Expansions by reciprocals of integers, rational approximations and absolutely normal sequences

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1 Introduction

Any real number $x \in [0,1)$ can be expressed as a finite or infinite sum:

$$x = \frac{1}{n_0} + \frac{1}{n_1} + \frac{1}{n_2} + \cdots , \tag{1.1}$$

$$\text{where } 2 \le n_0 < n_1 < n_2 < \cdots \text{ are integers,} \tag{1.2}$$

which is called an *expansion by reciprocals of integers* of x. This expansion is called *finite* or *infinite* according to whether the right hand is a finite sum or not. The empty sum means 0.

On the space of finite or infinite sequences of integers $n_0 n_1 n_2 \cdots$ satisfying (1.2), we define the *lexicographic order* $<_{lex}$ as follows:

$$n_0 n_1 n_2 \cdots <_{lex} n'_0 n'_1 n'_2 \cdots \tag{1.3}$$

if and only if there exists $k = 0, 1, \cdots$ such that $n_i = n'_i$ for any $i = 0, 1, \cdots, k-1$ and that either $n_k < n'_k$ or n'_k is not defined while n_k is defined. Note that n'_k is not defined is equivalent to say that $n'_k = \infty$. In this sence the empty sequence is largest.

Various expansions of this type are known, but in this paper, we specially consider the following 2 of them.

Greedy expansion (GE, for short), that is, among the expansion by reciprocals of integers (1.1) of x, $n_0 n_1 n_2 \cdots$ is lexicographically smallest.

Lazy expansion (LE, for short), that is, (1.1) is defined by the following algorithm:
Let $f : [0,1) \to [0,1)$ be

$$f(x) = \begin{cases} \left(\lceil \frac{1}{x} \rceil - 1\right)\left(x - \frac{1}{\lceil \frac{1}{x} \rceil}\right) & (x > 0) \\ \quad undefined & (x = 0) \end{cases} , \tag{1.4}$$

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and define $a_k : [0,1) \to \{2,3,\cdots\}$ $(k = 0,1,2,\cdots)$ by

$$a_k(x) = \lceil \frac{1}{f^k(x)} \rceil. \tag{1.5}$$

Then, the finite or infinite sequence (1.2) is given by

$$n_0 = a_0(x),\ n_1 = (a_0(x)-1)a_1(x),\ n_2 = (a_0(x)-1)(a_1(x)-1)a_2(x),\ \cdots. \tag{1.6}$$

The sequence $a_0(x), a_1(x), a_2(x), \cdots$ is called the *partial lazy quotient* of x.

GE also has an expression by a piecewise linear function. That is, define a piecewise linear function g as follows:

$$g(x) = \begin{cases} x - \frac{1}{\lceil \frac{1}{x} \rceil} & (x > 0) \\ undefined & (x = 0) \end{cases}, \tag{1.7}$$

and define

$$A_k(x) = \lceil \frac{1}{g^k(x)} \rceil \ (k = 0,1,2,\cdots). \tag{1.8}$$

This gives the GE of $x \in [0,1)$ with $n_k = A_k(x)$ $(k = 0,1,2,\cdots)$ in (1.1). The sequence $A_0(x), A_1(x), A_2(x), \cdots$ is called the *greedy quotient*.

GE and TE are well known as *Sylvester expansion* ([2], for example) and *modified Engel expansion* ([3], for example), but here just because of the comparison, we call them GE and LE.

We give 4 examples of finite GE and LE. In the first example, GE and LE coincide but in the other 3, they differ.

$$\frac{23}{30} = \frac{1}{2} + \frac{1}{4} + \frac{1}{60} = \frac{1}{2} + \frac{1}{4} + \frac{1}{60}$$
$$\frac{59}{120} = \frac{1}{3} + \frac{1}{7} + \frac{1}{65} + \frac{1}{10920} = \frac{1}{3} + \frac{1}{8} + \frac{1}{30}$$
$$\frac{19}{39} = \frac{1}{3} + \frac{1}{7} + \frac{1}{91} = \frac{1}{3} + \frac{1}{8} + \frac{1}{36} + \frac{1}{960} + \frac{1}{38130} + \frac{1}{2083200} + \frac{1}{304854000} + \frac{1}{45421200000} + \frac{1}{4208418013800000}$$
$$e - 2 = \frac{1}{2} + \frac{1}{5} + \frac{1}{55} + \frac{1}{9999} + \frac{1}{3620211523} + \frac{1}{25838201785967533906} + \frac{1}{340884736660545309114055821832202344 0765} + \cdots$$
$$= \frac{1}{2} + \frac{1}{5} + \frac{1}{56} + \frac{1}{2392} + \frac{1}{152100} + \frac{1}{19768320} + \frac{1}{5179299840} + \cdots,$$

where the Taylor expansion

$$e - 2 = \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots + \frac{1}{n!} + \cdots$$

is lexicographically larger than both of GE and LE.

The LE of 2/5 :

$$\frac{2}{5} = \frac{1}{3} + \frac{1}{16} + \frac{1}{252} + \frac{1}{5236} + \frac{1}{134946} + \frac{1}{59386236} + \frac{1}{352811938752} + \frac{1}{2157082653267360} + \frac{1}{2440447719006839068800} + \cdots$$

seems to be infinite, but we cannot prove (see Section 5).

The following theorems are well known, but we give proofs of some of them by the reasons of self-containedness and difficulty to get the paper.

