Infinitesimal geometry and superstationary factors of dynamical systems

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Abstract

Let $\mathbb{A}$ be a finite set with $\#\mathbb{A} \geq 2$ and $\mathbb{N} = \{0, 1, 2, \cdots\}$. A nonempty closed set $\Theta \subset \mathbb{A}^{\mathbb{N}}$ is called a superstationary set if for any infinite set $\{N_0 < N_1 < \cdots\} \subset \mathbb{N}$, we have $\{\omega(N_0)\omega(N_1)\cdots;\ \omega \in \Theta\} = \Theta$. That is, Θ remains invariant for any selective observations, say at $N_0 < N_1 < \cdots$.

More generally, let Σ be any infinite set and $\Omega \subset \mathbb{A}^{\Sigma}$ be a nonempty set. Let χ be a nonprincipal ultrafilter on Σ and let $\Omega[\chi^\infty]$ be the projective limit of $\Omega[\chi^k]$, where $\Omega[\chi^k]$ is the value at the product ultrafilter χ^k of the natural extension of the mapping $S \mapsto \Omega[S]$ from $S = (s_1, \cdots, s_k) \in \Sigma^k$ to a subset of $\mathbb{A}^k$ given by

$$\Omega[S] = \{\omega(s_1)\cdots\omega(s_k);\ \omega \in \Omega\}.$$

We prove that $\Omega[\chi^\infty]$ is a superstationary set, which we call a superstationary factor of Ω at χ.

In the study of dynamical systems with time parameter, quantities which are sensitive to the time scaling, such as entropy have been of exclusive interest. On the contrary, superstationary factors represent properties depending only on time order, and not on the spacing of time. These properties are shown to reflect local aspects of the geometry behind it.

We also discuss a stronger and constructive version of superstationary factors.

1 Introduction

In the study of symbolic dynamics with discrete time, quantities such as entropy and averages play an essential role. In this article, we continue our development of the notion of superstationarity. The emphasis then changes to properties which only depend on the order, and are in particular invariant under time scaling in a general sense. We shall explain how superstationary factors of symbolic systems are related to what we call "infinitesimal geometry" in the case of dynamics defined by discretizing continuous space geometrically using partitions.

Let us review the notion of superstationary set (so far, we have called it "super-stationary set" with "-") and a motivation to have introduced it.

Let $\mathbb{A}$ be a finite set having at least two elements. Let Σ be an arbitrary infinite set. An element $\omega \in \mathbb{A}^\Sigma$ is considered as a mapping $\Sigma \to \mathbb{A}$. For a subset $S \subset \Sigma$, let $\omega|_S$ be the restriction of the mapping $\omega : \Sigma \to \mathbb{A}$ to $S \subset \Sigma$. Define the *complexity* $p_\Omega(S)$ which is a function of finite subsets $S \subset \Sigma$ by $p_\Omega(S) := \#\Omega|_S = \#\{\omega|_S;\ \omega \in \Omega\}$, $\#$ denoting the number of elements in a set. The *maximal pattern complexity* of Ω is defined by $p^*_\Omega(k) := \sup_{S\subset\Sigma,\#S=k} p_\Omega(S)$ as a function of $k = 1, 2, \cdots$. If $p_\Omega(S)$ depends only on $\#S$, then we call Ω a *uniform set*. In this case, $p_\Omega(k) := p_\Omega(S)$ defined as a function of $k = \#S$ is called the *uniform complexity* of Ω.

Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $\omega \in \mathbb{A}^\mathbb{N}$. In this special case that $\Sigma = \mathbb{N}$, we use some conventional notations. We consider ω an infinite word $\omega(0)\omega(1)\omega(2)\cdots$ over $\mathbb{A}$ as well as the mapping $\omega : \mathbb{N} \to \mathbb{A}$. On the other hand, an element ξ in $\mathbb{A}^* := \cup_{k=0}^\infty \mathbb{A}^k$ is considered as a finite word $\xi_1\xi_2\cdots\xi_k$ over $\mathbb{A}$. For $a \in \mathbb{A}$, we denote $a^\infty = aa\cdots \in \mathbb{A}^\mathbb{N}$. We also denote $\mathbb{A}^+ := \cup_{k=1}^\infty \mathbb{A}^k$, that is, $\mathbb{A}^+ = \mathbb{A}^* \setminus \{\epsilon\}$, where ϵ is the empty word.

For $\omega \in \mathbb{A}^\mathbb{N}$ and $S \subset \mathbb{N}$, we denote $\omega[S] = \omega(s_1)\omega(s_2)\cdots\omega(s_k) \in \mathbb{A}^k$, where $s_1 < s_2 < \cdots < s_k$ are the elements in S arranged in order. Moreover, for an infinite set $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$, we denote $\omega[\mathcal{N}] = \omega(N_0)\omega(N_1)\omega(N_2)\cdots \in \mathbb{A}^\mathbb{N}$. Thus, we identify a subset of $\mathbb{N}$ with the increasing sequence of its elements. For $\Theta \subset \mathbb{A}^\mathbb{N}$ and S, $\mathcal{N}$ as above, we denote $\Theta[S] = \{\omega[S];\ \omega \in \Theta\}$ and $\Theta[\mathcal{N}] = \{\omega[\mathcal{N}];\ \omega \in \Theta\}$. A nonempty set $\Theta \subset \mathbb{A}^\mathbb{N}$ is called a *k-superstationary set* if $\Theta[S] = \Theta[\{0, 1, \cdots, k-1\}]$ holds for any $S \subset \mathbb{N}$ with $\#S = k$. It is called a *superstationary set* if it is closed and $\Theta[\mathcal{N}] = \Theta$ holds for any infinite subset $\mathcal{N}$ of $\mathbb{N}$. It is clear that a nonempty closed set $\Theta \subset \mathbb{A}^\mathbb{N}$ is superstationary if and only if it is k-superstationary for all $k = 1, 2, \cdots$. It is also clear that a superstationary set is a uniform set. For example, any one point set $\{w\}$ with $\omega \in \mathbb{A}^\mathbb{N}$ is a uniform set, while it is a superstationary set only if $\omega = a^\infty$ for some $a \in \mathbb{A}$.

The notion of superstationary set was introduced for the first time to classify pattern Sturmian words [2, 3]. It is known that for any infinite word $\omega \in \mathbb{A}^\mathbb{N}$, $p^*_{\overline{O}(\omega)}(k) \geq 2k$ holds for any $k = 1, 2, \cdots$ if and only if ω is not eventually periodic, where $\overline{O}(\omega)$ is the closure of $\{T^n\omega;\ n \in \mathbb{N}\}$ and T is the shift on $\mathbb{A}^\mathbb{N}$. An infinite word $\omega \in \mathbb{A}^\mathbb{N}$ with the property that $p^*_{\overline{O}(\omega)}(k) = 2k$ $(k = 1, 2, \cdots)$ is called a *pattern Sturmian word*. It is known that for a recurrent pattern Sturmian word $\omega \in \mathbb{A}^\mathbb{N}$, there exists an infinite set $\mathcal{N} \subset \mathbb{N}$ such that $\Theta := \overline{O}(\omega)[\mathcal{N}]$ is a superstationary set. Such Θ is called a superstationary factor of $\overline{O}(\omega)$.

