p^*_{\mathcal{Q}_1}(k) = k^2 - k + 2$ $(k = 1, 2, \cdots)$. *An optimal position for* $\mathcal{Q}_1$ *does not exist.*
(4) *We have* $p^*_{\mathcal{H}}(k) = k^2 - k + 2$ $(k = 1, 2, \cdots)$. *Moreover,* $\Theta \subset \mathbb{R}^2$ *with* $\#\Theta = \infty$ *is an optimal position for* $\mathcal{H}$ *if* Θ *is a subset of a circle with radius* δ *such that* $\delta > 0$.
(5) *We have* $p^*_{\mathcal{C}_\infty}(k) = 2^k$ $(k = 1, 2, \cdots)$. *Moreover,* $\Theta \subset \mathbb{R}^2$ *with* $\#\Theta = \infty$ *is an optimal position for* $\mathcal{C}_\infty$ *if and only if* Θ *is a subset of the boundary of a strictly convex set.*

Theorem 6. [13]
(1) *We have* $p^*_{\mathcal{D}}(k) \asymp k^3$ *as* $k \to \infty$ *in the sense that*

$$0 < \liminf_{k\to\infty} p^*_{\mathcal{D}}(k)/k^3 \le \limsup_{k\to\infty} p^*_{\mathcal{D}}(k)/k^3 < \infty.$$

(2) *We have* $p^*_{\mathcal{Q}}(k) \asymp k^3$ *as* $k \to \infty$.
(3) *We have* $p^*_{\mathcal{R}}(k) \asymp k^4$ *as* $k \to \infty$.
(4) *We have* $k^{2n} \prec p^*_{\mathcal{C}_n}(k) \prec k^{2n+1}$ *as* $k \to \infty$ *for any* $n = 3, 4, \cdots$ *in the sense that*

$$\liminf_{k\to\infty} p^*_{\mathcal{C}_n}(k)/k^{2n} > 0 \text{ and } \limsup_{k\to\infty} p^*_{\mathcal{C}_n}(k)/k^{2n+1} < \infty.$$

Remark 2. We can also considered the problem of maximizing the number of partition of Σ generated by the finite sets of patterns in Ω of a fixed size. This is the *dual* problem of the pattern recognition problem to attain $p^*_\Omega(k)$ and is also discussed in [13].

Remark 3. Some results in the above theorems (e.g. for $\mathcal{L}$, $\mathcal{H}$, $\mathcal{Q}$, $\mathcal{R}$) are well known in term of VC-dimension([2, 3]. But some exact values and the notions of duality and optimal position have not been discussed so far.

6 Uniform sets and uniform complexities

Let Σ be any infinite set. A nonempty closed subset Ω of $\mathbb{A}^\Sigma$ is called a *uniform set* if for any finite set $S \subset \Sigma$, $\#\Omega|_S$ depends only on $\#S$, where $\Omega|_S$ is the restriction of the mapping $\Omega : \Sigma \to \mathbb{A}$ to S. In this case, $\#\Omega|_S$ is denote by $p_\Omega(n)$ if $n = \#S$, and the function $p = p_\Omega : \mathbb{N} \to \mathbb{N}$ is called the *uniform complexity* of Ω. To study this function, we may assume without loss of generality that $\Sigma = \mathbb{N}$. The *entropy* $h(p)$ and the *degree* $d(p)$ of the uniform complexity p are defined as

$$h(p) = \lim_{n\to\infty} \frac{\log p(n)}{n}\ ,\ d(p) = \lim_{n\to\infty} \frac{\log p(n) - h(p)n}{\log n},$$

the limits being known to exist and are nonnegative integers ([10]).

Remark 4. The above $h(p)$ for any maximal pattern complexity is known to be nonnegative integer (with Wen Huan and Xiangdong Ye [11]). Therefore, the result here is just a special case, while it is not known the result on $d(n)$ for the maximal pattern complexity.

The infinitesimal Ramsey theorem ([4]) leads the first part of (1) of the following Theorem.