Theorem 1. *(A. Rényi [3])*
(1) A finite or infinite sequence of integers $b_0, b_1, b_2, \cdots$ *is the partial lazy quotient of some* $x \in [0,1)$ *if and only if*
(1-1) $b_0 \geq 2,\ b_{k+1} \geq b_k + 1\ (k = 0,1,2,\cdots)$, *and*
(1-2) either it is finite or the strict inequality holds infinitely often in (1-1).
(2) For any $x, y \in [0,1)$, $y < x$ *if and only if*

$$a_0(x)a_1(x)a_2(x)\cdots <_{lex} a_0(y)a_1(y)a_2(y)\cdots.$$

(3) The partial lazy quotient $\{a_k = a_k(x);\ k = 0,1,2,\cdots\}$ *define a Markov process under the Lebesgue measure on* $x \in [0,1)$ *such that*

$$\mathbb{P}(a_0 = n) = \frac{1}{(n-1)n} \text{ and} \tag{1.9}$$

$$\mathbb{P}(a_{k+1} = n \mid a_k = m) = \begin{cases} \frac{m}{(n-1)n} & n \geq m+1 \\ 0 & otherwise \end{cases} \tag{1.10}$$

for any $m = k+2, k+3, \cdots\ (k = 0,1,2,\cdots)$.

Theorem 2. *(A. Rényi [3])*
(1) Let $X_n = \mathbf{1}_{n \in \{a_0,a_1,a_2,\cdots\}}\ (n = 2,3,\cdots)$. *Then,* $X_2, X_3, \cdots$ *are independent random variables such that* $\mathbb{E}(X_n) = \mathbb{P}(X_n = 1) = 1/n\ (n = 2,3,\cdots)$.
(2) Conversely, the distribution of the random variables $a_0, a_1, a_2, \cdots$ *is characterized as above. That is, if* $X_2, X_3, \cdots$ *are independent random variables such that* $\mathbb{P}(X_n = 1) = 1/n\ (n = 2,3,\cdots)$ *and let* $a'_0 < a'_1 < a'_2 < \cdots$ *be such that* $\{a'_0, a'_1, a'_2, \cdots\} = \{i;\ X_i = 1\}$, *then*

$$(a'_0, a'_1, a'_2, \cdots) \sim_{\text{law}} (a_0, a_1, a_2, \cdots).$$

(3) For almost all $x \in [0,1)$ *with respect to the Lebesgue measure, it holds that*

$$\lim_{n\to\infty} \frac{\#\{i;\ a_i \leq n\}}{\log n} = 1.$$

Theorem 3. *(P. Erdős, A. Rényi and P. Szűsz [2])*
(1) The greedy quotient $\{A_k = A_k(x);\ k = 0,1,2,\cdots\}$ *define a Markov*

process under the Lebesgue measure on $x \in [0,1)$ such that

$$\mathbb{P}(A_0 = n) = \frac{1}{(n-1)n}, \quad \textit{and} \tag{1.11}$$

$$\mathbb{P}(A_{k+1} = n \mid A_k = m) = \begin{cases} \frac{(m-1)m}{(n-1)n} & n \geq D(m) \\ 0 & \textit{otherwise} \end{cases}. \tag{1.12}$$

for any $k = 0, 1, 2, \cdots$ and $m = D^k(2), D^k(2)+1, D^k(2)+2, \cdots$, where $D(m) = (m-1)m+1$ $(m = 2, 3, \cdots)$ and $D^2(m) = D(D(m)) = D((m-1)m+1) = (m-1)m((m-1)m+1)+1, \ \cdots$.
(2) For any rational number $x \in [0,1)$, the GE of x is finite. The length of the expansion is at most p, where $x = p/q$ is the irreducible fraction.
(3) For almost all $x \in [0,1)$ with respect to the Lebesgue measure,

$$\lim_{k \to \infty} \frac{\log A_k(x)}{2^k}$$

exists. This value may depends on $x \in [0,1)$.

Definition 1. An infinite sequence of integers $2 \leq b_0 < b_1 < b_2 < \cdots$ is said to be *absolutely normal* if for any $r = 2, 3, \cdots$, the sequence

$$b_0 b_1 b_2 \cdots \pmod{r}$$

is a r-adic normal number.

Theorem 4. *Almost all $A_0(x)A_1(x)A_2(x), \cdots$ and $a_0(x)a_1(x)a_2(x) \cdots$ with respect to the Lebesgue measure on $x \in [0,1)$ are absolutely normal.*

2 Proof of Theorem 1

Lemma 1. *Let $x \in [0,1)$ and $b_0 = a_0(x), b_1 = a_1(x), b_2 = a_2(x), \cdots$.*
(1) It holds $2 \leq b_0 < b_1 < b_2 < \cdots$ and that

$$1 \leq b_0 - 1 < \frac{1}{x} \leq b_0 \leq b_1 - 1 < \frac{1}{f(x)} \leq b_1 \leq \cdots$$
$$\leq b_{k-1} \leq b_k - 1 < \frac{1}{f^k(x)} \leq b_k \leq \cdots.$$

(2) For any $k = 1, 2, \cdots$, we have

$$x = \frac{1}{b_0} + \frac{1}{(b_0-1)b_1} + \cdots + \frac{1}{(b_0-1)\cdots(b_{k-2}-1)b_{k-1}} + \frac{f^k(x)}{(b_0-1)\cdots(b_{k-2}-1)(b_{k-1}-1)}$$
$$= \frac{1}{b_0} + \frac{1}{(b_0-1)b_1} + \cdots + \frac{1}{(b_0-1)\cdots(b_{k-2}-1)b_{k-1}} + \cdots,$$

hence (1.6) *implies* (1.1).
(3) For $y \in [0,1)$, $a_0(y) = b_0, a_1(y) = b_1, \cdots, a_k(y) = b_k$ *holds if and only if*

$$\frac{1}{b_0}+\frac{1}{(b_0-1)b_1}+\cdots+\frac{1}{(b_0-1)\cdots(b_{k-2}-1)b_{k-1}}+\frac{1}{(b_0-1)\cdots(b_{k-1}-1)b_k}$$
$$\le y < \frac{1}{b_0}+\frac{1}{(b_0-1)b_1}+\cdots+\frac{1}{(b_0-1)\cdots(b_{k-2}-1)b_{k-1}}$$
$$+\frac{1}{(b_0-1)\cdots(b_{k-1}-1)(b_k-1)}.$$