There are two types of recurrent pattern Sturmian words, rotation words $\omega_1 \in \{0,1\}^{\mathbb{N}}$ and Toeplitz words $\omega_2 \in \{0,1\}^{\mathbb{N}}$ defined as follows. Let α be an irrational number and a, b be real numbers such that $a < b < a+1$. Define $\omega_1 \in \{0,1\}^{\mathbb{N}}$ by

$$\omega_1(n) = \begin{cases} 0 & n\alpha \in [a,b) \quad\quad (mod\ 1) \\ 1 & n\alpha \in [b, a+1)\ (mod\ 1). \end{cases}$$

Then, ω_1 is a recurrent pattern Sturmian word. The set of superstationary factors of $\overline{O}(\omega_1)$ is $\{\Theta_1\}$, where

$$\Theta_1 = \{0^i 1^\infty;\ i \in \mathbb{N}\} \cup \{1^i 0^\infty;\ i \in \mathbb{N}\}.$$

For $n \in \mathbb{N}$, let $\tau(n) \in \mathbb{N}$ be the maximum $k \in \mathbb{N}$ such that 2^k divides n. Define $\omega_2 \in \{0,1\}^{\mathbb{N}}$ by

$$\omega_2(n) = \begin{cases} 0 & \text{if } \tau(n+1) \text{ is even} \\ 1 & \text{if } \tau(n+1) \text{ is odd.} \end{cases}$$

Then, ω_2 is a recurrent pattern Sturmian word of another type. In fact, the set of superstationary factors of $\overline{O}(\omega_2)$ is $\{\Theta_2, \Theta_2'\}$, where

$$\Theta_2 = \{0^\infty\} \cup \{0^i 1 0^\infty;\ i \in \mathbb{N}\} \cup \{0^i 1^\infty;\ i \in \mathbb{N}\},$$
$$\Theta_2' = \{1^\infty\} \cup \{1^i 0 1^\infty;\ i \in \mathbb{N}\} \cup \{1^i 0^\infty;\ i \in \mathbb{N}\}.$$

In this article, the notion of superstationary factors is generalized so that any nonempty set $\Omega \subset \mathbb{A}^\Sigma$ has a superstationary factor.

Superstationary sets were studied in [4, 5, 6, 7, 8]. For $\xi \in \mathbb{A}^k$ and $\omega \in \mathbb{A}^{\mathbb{N}}$, ξ is called a *super-subword* of ω if there exists $S \subset \mathbb{N}$ with $\#S = k$ such that $\omega[S] = \xi$. We use the following characterization of the superstationary sets. That is, the family of superstationary sets coincides with the family of sets expressed as $\mathcal{P}(\Xi)$, where Ξ is a finite (possibly, empty) subset of $\mathbb{A}^+$ satisfying the condition (#) (see [7]) and $\mathcal{P}(\Xi)$ is the set of infinite words where all words in Ξ are prohibited as super-subwords. For example, the above Θ_1, Θ_2 and Θ_2' are written as $\Theta_1 = \mathcal{P}(010, 101)$, $\Theta_2 = \mathcal{P}(101, 110)$ and $\Theta_2' = \mathcal{P}(010, 001)$. Using this characterization, we get rich properties of the uniform complexity, since the uniform complexity is always realized by a superstationary set [8].

Let Σ be any infinite set and $\beta\Sigma$ be the Stone-Čech compactification of Σ (see [1], for example). That is, $\beta\Sigma$ is a set of ultrafilters on Σ, where principal ultrafilters are identified with elements in Σ, and $\beta\Sigma \setminus \Sigma$ consists of all nonprincipal ultrafilters on Σ. It is known that $\beta\Sigma$ is a compact Hausdorff space and $\beta\Sigma \setminus \Sigma$ is a closed subset of it. For a subset U of Σ, let $\mathcal{U}(U) = \{\chi \in \beta\Sigma;\ U \in \chi\}$. Then, $\mathcal{U}(U)$ is a clopen subset of $\beta\Sigma$. Moreover, the family $\{\mathcal{U}(U);\ U \text{ is a subset of } \Sigma\}$ is a topological base of $\beta\Sigma$. Let Σ

be an infinite set and $\chi \in \beta\Sigma$. In Section 2, we define the k-times product $\chi^k = \chi \times \ldots \times \chi$ of it. It is an element of $\beta(\Sigma^k)$.

For a nonempty set $\Omega \subset \mathbb{A}^\Sigma$, define the superstationary factor $\Omega[\chi^\infty] \subset \mathbb{A}^\mathbb{N}$ at $\chi \in \beta\Sigma$ to be the projective limit of $\Omega[\chi^k]$ as $k \to \infty$, where $\Omega[\chi^k]$ is the value at the ultrafilter χ^k of the natural extension of the mapping $S \mapsto \Omega[S]$ from Σ^k to the family of subsets of $\mathbb{A}^k$, where for $S = (s_1, \ldots, s_k) \in \Sigma^k$, $\Omega[S] = \{\omega(s_1)\ldots\omega(s_k);\ \omega \in \Omega\}$. Then, the following theorems hold.

Theorem 1. *Let $\chi \in \beta\Sigma$ and $k = 1, 2, \ldots$.*
(1) *The restriction of $\Omega[\chi^{k+1}]$ to the first k-coordinates coincides with $\Omega[\chi^k]$.*
(2) *$\Omega[\chi^\infty]$ is a superstationary set.*
(3) *If χ is a principal ultrafilter, then $\Omega[\chi^\infty] = \{a^\infty;\ a \in \Omega[\chi]\}$.*

Theorem 2. *For any nonempty set $\Theta \subset \mathbb{A}^\mathbb{N}$ and $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$, Θ is a superstationary set if and only if $\Theta[\chi^\infty] = \Theta$.*

Let (Ω, T) be a symbolic dynamics. That is, Ω is a nonempty closed subset of $\mathbb{A}^\mathbb{N}$ such that $T\Omega \subset \Omega$. In the studies of dynamical systems with time parameter $\mathbb{N}$, quantities which are sensitive to the time scaling, like entropy have been of primary interest. Here in the contrary, the superstationary factors which are irrelevant to the time scaling are studied. In fact, a superstationary factor Θ of the underlying set Ω of a symbolic dynamics reflects properties of the system depending just on time order, but not on the quantity of time, since $\Theta[\mathcal{N}] = \Theta$ for any $\mathcal{N}$, that is, Θ is invariant under any choice of observing times $N_0 < N_1 < \ldots$. It is a world of eternity, where the notion of time length makes no sense. But, if the symbolic dynamics comes from a geometrical setting on a compact metric space, then the superstationary factor $\Omega[\chi^\infty]$ is closely related to the *infinitesimal geometry* behind it (Theorem 6), more precisely, to the infinitesimal move of the point in the geometrical space how it crosses the boundary of its partition used for the symbolic representation. If the boundary is complex e.g. for the Thue-Morse word, then it has the full space $\mathbb{A}^\mathbb{N}$ as its superstationary factor. If the dynamics is an irrational rotation on the circle partitioned by d intervals with $d \geq 3$, then the system has two different superstationary factors corresponding to the infinitesimal move crossing the boundary upward or downward. If the dynamics is the translation on $(\mathbb{R}/\mathbb{Z})^2$ partitioned by a disc of radius δ, then the system has two different superstationary factors corresponding to the radius of the infinitesimal move, whether smaller than δ or not. Thus, the world of eternity is closely related to the world of instant.