Theorem 7. [10])
(1) *For any uniform set, there exists a super-stationary set having the same complexity* p*, hence, there exists a finite set* $\Xi \subset \mathbb{A}^+$ *with the condition* (#) *such that* $p(n) = p_{\mathcal{P}(\Xi)}(n)$ $(\forall n \in \mathbb{N})$.
(2) *For any uniform complexity* p *over* $\mathbb{A}$ *with* $\#\mathbb{A} = d$*, either* $p(n) = d^n$ $(\forall n \in \mathbb{N})$ *or there exists a positive integer* $k \leq d-1$ *and polynomials* $r_1, \cdots, r_k$ *of* n *with rational coefficients with* $r_k \not\equiv 0$ *such that* $p(n) = \sum_{i=1}^k i^n r_i(n)$ *holds for any sufficiently large* $n \in \mathbb{N}$*. The former case,* $h(p) = \log d$ *and* $\deg(p) = 0$*, while the latter case,* $h(p) = \log k$ *and* $d(p) = \deg r_k$.

Theorem 8. [9]
For any uniform complexity p *over* $\mathbb{A}$ *with* $\#\mathbb{A} = 2$ *other than* $p(n) = 2^n$ $(\forall n \in \mathbb{N})$*, there exists a nonempty finite set* $\Xi \subset \mathbb{A}^+$ *such that* $p(n) = p_{\mathcal{Q}(\Xi)}(n)$ $(\forall n \in \mathbb{N})$*, hence* $p(n) = p_{\mathcal{P}(lcm(\Xi))}(n)$.

Example 5. [8] Let us list up below all the uniform complexities with entropy 0 and degree ≤ 1 in the case $\mathbb{A} = \{0,1\}$. If Ξ contains ξ with $|\xi| \geq 3$, then $p_{\mathcal{Q}(\Xi)}(k)$ is a polynomial of degree ≥ 2 since

$$p_{\mathcal{Q}(\Xi)}(k) \geq p_{\mathcal{Q}(\{\xi\})}(k) = \sum_{i=0}^{|\xi|-1} \binom{k}{i}.$$

Therefore by Theorem 5, the uniform complexities with degree ≤ 1 are realized by the unions of $\mathcal{P}(\{0\}), \mathcal{P}(\{1\}), \mathcal{P}(\{00\}), \mathcal{P}(\{01\}), \mathcal{P}(\{10\}), \mathcal{P}(\{11\})$. The above list contains all non-comparable unions of these 0-1-blocks up to the symmetry of exchanging 0 and 1. The 3rd column is complexity function $p_\Omega(k)$

with $\Omega = \mathcal{Q}(\Xi) = \mathcal{P}(lcm(\Xi))$, and the 4th column is the minimum k_0 such that these formulas hold for $k \geq k_0$. Here, the language trees of $\mathcal{Q}(\{11, 10\})$ and $\mathcal{Q}(\{11, 01\})$ naving the same uniform complexity are not isomorphic (Figure 1), while those of $\mathcal{Q}(\{11, 01\})$ and $\mathcal{Q}(\{10, 01\})$ are isomorphic.

Ξ	$lcm(\Xi)$	$p_\Omega(k)$	k_0
{1}	{1}	1	0
{1,0}	{10,01}	2	1
{11}	{11}	$k+1$	0
{10}	{10}		
{11,0}	{110,101,011}	$k+2$	2
{11,10}	{110,101}	$2k$	1
{11,01}	{101,011}		
{10,01}	{101,010}		
{11,00}	{1100,1010,1001,0110,0101,0011}	$2k+2$	3
{11,10,01}	{101,0101,0110}	$3k-2$	2
{11,10,00}	{1100,1010,1001,0110,0101}	$3k-1$	3
{11,10,01,00}	{1010,1001,0110,0101}	$4k-4$	2

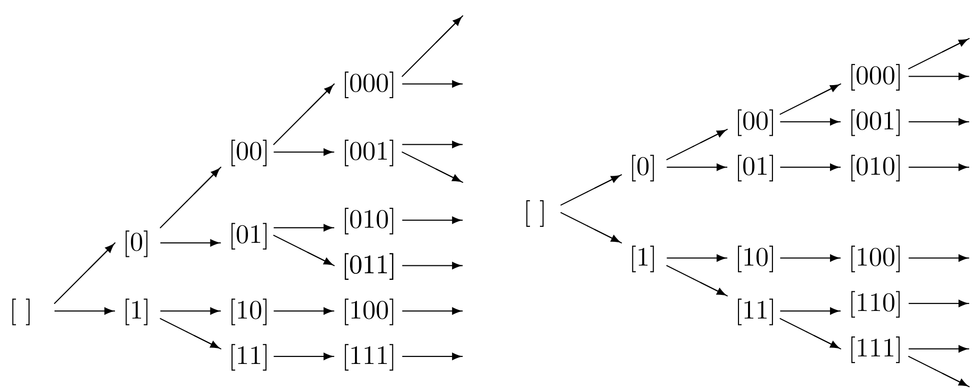


Figure 1: $\mathcal{Q}(\{11, 10\})$ (left), $\mathcal{Q}(\{11, 01\})$ (right)