Proof (1) If $x \in [0,1)$, then as long as $b_0 = a_0(x), b_1 = a_1(x), b_2 = a_2(x), \cdots$ are defined, it holds that

$$b_0 = \lceil \frac{1}{x} \rceil \ge 2 \text{ and } b_0 - 1 < \frac{1}{x} \le b_0. \tag{2.1}$$

Hence,

$$0 \le x - \frac{1}{b_0} < \frac{1}{b_0-1} - \frac{1}{b_0} = \frac{1}{(b_0-1)b_0}.$$

Therefore,

$$0 \le f(x) = (b_0-1)(x - \frac{1}{b_0}) < \frac{1}{b_0},$$

so that $b_0 < \frac{1}{f(x)}$ follows. Also, replacing x in (2.1) by $f(x)$, we have

$$b_1 - 1 = a_0(f(x)) - 1 < \frac{1}{f(x)} \le a_0(f(x)) = b_1.$$

From these inequalities, it follows that $b_0 < b_1$, and that

$$b_0 \le b_1 - 1 < \frac{1}{f(x)} \le b_1.$$

Replacing x by $f(x)$ in the above, we have

$$b_1 \le b_2 - 1 < \frac{1}{f^2(x)} \le b_2.$$

Repeating this, we have $2 \le b_0 < b_1 < b_2 < \cdots$ and

$$1 \le b_0 - 1 < \frac{1}{x} \le b_0 \le b_1 - 1 < \frac{1}{f(x)} \le b_1 \le \cdots$$
$$\le b_{k-1} \le b_k - 1 < \frac{1}{f^k(x)} \le b_k \le \cdots.$$

(2) Since $b_0 = a_0(x) = \lceil 1/x \rceil$ and $f(x) = (b_0-1)(x - \frac{1}{b_0})$,

$$x = \frac{1}{b_0} + \frac{f(x)}{b_0 - 1} \tag{2.2}$$

holds. Since $f(x) = \frac{1}{b_1} + \frac{f^2(x)}{b_1 - 1}$, we have

$$x = \frac{1}{b_0} + \frac{f(x)}{b_0 - 1} = \frac{1}{b_0} + \frac{\frac{1}{b_1} + \frac{f^2(x)}{b_1-1}}{b_0 - 1} = \frac{1}{b_0} + \frac{1}{(b_0 - 1)b_1} + \frac{f^2(x)}{(b_0 - 1)(b_1 - 1)}.$$

Repeating this, we have the first half of the equation in (2). The rest follows since the last term in the 2nd side of the equation is either 0 or tends to 0 as $k \to \infty$.

(3) Assume that $y \in [0, 1)$ satisfies that $a_0(y) = b_0, a_1(y) = b_1, \cdots, a_k(y) = b_k$. Then by (1), $b_k - 1 < \frac{1}{f^k(y)} \leq b_k$. Hence, by (2) with $k - 1$ in place of k, we have

$$\frac{1}{b_0} + \frac{1}{(b_0 - 1)b_1} + \cdots + \frac{1}{(b_0 - 1)\cdots(b_{k-2} - 1)b_{k-1}} + \frac{1}{(b_0 - 1)\cdots(b_{k-1} - 1)b_k}$$
$$\leq y < \frac{1}{b_0} + \frac{1}{(b_0 - 1)b_1} + \cdots + \frac{1}{(b_0 - 1)\cdots(b_{k-2} - 1)b_{k-1}} + \frac{1}{(b_0 - 1)\cdots(b_{k-1} - 1)(b_k - 1)}.$$

Moreover, $f^k(y)$ is linear in this interval with the image equal to $[\frac{1}{b_k}, \frac{1}{b_k - 1})$. Therefore, the above interval coincides the set of y with $a_0(y) = b_0, a_1(y) = b_1, \cdots, a_k(y) = b_k$, which proves (3). □

Lemma 2. *Let $2 \leq b_0 < b_1 < b_2 < \cdots$ be an arbitrary infinite sequence of integers. Then, we have*

$$\frac{1}{b_0} + \frac{1}{(b_0 - 1)b_1} + \frac{1}{(b_0 - 1)(b_1 - 1)b_2} + \frac{1}{(b_0 - 1)(b_1 - 1)(b_2 - 1)b_3} + \cdots \leq \frac{1}{b_0 - 1}$$

The equality holds if and only if $b_k = b_0 + k$ holds for any $k = 1, 2, \cdots$.

Proof Note that

$$\frac{1}{b_0} + \frac{1}{(b_0 - 1)b_1} + \frac{1}{(b_0 - 1)(b_1 - 1)b_2} + \frac{1}{(b_0 - 1)(b_1 - 1)(b_2 - 1)b_3} + \cdots$$

converges. Assume that $b_k = b_0 + k$ holds for any $k = 1, 2, \cdots$. Put $B = b_0$. Then, we have

$$\frac{1}{b_0} + \frac{1}{(b_0 - 1)b_1} + \frac{1}{(b_0 - 1)(b_1 - 1)b_2} + \frac{1}{(b_0 - 1)(b_1 - 1)(b_2 - 1)b_3} + \cdots$$
$$= \frac{1}{B} + \frac{1}{(B - 1)(B + 1)} + \frac{1}{(B - 1)B(B + 2)} + \frac{1}{(B - 1)B(B + 1)(B + 3)} + \cdots$$
$$= \frac{1}{B - 1} - \frac{1}{(B - 1)B} + \frac{1}{(B - 1)(B + 1)} + \frac{1}{(B - 1)B(B + 2)} + \frac{1}{(B - 1)B(B + 1)(B + 3)} + \cdots$$
$$= \frac{1}{B - 1} - \frac{1}{(B - 1)B(B + 1)} + \frac{1}{(B - 1)B(B + 2)} + \frac{1}{(B - 1)B(B + 1)(B + 3)} + \cdots$$

$$= \frac{1}{B-1} - \frac{1}{(B-1)B(B+1)(B+2)} + \frac{1}{(B-1)B(B+1)(B+3)} + \cdots = \cdots$$
$$= \frac{1}{B-1} - \frac{1}{(B-1)B(B+1)\cdots(B+k)} + \frac{1}{(B-1)B(B+1)\cdots(B+k+1)} + \cdots$$
$$= \frac{1}{B-1}.$$