We introduce basic notions and prove preliminary lemmas in Section 2. We also define a constructive, stronger version of superstationary factors in Section 2. We prove the main results in Section 3. We discuss the infinitesimal geometry in Section 4. We discuss seven examples in Section 5.

2 Basic notions and preliminary lemmas

Definition 1. Let Σ_i be infinite sets for $i = 1, 2$ and $\chi_i \in \beta\Sigma_i$ $(i = 1, 2)$. We define $\chi_1 \times \chi_2 \in \beta(\Sigma_1 \times \Sigma_2)$ as

$$\chi_1 \times \chi_2 = \{U \subset \Sigma_1 \times \Sigma_2;\ \{x \in \Sigma_1;\ U^x \in \chi_2\} \in \chi_1\},$$

where $U^x := \{y \in \Sigma_2;\ (x, y) \in U\}$.

Note that this product operation is not symmetric between the first entry and the second entry. That is, $U \in \chi_1 \times \chi_2$ does not mean $\{(y, x);\ (x, y) \in U\} \in \chi_2 \times \chi_1$. Moreover, the mapping $(\chi_1, \chi_2) \mapsto \chi_1 \times \chi_2$ is continuous in χ_1 but not continuous in χ_2. In fact, $\{(x, y) \in \mathbb{N} \times \mathbb{N};\ x < y\} \in \chi \times \chi$ and $\{(y, x) \in \mathbb{N} \times \mathbb{N};\ x < y\} \notin \chi \times \chi$ hold for any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$, since any cofinite set is in χ, but any finite set is not in χ.

Definition 2. For $\chi \in \beta\Sigma$ and $k \geq 1$, define χ^k inductively by $\chi^1 = \chi$ and $\chi^k = \chi^{k-1} \times \chi$ $(k \geq 2)$. Then, $\chi^k \in \beta(\Sigma^k)$.

Definition 3. For a finite sequence $(\sigma_1, \ldots, \sigma_k) \in \Sigma^k$, we define

$$\Omega[(\sigma_1, \ldots, \sigma_k)] := \{\omega(\sigma_1) \ldots \omega(\sigma_k) \in \mathbb{A}^k;\ \omega \in \Omega\} \subset \mathbb{A}^k.$$

This mapping $(\sigma_1, \ldots, \sigma_k) \mapsto \Omega[(\sigma_1, \ldots, \sigma_k)]$ has the natural extension as a mapping from $\beta(\Sigma^k)$ to the family of subsets of $\mathbb{A}^k$, in the sense that $\Omega[\gamma] = \Lambda$ for $\gamma \in \beta(\Sigma^k)$ if

$$\{(\sigma_1, \ldots, \sigma_k) \in \Sigma^k;\ \Omega[(\sigma_1, \ldots, \sigma_k)] = \Lambda\} \in \gamma.$$

Let $\chi \in \beta\Sigma$. Then, $\Omega[\chi^k]$ makes sense as a subset of $\mathbb{A}^k$.

Definition 4. For any $\chi \in \beta\Sigma$, we define $\Omega[\chi^\infty] \subset \mathbb{A}^{\mathbb{N}}$ as

$$\Omega[\chi^\infty] = \{\omega \in \mathbb{A}^{\mathbb{N}};\ \omega(0)\omega(1) \ldots \omega(k-1) \in \Omega[\chi^k] \text{ for any } k = 1, 2, \ldots\},$$

which turns out later to be the projective limit of $\Omega[\chi^k]$ as $k \to \infty$ by (1) of Theorem 1. We call $\Omega[\chi^\infty]$ for $\chi \in \beta\Sigma \setminus \Sigma$ the *superstationary factor* of Ω at χ. This naming will be justified by Theorem 1.

Definition 5. A superstationary set $\Theta \subset \mathbb{A}^{\mathbb{N}}$ is called a *strong superstationary factor* of Ω if there exist injections $\varphi_k : \mathbb{N} \to \Sigma$ for any $k = 1, 2, \ldots$ such that
(1) $\varphi_1(\mathbb{N}) \supset \varphi_2(\mathbb{N}) \supset \ldots$, and
(2) $\Omega \circ \varphi_k$ is a k-superstationary set satisfying that

$$\Omega \circ \varphi_k[\{0, 1, \ldots, k-1\}] = \Theta[\{0, 1, \ldots, k-1\}].$$

Moreover, a strong superstationary factor Θ of Ω is called *attainable* if in the above, φ_k does not depend on $k = 1, 2, \ldots$.

Definition 6. For a nonempty subset G of a compact Hausdorff space, the set of accumulating points of $\overline{G}$ (the closure of G) is denote by G'. Clearly, G' is a closed (possibly, empty) set. We denote $G^{(0)} = \overline{G}$ and $G^{(i)} = (G^{(i-1)})'$ for $i = 1, 2, \ldots$. The *accumulation degree* of G, denoted by $\operatorname{acdeg} G$ is defined to be $\inf\{d;\ G^{(d+1)} = \emptyset\}$, where we define $\operatorname{acdeg} G = \infty$ if $G^{(d+1)} \neq \emptyset$ for any $d \in \mathbb{N}$.

Theorem 3. ([7]) *If Ω is a uniform set, then all strong superstationary factors of Ω are attainable.*

Theorem 4. ([5]) *If $\#\mathbb{A} = 2$ and $\operatorname{acdeg} \Omega < \infty$, then there exists an attainable strong superstationary factor of Ω.*

We'll prove the following theorem in Section 3.

Theorem 5.
(1) *Let $\chi \in \beta\Sigma \setminus \Sigma$. A strong superstationary factor of Ω such that* (1), (2) *in* Definition 5 *hold with $\varphi_k(\mathbb{N}) \in \chi$ for any $k = 1, 2, \ldots$ is unique and coincides with $\Omega[\chi^\infty]$, if it exists.*
(2) *The set of $\chi \in \beta\Sigma \setminus \Sigma$ such that there exists a strong superstationary factor of Ω as above with χ is dense in $\beta\Sigma \setminus \Sigma$.*

Definition 7. We denote $\Omega((\chi)) = \Theta$ if Θ is the strong superstationary factor of Ω as in (1) of Theorem 5 with $\chi \in \beta\Sigma \setminus \Sigma$. The set of χ such that $\Omega((\chi))$ exists is denoted by $DS(\Omega)$. Then, $DS(\Omega)$ is the domain of the mapping $\chi \mapsto \Omega((\chi))$ and is dense in $\beta\Sigma \setminus \Sigma$ by Theorem 5.