7 Infinitesimal geometry at infinities

Let X, Y be infinite sets and $F : X \to Y$ be a mapping. For $\chi \in \beta X$, define $F(\chi) \in \beta Y$ by

$$F(\chi) = \{U \subset Y;\ F^{-1}U \in \chi\}.$$

Let X be a compact metric space with metric ρ_X and $\#X = \infty$. Let $\chi \in \beta X \setminus X$. Then χ determine a point in X denoted by $\lim \chi$ so that

$$x = \lim \chi \text{ if and only if } \{y \in X;\ \rho_X(x, y) < \epsilon\} \in \chi \text{ for any } \epsilon > 0.$$

Moreover, let X be a compact Riemanian manifold and $\chi \in \beta X \setminus X$. Let W be a small neighborhood of $\lim \chi$ belonging to χ. For distinct points $x_1, x_2 \in W$, let $\varphi(x_1, x_2)$ be the unit vector $\vec{x_2 x_1}/||\vec{x_2 x_1}||$ with respect to a local ortho-normal coordinate. Then, $v := \lim \varphi(\chi^2)$ is a unit tangent vector of X at $\lim \chi$ which is called the *tangent vector* of χ. Actually, it satisfied that

$$\{(x_1, x_2) \in X \times X;\ ||\ v - \vec{x_2 x_1}/||\vec{x_2 x_1}||\ || < \epsilon\} \in \chi^2$$

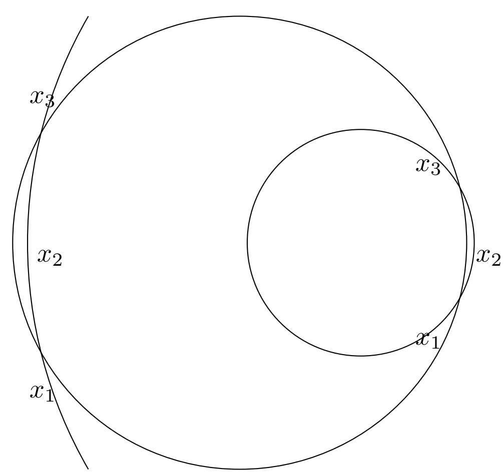
for any $\epsilon > 0$. Furthermore, in the case $\dim X = 2$, we define the *radius* of χ as $\lim \psi(\chi^3)$, where $\psi(x_1, x_2, x_3)$ with distinct $x_1, x_2, x_3 \in W$ is the radius of the circle passing these points in this order. The radius of χ is denoted by $Rad(\chi)$. In this way, we can define local geometric quantities at $\chi \in \beta X \setminus X$.

Example 6. [15] Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ be an irrational vector, that is, $1, \alpha_1, \alpha_2$ are linearly independent over the rational field. Let $x \mapsto x + \alpha$ be the rotation of $(\mathbb{R}/\mathbb{Z})^2$ by α. For $0 < \delta < 1/4$, let $\mathcal{D}$ be the closed disc with radius δ and center at the origin. For $x \in (\mathbb{R}/\mathbb{Z})^2$, define $\omega_0 \in \{0,1\}^{\mathbb{N}}$ by $\omega_x(n) = 1$ if and only if $n\alpha \in \mathcal{D}$ $(mod\ 1)$. Let $\Omega \subset \{0,1\}^{\mathbb{N}}$ be the closure of $\{T^n \omega_0;\ n \in \mathbb{N}\}$. Then, for any $\chi \in \beta(\mathbb{R}/\mathbb{Z})^2 \setminus (\mathbb{R}/\mathbb{Z})^2$, we have

$$\Omega[\chi^\infty] = \begin{cases} \mathcal{P}(101) & Rad(\chi) \geq \delta \\ \mathcal{P}(0101, 1010) & Rad(\chi) < \delta \end{cases}.$$

These remote moves are attainable.

In the picture below, the disc surrounded by the central circle is $\mathcal{D}$. The right hand side shows the case $Rad(\chi) < \delta$, where subword 101 is possible under the condition that both before and after it should be 1. The left hand side shows the case $Rad(\chi) > \delta$, where 101 is impossible.



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(All the above papers except for [1][2][3][4][16] can be downloaded from the site: http://www14.plala.or.jp/kamae)