If $b_k = b_0 + k$ fails at $k = k_0$ for the first place. Then,

$$\frac{1}{(b_0-1)(b_1-1)\cdots(b_{k-1}-1)b_k} = \frac{1}{(B-1)B(B+1)\cdots(B+k-2)(B+k)}$$

holds for $k = 0, 1, \cdots, k_0 - 1$, but

$$\frac{1}{(b_0-1)(b_1-1)\cdots(b_{k-1}-1)b_k} < \frac{1}{(B-1)B(B+1)\cdots(B+k-2)(B+k)}$$

holds for $k = k_0, k_0+1, \cdots$. Thus, we have

$$\frac{1}{b_0} + \frac{1}{(b_0-1)b_1} + \frac{1}{(b_0-1)(b_1-1)b_2} + \frac{1}{(b_0-1)(b_1-1)(b_2-1)b_3} + \cdots < \frac{1}{b_0-1}.$$

□

Proof of Theorem 1:

Let $2 \le b_0 < b_1 < b_2 < \cdots$ be an arbitrary sequence of integers such that either it is finite or there does not exist k such that $b_{k+n} = b_k + n$ for any $n = 1, 2, \cdots$. Define

$$x = \frac{1}{b_0} + \frac{1}{(b_0-1)b_1} + \frac{1}{(b_0-1)(b_1-1)b_2} + \frac{1}{(b_0-1)(b_1-1)(b_2-1)b_3} + \cdots .$$

Then by Lemma 2, $\frac{1}{b_0} \le x < \frac{1}{b_0-1}$ holds. Hence, $b_0 - 1 < \frac{1}{x} \le b_0$ and $a_0(x) = b_0$. It also follows that

$$f(x) = (b_0-1)(y - \frac{1}{b_0}) = \frac{1}{b_1} + \frac{1}{(b_1-1)b_2} + \frac{1}{(b_1-1)(b_2-1)b_3} + \cdots .$$

Repeating this, we have $b_1 = \lceil 1/f(x) \rceil, b_2 = \lceil 1/f^2(x) \rceil, \cdots$, which proves the "if" part of (1).

Suppose that an infinite sequence $2 \le b_0 < b_1 < b_2 < \cdots$ of integers satisfies that there exists k such that $b_{k+n} = b_k + n$ for any $n = 1, 2, \cdots$ and that it is the partial lazy quotient of some $x \in [0, 1)$. Take the smallest k as this. Then, either $k = 0$ or $k > 0$ and $b_{k-1} \le b_k - 2$. Let $n_0, n_1, n_2, \cdots$ be defined by (1.6) with these $b_0, b_1, b_2, \cdots$ instead of $a_0(x), a_1(x), a_2(x), \cdots$. Then, putting

$$C = (b_0-1)(b_1-1)\cdots(b_{k-1}-1) \text{ and } K = b_k,$$

we have

$$\begin{aligned}
&\frac{1}{n_k}+\frac{1}{n_{k+1}}+\frac{1}{n_{k+2}}+\cdots\\
&=\frac{1}{C}\left(\frac{1}{K}+\frac{1}{(K-1)(K+1)}+\frac{1}{(K-1)K(K+2)}+\cdots\right)
\end{aligned}$$

$$\begin{aligned}
&=\frac{1}{C}\left(\frac{1}{K-1}-\frac{1}{(K-1)K}+\frac{1}{(K-1)(K+1)}+\frac{1}{(K-1)K(K+2)}+\cdots\right)\\
&=\frac{1}{C}\left(\frac{1}{K-1}-\frac{1}{(K-1)K(K+1)}+\frac{1}{(K-1)K(K+2)}+\cdots\right)\\
&=\frac{1}{C}\left(\frac{1}{K-1}-\frac{1}{(K-1)K(K+1)(K+2)}+\cdots\right)=\cdots=\frac{1}{C}\frac{1}{K-1}\\
&=\frac{1}{(b_0-1)(b_1-1)\cdots(b_{k-1}-1)(b_k-1)}
\end{aligned}$$

Hence, we have

$$x=\frac{1}{b_0}+\frac{1}{(b_0-1)b_1}+\cdots+\frac{1}{(b_0-1)(b_1-1)\cdots(b_k-1)}$$

with $2\le b_0<b_1<\cdots<b_{k-1}<b_k-1$. Therefore, x has the partial lazy quotient $b_0,b_1,\cdots,b_{k-1},b_k-1$, which contradicts with the assumption that the partial lazy quotient of x is $b_0,b_1,\cdots,b_{k-1},b_k,b_{k+1},\cdots$.

(2) follows from Lemma 2 and (1).