Lemma 1. *For any $\chi \in \beta\Sigma$ and positive integers k, l, we have $\chi^{k+l} = \chi^k \times \chi^l$.*

Proof We use induction on $k + l$. If $k + l = 2$, our statement is clear. Assume that $k + l \geq 3$ and our statement holds for the case $k + l - 1$. If $l = 1$, then our statement follows from Definition 2. If $l \geq 2$, then we have

$$\chi^{k+l} = \chi^{k+l-1} \times \chi = (\chi^k \times \chi^{l-1}) \times \chi$$

by induction hypothesis. Hence, $U \in \chi^{k+l}$ if and only if

$$\{(x, y) \in \Sigma^k \times \Sigma^{l-1};\ \{z \in \Sigma;\ (x, y, z) \in U\} \in \chi\} \in \chi^k \times \chi^{l-1}.$$

By the definition of $\chi^k \times \chi^{l-1}$, this is equivalent to

$$\{x \in \Sigma^k;\ \{y \in \Sigma^{l-1};\ \{z \in \Sigma;\ (x, y, z) \in U\} \in \chi\} \in \chi^{l-1}\} \in \chi^k.$$

Then, by Definition 2, this is equivalent to

$$\{x \in \Sigma^k;\ \{(y, z) \in \Sigma^{l-1} \times \Sigma;\ (x, y, z) \in U\} \in \chi^l\} \in \chi^k,$$

which turns out to be equivalent to $U \in \chi^k \times \chi^l$.
Thus, we have $\chi^{k+l} = \chi^k \times \chi^l$. □

Lemma 2. *For any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$, $U \in \chi$ and a positive integer k, we have*

$$\Delta_k^U := \{(x_1, \ldots, x_k) \in U^k;\ x_1 < \ldots < x_k\} \in \chi^k.$$

Particularly,

$$\Delta_k = \{(x_1, \ldots, x_k) \in \mathbb{N}^k;\ x_1 < \ldots < x_k\} \in \chi^k.$$

Proof We prove by induction in k. For $k = 1$, our result is clear since $\Delta_1^U = U \in \chi$. Let $k \geq 2$ and assume that our result holds for $k - 1$. Take any $x = (x_1, \ldots, x_{k-1}) \in \Delta_{k-1}^U$. Since

$$(\Delta_k^U)^x = \{y \in U;\ y > x_{k-1}\} = U \setminus \{z \in \mathbb{N};\ z \leq x_{k-1}\},$$

we have $(\Delta_k^U)^x \in \chi$. Since this holds for any $x \in \Delta_{k-1}^U$, $\Delta_k^U \in \chi^k$ holds by induction hypothesis. Thus, $\Delta_k^U \in \chi^k$ for any positive integer k. □

Recall that we identify a subset $S = \{s_1 < \ldots < s_k\} \subset \mathbb{N}$ with the increasing sequence $S = (s_1, \ldots, s_k) \in \mathbb{N}^k$.

Lemma 3. *For any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$, $U \in \chi$ and $k = 1, 2, \ldots$, there exists $S = \{x_1 < \ldots < x_k\} \subset U$ such that $\Omega[\chi^k] = \Omega[S]$.*

Proof Take any $V \subset \mathbb{N}^k$ with $V \in \chi^k$ satisfying that $\Omega[x] = \Omega[\chi^k]$ for any $x \in V$. Since χ^k is a ultrafilter and $\Delta_k^U \in \chi^k$ by Lemma 2, $V \cap \Delta_k^U \in \chi^k$ and is a nonempty set. Take any $(x_1, \ldots, x_k) \in V \cap \Delta_k^U$ and let $S := \{x_1 < \ldots < x_k\} \subset U$. Then, we have $\Omega[\chi^k] = \Omega[(x_1, \ldots, x_k)] = \Omega[S]$. □

Let $S = \{s_1 < \ldots < s_k\} \subset \{1, 2, \ldots, l\}$ with positive integers $k \leq l$. The projection $\pi_{l,S} : \Sigma^l \to \Sigma^k$ is defined by

$$\pi_{l,S}(x_1, x_2, \ldots, x_l) = (x_{s_1}, \ldots, x_{s_k})$$

for any $(x_1, x_2, \ldots, x_l) \in \Sigma^l$.

Lemma 4. *Let $\chi \in \beta\Sigma$ and $S = \{s_1 < \ldots < s_k\} \subset \{1, 2, \ldots, l\}$.*
(1) *For any $V \in \chi^k$, we have $\pi_{l,S}^{-1}(V) \in \chi^l$.*
(2) *For any $U \in \chi^l$, we have $U[S] = \pi_{l,S}(U) \in \chi^k$.*

Proof (1) We use induction on l. If $l = 1$, our statement is clear. Assume that $l \geq 2$ and our statement holds for $l - 1$.
Case 1: $s_1 \neq 1$. Since $\pi_{l,S}^{-1}(V) = \Sigma \times \pi_{l-1,S-1}^{-1}(V)$ and $\pi_{l-1,S-1}^{-1}(V) \in \chi^{l-1}$ by induction hypothesis, we have $\pi_{l,S}^{-1}(V) \in \chi^l$, where $S - 1 = \{s_1 - 1, \ldots, s_k - 1\}$.
Case 2: $s_1 = 1$. For $x \in \Sigma$, we have $\pi_{l,S}^{-1}(V)^x = \pi_{l-1,S'-1}^{-1}(V^x)$, where $S' = \{s_2 < \ldots < s_k\}$. By induction hypothesis, $\pi_{l-1,S'-1}^{-1}(V^x) \in \chi^{l-1}$ if $V^x \in \chi^{k-1}$. Since the set of $x \in \Sigma$ such that $V^x \in \chi^{k-1}$ is in χ, the set of $x \in \Sigma$ such that $\pi_{l,S}^{-1}(V)^x \in \chi^{l-1}$ is in χ. This implies that $\pi_{l,S}^{-1}(V) \in \chi^l$.