(3) It follows from Lemma 1 that for any integers $2\le b_0<b_1<\cdots<b_{k+1}$,

$$\begin{aligned}
&\mathbb{P}(a_0=b_0,a_1=b_1,\cdots,a_k=b_k)\\
&=\left(\frac{1}{b_0}+\frac{1}{(b_0-1)b_1}+\cdots+\frac{1}{(b_0-1)\cdots(b_{k-2}-1)b_{k-1}}\right.\\
&\qquad\left.+\frac{1}{(b_0-1)\cdots(b_{k-1}-1)(b_k-1)}\right)\\
&\quad-\left(\frac{1}{b_0}+\frac{1}{(b_0-1)b_1}+\cdots+\frac{1}{(b_0-1)\cdots(b_{k-2}-1)b_{k-1}}\right.\\
&\qquad\left.+\frac{1}{(b_0-1)\cdots(b_{k-1}-1)b_k}\right)\\
&\quad=\frac{1}{(b_0-1)\cdots(b_{k-1}-1)(b_k-1)b_k}.
\end{aligned}$$

In the same way, we have

$$\begin{aligned}
&\mathbb{P}(a_0=b_0,a_1=b_1,\cdots,a_k=b_k,a_{k+1}=b_{k+1})\\
&\qquad=\frac{1}{(b_0-1)\cdots(b_{k-1}-1)(b_k-1)(b_{k+1}-1)b_{k+1}}.
\end{aligned}$$

Hence, we have

$$\mathbb{P}(a_0 = b_0) = \frac{1}{(b_0-1)b_0}$$
$$\mathbb{P}(a_{k+1} = b_{k+1} \mid a_0 = b_0, a_1 = b_1, \cdots, a_k = b_k) = \frac{b_k}{(b_{k+1}-1)b_{k+1}},$$

which proves (3).

3 Proof of Theorem 2

(1) We first prove that

$$\mathbb{E}(X_n) = 1/n \tag{3.1}$$

by the induction on n. Since

$$\mathbb{E}(X_2) = \mathbb{P}(a_0 = 2) = \frac{1}{2},$$

(3.1) holds for $n = 2$. Let $n \geq 3$ and assume that (3.1) holds up to $n-1$. Then, using the first half of Theorem 2 and the introduction hypothesis, we have

$$\begin{aligned}\mathbb{E}(X_n) &= \mathbb{P}(n \in \{a_k;\ k = 0, 1, 2, \cdots\}) \\ &= \mathbb{P}(a_0 = n) + \sum_{m=2}^{n-1}\sum_{k=1}^{n-2} \mathbb{P}(a_k = n,\ a_{k-1} = m)\end{aligned}$$

$$\begin{aligned}&= \frac{1}{(n-1)n} + \sum_{m=2}^{n-1}\sum_{k=1}^{n-2} \frac{m}{(n-1)n}\mathbb{P}(a_{k-1} = m) \\ &= \frac{1}{(n-1)n} + \sum_{m=2}^{n-1} \frac{m}{(n-1)n}\mathbb{P}(m \in \{a_0, a_1, a_2, \cdots\}) \\ &= \frac{1}{(n-1)n} + \sum_{m=2}^{n-1} \frac{m}{(n-1)n}\ \frac{1}{m} = \frac{1}{n},\end{aligned}$$

which proves (3.1).

Now let us prove the independence of $X_2, X_3, \cdots$. It is sufficient to prove that for any $n \geq 1$ and $2 \leq i_1 < i_2 < \cdots < i_n$,

$$\mathbb{P}(X_{i_1}X_{i_2}\cdots X_{i_n} = 1) = \frac{1}{i_1 i_2 \cdots i_n}. \tag{3.2}$$

(3.2) holds for $n = 1$. Assume that $n \geq 2$ and (3.2) holds for $n - 1$. Since

$$\begin{aligned}
&\mathbb{P}(X_{i_1} X_{i_2} \cdots X_{i_n} = 1) \\
=&\mathbb{P}(X_{i_1} X_{i_2} \cdots X_{i_{n-1}} = 1)\mathbb{P}(X_{i_n} = 1 | X_{i_1} X_{i_2} \cdots X_{i_{n-1}} = 1) \\
=&\mathbb{P}(X_{i_1} X_{i_2} \cdots X_{i_{n-1}} = 1)\mathbb{P}(X_{i_n} = 1 | X_{i_{n-1}} = 1) \\
=&\frac{1}{i_1 i_2 \cdots i_{n-1}} \mathbb{P}(X_{i_n} = 1 | X_{i_{n-1}} = 1)
\end{aligned}$$

by the Markov property of $(a_n;\ n = 2, 3, \cdots)$, it is sufficient to prove that for any $2 \leq m < n$,

$$\mathbb{P}(X_n = 1 | X_m = 1) = \frac{1}{n}. \tag{3.3}$$

We use the induction on $n - m$. If $n - m = 1$, then (3.3) follows from (1.10). Assume that $n - m \geq 2$ and (4.3) holds for any smaller $n - m$. Let E be the event that $X_{m+1} = X_{m+2} = \cdots = X_{n-1} = 0$. Then by (1.10), we have

$$\begin{aligned}
&\mathbb{P}(X_n = 1 | X_m = 1) = \mathbb{P}(E \wedge X_n = 1 | X_m = 1) + \mathbb{P}(E^c \wedge X_n = 1 | X_m = 1) \\
=&\frac{m}{(n-1)n} + \mathbb{P}(X_n = 1 | E^c \wedge X_m = 1)\mathbb{P}(E^c | X_m = 1) \\
=&\frac{m}{(n-1)n} + \frac{1}{n} \sum_{k=m+1}^{n-1} \frac{m}{(k-1)k} = \frac{m}{(n-1)n} + \frac{m}{n}\left(\frac{1}{m} - \frac{1}{n-1}\right) = \frac{1}{n},
\end{aligned}$$

which completes the proof of (2).

(2) is clear since the distribution of $(a'_2, a'_3, a'_4, \cdots)$ is uniquely determined.

(3) Let

$$Y_n = \frac{X_2 + X_3 + \cdots + X_n}{\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}} \quad (n = 2, 3, \cdots).$$

Then, since

$$\mathbb{E}(X_i) = \frac{1}{i} \quad \text{and} \quad \mathbb{V}(X_i) = \frac{1}{i}\left(1 - \frac{1}{i}\right) \leq \frac{1}{i} \quad (i = 2, 3, \cdots)$$

and $X_2, X_3, \cdots$ are independent, it holds that

$$\mathbb{E}(Y_n) = 1 \quad \text{and} \quad \mathbb{V}(Y_n) \leq \frac{1}{\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}} = O((\log n)^{-1}).$$

Since

$$\sum_{k=1}^{\infty} \mathbb{V}(Y_{2^{k^2}}) = O(\sum_{k=1}^{\infty} k^{-2}) < \infty,$$

by the usual method using Chebyshev's inequality and Borel -Canteli Lemma, we have

$$\lim_{k \to \infty} Y_{2^{k^2}} = 1$$

with probability 1. Let $2^{k^2} \le n < 2^{(k+1)^2}$. Then since

$$\frac{\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^{k^2}}}{\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^{(k+1)^2}}} Y_{2^{k^2}} \le Y_n < \frac{\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^{(k+1)^2}}}{\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^{k^2}}} Y_{2^{(k+1)^2}},$$

we have $\lim_{n\to\infty} Y_n = 1$ with probability 1. Thus, (4) follows.

4 Proof of Theorem 4

Lemma 3. *For any $k = 0, 1, 2, \cdots$ and $b \ge k+2$, it holds that*

$$\mathbb{P}(a_{k+1} > 2b \mid a_k = b) = \frac{1}{2}.$$

Hence, the random variables $\mathbf{1}_{a_{k+1}/a_k > 2}$ $(k = 0, 1, 2, \cdots)$ are independent and identically distributed with distribution $(1/2, 1/2)$.

Proof By Theorem 2,

$$\begin{aligned}\mathbb{P}(a_{k+1} \le 2b \mid a_k = b) &= \sum_{\xi=b+1}^{2b} \mathbb{P}(a_{k+1} = \xi \mid a_k = b) \\ &= b\left(\frac{1}{b(b+1)} + \frac{1}{(b+1)(b+2)} + \cdots + \frac{1}{(2b-1)2b}\right) \\ &= b\left(\frac{1}{b} - \frac{1}{2b}\right) = \frac{1}{2}.\end{aligned}$$

Hence, $\mathbb{P}(\mathbf{1}_{a_{k+1}/a_k > 2} \mid a_k = b) = \frac{1}{2}$. Since this probability is indifferent to b and the sequence of random variables $a_0, a_1, a_2, \cdots$ is Markov, the random variables $\{\mathbf{1}_{a_{k+1}/a_k > 2},\ k = 0, 1, 2, \cdots\}$ are i.i.d. with distribution $(1/2, 1/2)$. □

Lemma 4. *It holds with probability 1 that for any $\epsilon > 0$, $a_k > 2^{(1-\epsilon)k/2}$ holds for any sufficiently large k.*

Proof Let $\epsilon > 0$ be given. By the law of large numbers (W.Feller [5]) and Lemma 3,

$$\sum_{i=0}^{k-1} \mathbf{1}_{a_{i+1}/a_i > 2} > (1-\epsilon)k/2, \text{ and hence } a_k > 2^{(1-\epsilon)k/2}$$

holds for any sufficiently large k with probability 1. □

Lemma 5. *For any $k = 0, 1, 2, \cdots$ and $c_1c_2\cdots c_h \in \{0, 1, \cdots, r-1\}^h$, it holds that*

$$\begin{aligned}
&\mathbb{P}(a_{k+1}a_{k+2}\cdots a_{k+h} \equiv c_1c_2\cdots c_h (\mathrm{mod}\ r) \mid a_k = b) \\
&= \sum_{\substack{b_1,\cdots,b_h \\ b<b_1<\cdots<b_h \\ b_1\equiv c_1,\cdots,b_k\equiv c_h (\mathrm{mod}\ r)}} \prod_{i=1}^{h} \left(\frac{b_{i-1}}{b_i - 1} - \frac{b_{i-1}}{b_i} \right) \quad (b_0 := b) \\
&= \sum_{\substack{b_1=b+1 \\ b_1\equiv c_1 (\mathrm{mod}\ r)}}^{\infty} \left(\frac{b}{b_1 - 1} - \frac{b}{b_1} \right) \sum_{\substack{b_2=b_1+1 \\ b_2\equiv c_2 (\mathrm{mod}\ r)}}^{\infty} \left(\frac{b_1}{b_2 - 1} - \frac{b_1}{b_2} \right) \\
&\qquad \cdots \sum_{\substack{b_h=b_{h-1}+1 \\ b_h\equiv c_h (\mathrm{mod}\ r)}}^{\infty} \left(\frac{b_{h-1}}{b_h - 1} - \frac{b_{h-1}}{b_h} \right).
\end{aligned}$$

Proof By Theorem 2, we have

$$\begin{aligned}
&\mathbb{P}(a_{k+1}a_{k+2}\cdots a_{k+h} = b_1b_2\cdots b_h \mid a_k = b) \\
&= \mathbb{P}(a_{k+1} = b_1 \mid a_k = b)\mathbb{P}(a_{k+2} = b_2 \mid a_{k+1} = b_1)\cdots\mathbb{P}(a_{k+h} = b_h \mid a_{k+h-1} = b_{h-1}) \\
&= \frac{b}{(b_1-1)b_1}\frac{b_1}{(b_2-1)b_2}\cdots\frac{b_{h-1}}{(b_h-1)b_h} = \prod_{i=1}^{h}\left(\frac{b_{i-1}}{b_i - 1} - \frac{b_{i-1}}{b_i} \right),
\end{aligned}$$

which implies our Lemma. □

Lemma 6. *For any $k = 0, 1, 2, \cdots$ and $c_1c_2\cdots c_h \in \{0, 1, \cdots, r-1\}^h$,*

$$|\mathbb{P}(a_{k+1}a_{k+2}\cdots a_{k+h} \equiv c_1c_2\cdots c_h (\mathrm{mod}\ r) \mid a_k = b) - \frac{1}{r^h}| \le \frac{h}{b+1}$$