(2) We use induction on l. If $l = 1$, our statement is clear. Assume that $l \geq 2$ and our statement holds for $l-1$. Let $U \in \chi^l$.
Case 1: $s_1 \neq 1$. Since $\{x \in \Sigma;\ U^x \in \chi^{l-1}\} \in \chi$, there exists $x \in \Sigma$ such that $U^x \in \chi^{l-1}$. Since $U^x[S-1] \subset U[S]$ and $U^x[S-1] \in \chi^k$ by induction hypothesis, we have $U[S] \in \chi^k$.
Case 2: $s_1 = 1$. In this case, for $x \in \Sigma$, we have $U[S]^x = U^x[S'-1]$, where $S' = \{s_2 < \ldots < s_k\}$. By induction hypothesis, $U^x[S'-1] \in \chi^{k-1}$ if $U^x \in \chi^{l-1}$. Since the set of $x \in \Sigma$ such that $U^x \in \chi^{l-1}$ is in χ, the set of $x \in \mathbb{N}$ such that $U^x[S'-1] \in \chi^{k-1}$ is in χ. Since $U[S]^x = U^x[S'-1]$, this implies that $U[S] \in \chi^k$. □

Lemma 5. *Let $\chi \in \beta\Sigma$ and l be a positive integer. For any $k = 1, 2, \ldots, l$, let $V_k \in \chi^k$. Then, there exists $U \in \chi^l$ such that $U[S] \subset V_k$ for any $k = 1, 2, \ldots, l$ and $S \subset \{1, 2, \ldots, l\}$ with $\#S = k$.*

Proof By Lemma 4, $\pi_{l,S}^{-1}(V_k) \in \chi^l$ holds for any $k = 1, 2, \ldots, l$ and $S \subset \{1, 2, \ldots, l\}$ with $\#S = k$. Hence,

$$U := \bigcap_{k=1}^{l} \bigcap_{S \subset \{1,2,\ldots,l\}, \#S=k} \pi_{l,S}^{-1}(V_k)$$

has the desired property. □

3 Proofs of the main results

We always assume that $\Omega \subset \mathbb{A}^\Sigma$ is a nonempty set.

Proof of Theorem 1

(1) Let $\chi \in \beta\Sigma$ and $k = 1, 2, \ldots$. Then, there exists $U \in \chi^{k+1}$ such that $\Omega[S] = \Omega[\chi^{k+1}] =: \Xi$ for any $S \in U$. Let $\pi : \Sigma^{k+1} \to \Sigma^k$ be the projection to the first k coordinates. Then, by Lemma 4, $\pi(U) \in \chi^k$ and $\Omega[\pi(S)] = \Xi'$ holds for any $S \in U$, where Ξ' is the restriction of Ξ to the first k coordinates. Since $\{\pi(S);\ S \in U\} = \pi(U) \in \chi^k$, this implies that $\Xi' = \Omega[\chi^k]$.

(2) By (1), the restriction of $\Omega[\chi^\infty]$ to the first k coordinates coincides with $\Omega[\chi^k]$. Take any $S = \{s_1 < \ldots < s_k\} \subset \mathbb{N}$. Take $K > s_k$. Then, there exists $U \in \chi^K$ such that $\Omega[x] = \Omega[\chi^K]$ for any $x \in U$. Then, we have $\Omega[\chi^\infty][S] = \Omega[\chi^K][S] = \Omega[x][S] = \Omega[x \circ S]$ for any $x = (x_0, \ldots, x_{K-1}) \in U$, where $x \circ S = (x_{s_1}, \ldots, x_{s_k}) \in \Sigma^k$. Let $U \circ S = \{x \circ S;\ x \in U\}$. Then by Lemma 4, $U \circ S \in \chi^k$. Therefore, we have $\Omega[\chi^\infty][S] = \Omega[\chi^k] = \Omega[\chi^\infty][\{0, 1, \ldots, k-1\}]$, and hence, $\Omega[\chi^\infty]$ is k-superstationary. Since $k = 1, 2, \ldots$ is arbitrary and $\Omega[\chi^\infty]$ is a closed set, $\Omega[\chi^\infty]$ is superstationary.

(3) If $\chi \in \beta\Sigma$ is a principal ultrafilter considered as an element in Σ, then $\chi^k = (\chi, \ldots, \chi) \in \Sigma^k$ and

$$\Omega[\chi^k] = \Omega[(\chi, \ldots, \chi)] = \{a^k;\ a \in \Omega[\chi]\}.$$

Thus, $\Omega[\chi^\infty] = \{a^\infty;\ a \in \Omega[\chi]\}$. $\square$

Proof of Theorem 2

If $\Theta = \Theta[\chi^\infty]$, then by Theorem 1, Θ is a superstationary set.

Conversely, assume that Θ is a superstationary set. Take any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$, $U \in \chi$ and $k = 1, 2, \ldots$. Then by Lemma 3, there exists $S = \{x_1 < \ldots < x_k\} \subset U$ such that $\Theta[\chi^k] = \Theta[S]$. Then, we have

$$\Theta[\chi^\infty][\{0, 1, \ldots, k-1\}] = \Theta[\chi^k] = \Theta[S] = \Theta[\{0, 1, \ldots, k-1\}],$$

where the last equality follows from the fact that Θ is k-superstationary. Since this holds for any $k = 1, 2, \ldots$ and Θ is a closed set, we have $\Theta[\chi^\infty] = \Theta$. $\square$

Proof of Theorem 5

(1) Let Θ be a strong superstationary factor of Ω such that $\varphi_k : \mathbb{N} \to \Sigma$ satisfying the conditions (1), (2) of Definition 5 together with $\varphi_k(\mathbb{N}) \in \chi$ for any $k = 1, 2, \ldots$ with $\chi \in \beta\Sigma \setminus \Sigma$.

To prove $\Omega[\chi^\infty] = \Theta$, it is sufficient to prove $\Omega[\chi^k] = \Theta[\{0, 1, \ldots, k-1\}]$ for any $k = 1, 2, \ldots$. Define $\Delta_i^{\varphi_k} \subset \Sigma^i$ $(i = 1, \ldots, k)$ by

$$\Delta_i^{\varphi_k} = \{(\varphi_k(x_1), \ldots, \varphi_k(x_i));\ \{x_1 < \ldots < x_i\} \subset \mathbb{N}\}.$$

We have $\Delta_1^{\varphi_k} = \varphi_k(\mathbb{N}) \in \chi$. Since for any $\xi = (\varphi_k(x_1), \ldots, \varphi_k(x_{i-1})) \in \Delta_{i-1}^{\varphi_k}$, we have

$$\{\sigma \in \Sigma;\ (\xi, \sigma) \in \Delta_i^{\varphi_k}\} = \{\varphi_k(x);\ x > x_{i-1}\} \in \chi$$

as $\{\varphi_k(x);\ x > x_{i-1}\} = \Delta_1^{\varphi_k} \setminus F$ with a finite set F. Hence, $\Delta_i^{\varphi_k} \in \chi^i$ follows from $\Delta_{i-1}^{\varphi_k} \in \chi^{i-1}$. Thus, we have $\Delta_k^{\varphi_k} \in \chi^k$ by induction on i.

Since

$$\begin{aligned}&\Omega[(\varphi_k(x_1), \ldots, \varphi_k(x_k))] = \Omega \circ \varphi_k[\{x_1, \ldots x_k\}] \\ &= \Omega \circ \varphi_k[\{0, 1, \ldots, k-1\}] = \Theta[\{0, 1, \ldots, k-1\}]\end{aligned}$$

holds for any $\{x_1 < \ldots < x_k\} \subset \mathbb{N}$, we have

$$\{(\sigma_1, \ldots, \sigma_k) \in \Sigma^k;\ \Omega[(\sigma_1, \ldots, \sigma_k)] = \Theta[\{0, 1, \ldots, k-1\}]\} \supset \Delta_k^{\varphi_k} \in \chi^k.$$

Hence, $\Omega[\chi^k] = \Theta[\{0, 1, \ldots, k-1\}]$ holds for any $k = 1, 2, \ldots$, which implies that $\Omega[\chi^\infty] = \Theta$.

(2) Let $\phi : \mathbb{N} \to \Sigma$ be any injection. Then, $\Omega \circ \phi$ is a nonempty subset of $\mathbb{A}^{\mathbb{N}}$. By Lemma 2 in [7], there exist a superstationary set Θ and a sequence $\mathcal{N}^1 \supset \mathcal{N}^2 \supset \ldots$ of infinite subsets of $\mathbb{N}$ such that $\Omega \circ \phi[\mathcal{N}^k]$ $(k = 1, 2, \ldots)$ is a k-superstationary set satisfying that

$$\Omega \circ \phi[\mathcal{N}^k][\{0, 1, \ldots, k-1\}] = \Theta[\{0, 1, \ldots, k-1\}].$$

Actually, $\mathcal{N}^1 \supset \mathcal{N}^2 \supset \ldots$ such that $\Omega \circ \phi[\mathcal{N}^k]$ $(k = 1, 2, \ldots)$ is a k-superstationary set exist by the Infinitary Ramsey Theorem [9], and Θ is defined as

$$\Theta = \bigcap_{l=1}^{\infty} \overline{\bigcup_{k=l}^{\infty} \Omega \circ \phi[\mathcal{N}^k]}.$$

Let $\varphi_k : \mathbb{N} \to \Sigma$ be the injection such that $\varphi_k(n) = \phi(N_n^k)$ $(\forall n \in \mathbb{N})$, where $\mathcal{N}^k = \{N_0^k < N_1^k < \ldots\}$. Then, it follows that φ_k $(k = 1, 2, \ldots)$ are injections satisfying (1), (2) of Definition 5. Let χ be any element in $\beta\Sigma \setminus \Sigma$ such that $\varphi_k(\mathbb{N}) \in \chi$ for any $k = 1, 2, \ldots$. Such χ exists since $\varphi_k(\mathbb{N})$ $(k = 1, 2, \ldots)$ is a decreasing sequence of infinite subsets of Σ. By (1), this implies that $\Theta = \Omega[\chi^\infty]$ is a strong superstationary factor of Ω. Moreover, since $\varphi_k(\mathbb{N}) \subset \phi(\mathbb{N})$ for any $k = 1, 2, \ldots$ and $\varphi_k(\mathbb{N}) \in \chi$, we have $\phi(\mathbb{N}) \in \chi$. Since $\phi(\mathbb{N})$ can be any countably infinite subset of Σ, the set of χ as this is dense in $\beta\Sigma \setminus \Sigma$. □

4 Infinitesimal geometry and dynamical imbedding

Let X, Y be infinite sets and $F : X \to Y$ be a mapping. For $\chi \in \beta X$, define $F(\chi) \in \beta Y$ by

$$F(\chi) = \{U \subset Y;\ F^{-1}U \in \chi\}. \tag{4.1}$$

Lemma 6. *Let $X_i,\ Y_i$ $(i = 1, 2)$ be infinite sets and $F_i : X_i \to Y_i$ $(i = 1, 2)$ be mappings. Let $\chi_i \in \beta X_i$ $(i = 1, 2)$. Then, we have $(F_1 \times F_2)(\chi_1 \times \chi_2) = F_1(\chi_1) \times F_2(\chi_2)$.*

Proof By definition, $U \in (F_1 \times F_2)(\chi_1 \times \chi_2)$ if and only if

$$\{(x_1, x_2) \in X_1 \times X_2;\ (F_1(x_1), F_2(x_2)) \in U\} \in \chi_1 \times \chi_2.$$

This is equivalent to

$$\{x_1 \in X_1;\ \{x_2 \in X_2;\ (F_1(x_1), F_2(x_2)) \in U\} \in \chi_2\} \in \chi_1,$$

and hence,

$$\{x_1 \in X_1;\ \{x_2 \in X_2;\ F_2(x_2) \in U^{F_1(x_1)}\} \in \chi_2\} \in \chi_1.$$

This holds if and only if $\{x_1 \in X_1;\ F_2^{-1} U^{F(x_1)} \in \chi_2\} \in \chi_1$, and hence,

$$\{x_1 \in X_1;\ U^{F_1(x_1)} \in F_2(\chi_2)\} \in \chi_1.$$

This is equivalent to

$$\{x_1 \in X_1;\ F_1(x_1) \in \{t \in Y_1;\ U^t \in F_2(\chi_2)\}\} \in \chi_1,$$

and hence,

$$\{t \in Y_1;\ U^t \in F_2(\chi_2)\} \in F_1(\chi_1).$$

Thus, $U \in (F_1 \times F_2)(\chi_1 \times \chi_2)$ if and only if $U \in F_1(\chi_1) \times F_2(\chi_2)$. □

Corollary 1. *Let X, Y be infinite sets, $F : X \to Y$ be a mapping and $\chi \in \beta X$. Then, we have $(F^k)(\chi^k) = (F(\chi))^k$ for $k = 2, 3, \ldots$.*

Let X be a compact metric space with metric ρ_X and $\#X = \infty$. Let $\chi \in \beta X$. Let φ be a mapping $W_k \to Y$, where $k = 1, 2, \ldots$, $W_k \in \chi^k$ and Y is a compact metric space. Then, $p \in Y$ is determined so that

$$\{(x_1, \ldots, x_k) \in W_k;\ \rho_Y(\varphi(x_1, \ldots, x_k), p) < \varepsilon\} \in \chi^k$$

for any $\varepsilon > 0$. This p is denoted by $\varphi(\chi^k)$. These values $\varphi(\chi^k)$ for various $k = 1, 2, \ldots$, W_k, Y and φ are called the *infinitesimal geometry* of X at χ. For example, let $\varphi = id_X : X \to X$ be such that $\varphi(x) = x$ ($\forall x \in X$). Then, $id_X(\chi)$ is the point $x_0 \in X$ such that χ *converges* to it in the sense that $\{x \in X;\ \rho_X(x, x_0) < \varepsilon\} \in \chi$ for any $\varepsilon > 0$. If $\varphi : W_k \to Y$ with $(x_0, \ldots, x_0) \in W_k \in \chi^k$ is continuous at $(x_0, \ldots, x_0)$, where $x_0 = id_X(\chi)$, then it is clear that $\varphi(\chi^k) = \varphi(x_0, \ldots, x_0)$. On the other hand, if φ is not continuous at $(x_0, \ldots, x_0)$, then $\varphi(\chi^k)$ takes one of the limiting values of φ as $(x_1, \ldots, x_k) \to (x_0, \ldots, x_0)$. In particular, if Y is a finite set, then $\varphi(\chi^k) = y$ is equivalent to $\{(x_1, \ldots, x_k) \in W_k;\ \varphi(x_1, \ldots, x_k) = y\} \in \chi^k$.