Proof Note that for $i = 1, 2, \cdots, h$, it holds that

$$\sum_{b_i=b_{i-1}+1}^{\infty} \left(\frac{b_{i-1}}{b_i - 1} - \frac{b_{i-1}}{b_i} \right) = 1 \tag{4.1}$$

and that the summand is decreasing in b_i, where $b_0 = b$. Therefore for any $c, d \in \{0, 1, \cdots, r-1\}$, it holds that

$$\begin{aligned}
&\left| \sum_{\substack{b_i=b_{i-1}+1 \\ b_i\equiv c (\mathrm{mod}\ r)}}^{\infty} \left(\frac{b_{i-1}}{b_i - 1} - \frac{b_{i-1}}{b_i} \right) - \sum_{\substack{b_i=b_{i-1}+1 \\ b_i\equiv d (\mathrm{mod}\ r)}}^{\infty} \left(\frac{b_{i-1}}{b_i - 1} - \frac{b_{i-1}}{b_i} \right) \right| \\
&\le \text{the first term of (4.1)} = \frac{1}{b_{i-1}+1} \le \frac{1}{b+1}.
\end{aligned}$$

It follows that

$$\left| \sum_{\substack{b_i=b_{i-1}+1 \\ b_i \equiv c (\mathrm{mod}\ r)}}^{\infty} \left(\frac{b_{i-1}}{b_i - 1} - \frac{b_{i-1}}{b_i} \right) - \frac{1}{r} \right| \leq \frac{1}{b+1}$$

for any $c \in \{0, 1, \cdots, r-1\}$. Let ε_i be the term inside the absolutely value symbol $|\quad|$ in the above inequality with $c = c_i$. Then by Lemma 5,

$$P := \mathbb{P}(a_{k+1}a_{k+2}\cdots a_{k+h} \equiv c_1c_2\cdots c_h (\mathrm{mod}\ r) \mid a_k = b) = \prod_{i=1}^{h} \left(\frac{1}{r} + \varepsilon_i \right).$$

Therefore,

$$\left| P - \frac{1}{r^h} \right| = \left| \prod_{i=1}^{h} \left(\frac{1}{r} + \varepsilon_i \right) - \frac{1}{r^h} \right| \leq \left| \left(\frac{1}{r} + \frac{1}{b+1} \right)^h - \frac{1}{r^h} \right|$$
$$\leq h \left(\frac{1}{r} + \frac{1}{b+1} \right)^{h-1} \frac{1}{b+1} \leq \frac{h}{b+1}$$

□

Proof of Theorem 4 for $a_0(x)a_1(x)a_2(x)\cdots$:

A sequence $\xi = \xi_1\xi_2\cdots\xi_L \in \{0, 1, \cdots, r-1\}^L$ is called an *h-ε-normal sequence* of size L if for any $\eta \in \{0, 1, \cdots, r-1\}^h$, it holds that

$$\left| \frac{1}{L-h+1} \#\{i \in \{1, 2, \cdots, L-h+1\};\ \xi_i\xi_{i+1}\cdots\xi_{i+h-1} = \eta\} - \frac{1}{r^h} \right| < \varepsilon.$$

An infinite sequence $\xi = \xi_1\xi_2\cdots \in \{0, 1, \cdots, r-1\}^\infty$ is called *h-ε-normal* if there exists L_0 such that $\xi_1\xi_2\cdots\xi_L$ is an h-ε-normal sequence of size L for any $L \geq L_0$.

Let $\mathcal{N}_{h,\varepsilon,L}$ be the set of h-ε-normal sequences of size L. Then, by the large deviation theory (H. Cramér [1]), there exists $0 < H < 1$ and L_0 such that for any $L \geq L_0$,

$$\frac{\#\mathcal{N}_{h,\varepsilon,L}}{r^L} > 1 - H^L$$

holds. Then by Lemma 6, it holds that

$$\mathbb{P}(a_{k+1}a_{k+2}\cdots a_{k+L} \in \mathcal{N}_{h,\varepsilon,L}\ (\mathrm{mod}\ r) \mid a_k = b) \geq \#\mathcal{N}_{h,\varepsilon,L} \left(\frac{1}{r^L} - \frac{L}{b+1} \right)$$
$$> (1 - H^L) r^L \left(\frac{1}{r^L} - \frac{L}{b+1} \right) \geq 1 - H^L - \frac{Lr^L}{b+1}.$$

By Lemma 4, there exists $\delta (:= 2^{(1-\epsilon)/2}) > 1$ such that $a_k > \delta^k$ holds for any sufficiently large k with probability 1. Let $k_0 < k_1 < k_2 < \cdots$ be sequence of integers such that

$$k_0 = 0,\ k_1 = k_0 + L_0,\ k_2 = k_1 + (L_0 + 1),\ k_3 = k_2 + (L_0 + 2),\ \cdots.$$

Then, it holds for any sufficiently large n with probability 1 that

$$\mathbb{P}(a_{k_n+1}a_{k_n+2}\cdots a_{k_n+L_0+n} \notin \mathcal{N}_{h,\varepsilon,L_0+n} \pmod r \mid a_{k_n})$$
$$< H^{L_0+n} + \frac{(L_0+n)r^{L_0+n}}{\delta^{k_n}} = H^{L_0+n} + \frac{(L_0+n)r^{L_0+n}}{\delta^{nL_0+\frac{n(n-1)}{2}}}.$$

Since

$$\sum_{n=0}^{\infty}\left(H^{L_0+n} + \frac{(L_0+n)r^{L_0+n}}{\delta^{nL_0+\frac{n(n-1)}{2}}}\right) < \infty,$$

it holds with probability 1 that

$$a_{k_n+1}a_{k_n+2}\cdots a_{k_n+L_0+n} \in \mathcal{N}_{h,\varepsilon,L_0+n} \pmod r$$

holds except for a finitely many n, which implies that the infinite sequence $a_0a_1a_2\cdots \pmod r$ is h-2ε-normal with probability 1.