For another example, let X be a compact Riemanian manifold and $\chi \in \beta X \setminus X$. Let W be a small neighborhood of $id_X(\chi)$. For distinct points $x_1, x_2 \in W$, let $\varphi(x_1, x_2)$ be the unit vector $\vec{x_2 x_1}/||\vec{x_2 x_1}||$ with respect to a local ortho-normal coordinate. Then, a unit vector $v = \varphi(\chi^2)$ in the tangent space of X at $id_X(\chi)$ is determined so that

$$\{(x_1, x_2) \in X \times X;\ ||\ v - \vec{x_2 x_1}/||\vec{x_2 x_1}||\ || < \varepsilon\} \in \chi^2$$

for any $\varepsilon > 0$, which we call the *tangent* vector of χ.

Let $f : X \to X$ be a continuous map so that (X, f) is a topological dynamical system. Let $x_0 \in X$ satisfy that $\{f^n(x_0);\ n \in \mathbb{N}\}$ is dense in X. Let $\kappa : X \to \mathbb{A}$ and $\omega_0 \in \mathbb{A}^{\mathbb{N}}$ be such that $\omega_0(n) = \kappa(f^n(x_0))$ ($\forall n \in \mathbb{N}$). Let $\Omega \subset \mathbb{A}^{\mathbb{N}}$ be the closure of $\{T^n \omega_0;\ n \in \mathbb{N}\}$, where T is the shift. The symbolic dynamics (Ω, T) is called the *symbolic representation* of (X, f) through κ and x_0. An imbedding $F : \mathbb{N} \to X$ is defined by $F(n) = f^n(x_0)$, which we call a *dynamical imbedding* of $\mathbb{N}$ into (X, f). Let $\chi \in \beta\mathbb{N}$ and $\tilde{\chi} := F(\chi) \in \beta X$. By Corollary 1, we have $\tilde{\chi}^k = (F^k)(\chi^k)$ for $k = 1, 2, \ldots$. Then, the superstationary factor $\Omega[\chi^\infty]$ is closely related to the infinitesimal geometry of X at $\tilde{\chi}$. In fact, we have

Theorem 6. *In the setting as above and for $k = 1, 2, \ldots$, let*

$$\varphi(x_1, \ldots, x_k) = \{(\kappa(f^n(x_1)), \ldots, \kappa(f^n(x_k)));\ n \in \mathbb{N}\} \subset \mathbb{A}^k.$$

Then, we have $\Omega[\chi^k] = \varphi(\tilde{\chi}^k)$. Moreover, if the family $\{f^n;\ n \in \mathbb{N}\}$ is equicontinuous and κ is continuous, then $\varphi(\tilde{\chi}^k) \subset \{a^k;\ a \in \mathbb{A}\}$.

Proof Since the image of φ is a finite set, we have

$$\{(x_1,\dots,x_k)\in X^k;\ \varphi(x_1,\dots,x_k)=\varphi(\tilde{\chi}^k)\}\in\tilde{\chi}^k.$$

This is equivalent to

$$\{(n_1,\dots,n_k)\in\mathbb{N}^k;\ \varphi(F(n_1),\dots,F(n_k))=\varphi(\tilde{\chi}^k)\}\in\chi^k.$$

On the other hand, since

$$\begin{aligned}
&\varphi(F(n_1),\dots,F(n_k))\\
&=\{(\kappa(f^{n+n_1}(x_0)),\dots,\kappa(f^{n+n_k}(x_0)));\ n\in\mathbb{N}\}\\
&=\{\omega_0(n+n_1)\dots\omega_0(n+n_k);\ n\in\mathbb{N}\}\\
&=\{\omega(n_1)\dots\omega(n_k);\ \omega\in\{T^n\omega_0;\ n\in\mathbb{N}\}\}\\
&=\{\omega(n_1)\dots\omega(n_k);\ \omega\in\Omega\}\\
&=\Omega[(n_1,\dots,n_k)],
\end{aligned}$$

we have $\Omega[\chi^k]=\varphi(\tilde{\chi}^k)$.

Assume that the family $\{f^n;\ n\in\mathbb{N}\}$ is equicontinuous and κ is continuous. Since the image of κ is a finite set, there exists $\delta>0$ such that if $\rho_X(x_1,x_2)<\delta$, then $\kappa(f^n(x_1))=\kappa(f^n(x_2))$ holds for any $n\in\mathbb{N}$. Let $V_\delta=\{x\in X;\ \rho(x,id_X(\tilde{\chi}))<\delta)$. Then, $V_\delta^k\in\tilde{\chi}^k$ holds. Since $\kappa(f^n(x_1))=\dots=\kappa(f^n(x_k))$ holds for any $n\in\mathbb{N}$ if $(x_1,\dots,x_k)\in V_\delta^k\in\tilde{\chi}^k$, we have $\varphi(\tilde{\chi}^k)\subset\{a^k;\ a\in\mathbb{A}\}$. □

In Example 7 of the next section, we consider (X,f) with $X=(\mathbb{R}/\mathbb{Z})^2$ and $f(x)=x+\vec{f}\ (mod\ 2)$, where $\vec{f}=(f_1,f_2)\in\mathbb{R}^2$ satisfies that $1,f_1,f_2$ are linearly independent over the rational field. We imbed $\mathbb{N}$ into (X,f) as $F(n)=(nf_1,nf_2)\ (mod\ 1)$. For three distinct points $x_1,x_2,x_3\in X$ which are close enough to each other, let $R(x_1,x_2,x_3)$ be the radius of the circle passing x_1,x_2,x_3, possibly ∞. Then, $\chi\in\beta X\setminus X$ determines $q\in(0,\infty]$ possibly with ±0 such that

$$\{(x_1,x_2,x_3)\in X^3;\ q<R(x_1,x_2,x_3)<q+\varepsilon\}\in\chi^3$$

for any $\varepsilon>0$,

$$\{(x_1,x_2,x_3)\in X^3;\ R(x_1,x_2,x_3)=q\}\in\chi^3,$$

or

$$\{(x_1,x_2,x_3)\in X^3;\ q-\varepsilon<R(x_1,x_2,x_3)<q\}\in\chi^3$$

for any $\varepsilon>0$. We define $R(\chi^3)$ to be $q+0$, q or $q-0$ corresponding to the three cases, and call it the *radius* of χ. Moreover, the *osculating circle* of χ is determined as $\varphi(\chi^3)$ for $\varphi(x_1,x_2,x_3)$ which is, by definition, the circle determined by three distinct points $x_1,x_2,x_3\in X$. Then, the osculating circle of χ passes $id_X(\chi)$ and has radius q as above.