Taking the intersection in $h \to \infty$ and $\varepsilon \to 0$, we get the conclusion that $a_0a_1a_2\cdots \pmod r$ is normal with probability 1. Taking the intersection in r again, we complete the proof of Theorem 4 for $a_0(x), a_1(x), a_2(x), \cdots$

The proof for $A_0(x), A_1(x), A_2(x), \cdots$ is similar and rather easier. The following Lemma for $A_0(x), A_1(x), A_2(x), \cdots$ corresponds to Lemma 3, the proof of which is just similar.

Lemma 7. *For any $k = 0, 1, 2, \cdots$ and $b \geq D^k(2)$, it holds that*

$$\mathbb{P}(A_{k+1} \geq 2D(b) - 1 \mid A_k = b) = \frac{1}{2}.$$

Hence, the random variables $\mathbf{1}_{A_{k+1}/A_k \geq 2D(b)-1}$ $(k = 0, 1, 2, \cdots)$ are independent and identically distributed with distribution $(1/2, 1/2)$.

Since $2(D(b) - 1) > 2$ for any $b \geq 2$, we have Lemma 4 for A_k instead of a_k.

The following Lemma for $A_0(x), A_1(x), A_2(x), \cdots$ corresponds to Lemma 3, the proof of which is just similar.

Lemma 8. *For any $k = 0, 1, 2, \cdots$ and $c_1c_2\cdots c_h \in \{0, 1, \cdots, r-1\}^h$, it holds that*

$$\mathbb{P}(A_{k+1}A_{k+2}\cdots A_{k+h} \equiv c_1c_2\cdots c_h \pmod r \mid A_k = b)$$
$$= \sum_{\substack{b_1=D(b)\\ b_1\equiv c_1 \pmod r}}^{\infty}\left(\frac{D(b)-1}{b_1-1} - \frac{D(b)-1}{b_1}\right)\sum_{\substack{b_2=D(b_1)\\ b_2\equiv c_2 \pmod r}}^{\infty}\left(\frac{D(b_1)-1}{b_2-1} - \frac{D(b_1)-1}{b_2}\right)$$
$$\cdots \sum_{\substack{b_h=D(b_{h-1})\\ b_h\equiv c_h \pmod r}}^{\infty}\left(\frac{D(b_{h-1})-1}{b_h-1} - \frac{D(b_{h-1})-1}{b_h}\right).$$

It follows from this lemma, we have Lemma 6 for $A_0A_1A_2\cdots$ instead of $a_0a_1a_2\cdots$ and $\frac{h}{D(b)}$ instead of $\frac{h}{b+1}$. Finally, this together with Lemma 7 implies that $A_0A_1A_2\cdots$ is an absolute normal sequence almost surely just same as $a_0a_1a_2\cdots$.

5 Is $a_0(2/5)a_1(2/5)a_2(2/5)\cdots$ an absolutely normal sequence?

We do not know even whether there is a rational number having the infinite LE, while the majority of rational numbers seem to have by numerical calculation. In spite of such an ignorant situation, we dare to conjecture that $\frac{2}{5}$ not only has an infinite LE, but also generates an absolutely normal sequence. We also calculate $\frac{\#\{k;\ a_k(2/5)\leq n\}}{\log n}$, the result of which is far from convincing us that it converges to 1.

n	e^{10}	e^{50}	e^{100}	e^{500}	e^{1000}	e^{1500}	e^{2000}
$\frac{\#\{k;\ a_k(2/5)\leq n\}}{\log n}$	0.8	0.94	0.96	0.944	0.946	0.946	0.9585

Here, we write down some numerical calculations of χ^2-test concerning the absolute normality.

$\underline{a_0(2/5)a_1(2/5)\cdots a_{299}(2/5)\ (\mathrm{mod}\ 2)}$

=100011011100100001100100010010100000111000011111000000101111111
0100000010101000000111000111100100111100100100101110100111010000
0011111011001100110111101110110001000101110010011001100011101011
0111111010101101001011111101001010111010000010111000011010110010
100000111111110001111001101001100010001011 01,

3-digits distribution: $(40, 38, 35, 35, 38, 33, 35, 44)$
(i.e. numbers of occurrences of 000 is 40, 100 is 38, 010 is 35, ..., 111 is 44)
giving χ^2_7-value 2.35 (I. Guttman & S.S.Wilks [4])

4-digits distribution: $(21, 19, 19, 18, 19, 16, 15, 20, 19, 19, 16, 17, 19, 16, 20, 24)$
giving χ^2_{15}-value 4.09

$\underline{a_0(2/5)a_1(2/5)\cdots a_{299}(2/5)\ (\mathrm{mod}\ 3)}$

=0201012111222211111122212101010120110112010001201102122012020102
0210222100110212212020020221112002100202010102221200121011001201
1102012011211101200202000122011110012220200011021102120201201210
1000221022111112021221012220001120211010001200022120110112100112
2011100022001210002111120022101100010122021 1,

3-digits distribution: $(10, 11, 10, 10, 13, 11, 12, 17, 7, 12, 14, 15, 15, 16, 9, 10, 7, 11, 9, 9, 11, 16, 10, 8, 8, 10, 7)$
giving χ^2_{26}-value 19.66

$\underline{a_0(2/5)a_1(2/5)\cdots a_{299}(2/5) \pmod 5}$

=333222400044422323014233442142120131342104331340133212123024044 1
212332024141214143012401034031110134432314443013411304001110414 1
301310222213330441313323240033242103432103433113043143431313114 1
141112033410424001414124100240114002020041043202333243203422141 0
3120001234224442020243014202300214214403001 3,

2-digits distribution: $(12, 11, 11, 11, 11, 14, 11, 13, 12, 16, 11, 11, 8, 12, 12, 9, 16, 10, 14, 12, 10, 17, 12, 12, 11)$
giving χ^2_{24}-value 8.94

All the distributions are enough uniform so that the χ^2-values are inside the probability level 0.2 from 0.

Acknowledgment: The author thanks Dr. Hiroaki Ito for giving him useful informations on the subject.

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