In Examples 3 and 6, we consider (X, f) with $X = \mathbb{Z}_2$ and $f(x) = x + 1$, where $\mathbb{Z}_2$ is the 2-adic compactification of $\mathbb{Z}$ as the additive group. It is identified with $\{0,1\}^{\mathbb{N}}$, and $n \in \mathbb{N}$ is imbedded into $\{0,1\}^{\mathbb{N}}$ as the element $e_0(n)e_1(n)e_2(n)\ldots \in \{0,1\}^{\mathbb{N}}$ satisfying that $n = \sum_{i=0}^{\infty} e_i(n)2^i$, while a negative integer $-n-1$ with $n \in \mathbb{N}$ is imbedded as the element $(1-e_0(n))(1-e_1(n))(1-e_2(n))\ldots \in \{0,1\}^{\mathbb{N}}$. We denote this imbedding by $F : \mathbb{Z} \to \{0,1\}^{\mathbb{N}}$. The multiplication by 2^n for $n \in \mathbb{N}$ also makes sense in $\{0,1\}^{\mathbb{N}}$. For $\theta \in \{0,1\}^{\mathbb{N}} \setminus \{0\}$, $\tau(\theta)$ denotes the maximum $n \in \mathbb{N}$ such that 2^n is a multiple of θ, that is, 0^n1 is a prefix of θ. To be complete, define $\tau(0^\infty) = \infty$.

Definition 8. For $\omega \in \{0,1\}^{\mathbb{N}}$ and $S = \{s_1 < s_2 < \ldots\} \subset \mathbb{N}$ with $\#S < \infty$ or $\#S = \infty$, define

$$\lambda(S, \omega) = \sup_{m\in\mathbb{N}} \sum_i (\omega(s_i + m) - \omega(s_{i+1} + m))^2,$$

where $\displaystyle\sum_i$ denotes $\displaystyle\sum_{i=1}^{k-1}$ if $\#S = k < \infty$ and $\displaystyle\sum_{i=1}^{\infty}$ if $\#S = \infty$.

Theorem 7. *Let $\omega_0 \in \{0,1\}^{\mathbb{N}}$ and Ω be $\{T^n\omega_0;\ n \in \mathbb{N}\} \subset \{0,1\}^{\mathbb{N}}$ or its closure, where $T : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is the shift. For any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$, the following statements hold.*
(1) $\Omega[\chi^\infty] = \{0,1\}^{\mathbb{N}}$ holds if there exists a sequence $(U_k \in \chi^k;\ k = 1, 2, \ldots)$ such that

$$\lim_{k\to\infty} \inf_{S\in U_k\cap\Delta_k} \lambda(S, \omega_0) = \infty,$$

where

$$\Delta_k := \{(s_1, \ldots, s_k) \in \mathbb{N}^k;\ s_1 < \ldots < s_k\}$$

and $(s_1, \ldots, s_k) \in U_k \cap \Delta_k$ is identified with $\{s_1 < \ldots < s_k\} \subset \mathbb{N}$.

(2) $\Omega[\chi^\infty] \neq \{0,1\}^{\mathbb{N}}$ holds if there exists $U \in \chi$ such that $\lambda(U, \omega_0) < \infty$.

Proof Let $\omega_0 \in \{0,1\}^{\mathbb{N}}$ and $\Omega = \{T^n\omega_0;\ n \in \mathbb{N}\}$. The proof for the closure of Ω is the same as the proof for Ω.

(1) Assume that $\Omega[\chi^\infty] \neq \{0,1\}^{\mathbb{N}}$. Then, there exists a nonempty finite set $\Xi \subset \{0,1\}^+$ such that $\Omega[\chi^\infty] = \mathcal{P}(\Xi)$. Take $\xi \in \Xi$ such that $\xi \in \{0,1\}^l$ for some $l \in \mathbb{N}$.

Take an arbitrary sequence $(U_k \in \chi^k;\ k = 1, 2, \ldots)$. Then, for any $k = 1, 2, \ldots$, there exists $S \in U_k \cap \Delta_k$ such that $\Omega[\chi^k] = \Omega[S]$. Take such $S = (s_1 < \ldots < s_k) \in U_k$. If $\lambda(S, \omega_0) \geq 2l$, then there exists $m \in \mathbb{N}$ such that

$$\lambda(S, B) = \sum_{i=1}^{k-1} (\omega_0(s_i + m) - \omega_0(s_{i+1} + m))^2 \geq 2l.$$

This implies that there exist

$$1 \le i_1 < i_1 + 1 < i_2 < i_2 + 1 < \ldots < i_l < i_l + 1 \le 2l$$

such that $\{\omega_0(s_{i_j} + m),\ \omega_0(s_{i_j+1} + m)\} = \{0,1\}$ $(\forall j = 1,\ldots,l)$. Therefore, any element in $\{0,1\}^l$ is a super-subword of $\omega_0(s_1 + m)\ldots\omega_0(s_k + m)$. Hence, $(T^m\omega_0)[S]$ contains ξ as a super-subword. Since $\Omega[S] = \Omega[\chi^k]$, this contradicts with $\Omega[\chi^\infty] = \mathcal{P}(\Xi)$. This implies that for any $k = 1,2,\ldots$, there exists $S \in U_k \cap \Delta_k$ such that $\lambda(S,\omega_0) < 2l$. Thus,

$$\lim_{k\to\infty} \inf_{S\in U_k\cap\Delta_k} \lambda(S,\omega_0) < 2l$$

for any sequence $(U_k \in \chi^k;\ k = 1,2,\ldots)$, which implies (1).

(2) Assume that there exists $U \in \chi$ such that $\lambda(U,\omega_0) =: l < \infty$. By Lemma 3, for any $k = 1,2,\ldots$, there exists $S = \{s_1 < \ldots < s_k\} \subset U$ such that $\Omega[\chi^k] = \Omega[S]$. Let $k = 2(l + 1)$. Since

$$\Omega[S] = \{\omega_0(s_1 + m)\ldots\omega_0(s_k + m) \in \{0,1\}^k;\ m \in \mathbb{N}\},$$

and $\lambda(S,\omega_0) \le \lambda(U,\omega_0) = l$, $(01)^{l+1} \notin \Omega[S] = \Omega[\chi^k]$. Thus, $\Omega[\chi^\infty] \neq \{0,1\}^{\mathbb{N}}$. □

5 Examples

We give seven examples of dynamical systems, where the superstationary factors are obtained. They were more or less discussed in earlier articles.

Example 1. ([4]) Let $\mathbb{Z}$ be the set of integers. We call $\omega \in \{0,1\}^{\mathbb{Z}}$ increasing if $\omega(n) \le \omega(m)$ holds for any $n,m \in \mathbb{Z}$ with $n \le m$. Let

$$\Omega = \{\omega \in \{0,1\}^{\mathbb{Z}};\ \omega \text{ is either increasing or } \sum_{n\in\mathbb{Z}} \omega(n) \le 1\}.$$

Then, we have $DS(\Omega) = \beta\mathbb{Z} \setminus \mathbb{Z}$ and

$$\Omega[\chi^\infty] = \Omega((\chi)) = \begin{cases} \mathcal{P}(101,110) & \text{if } \chi \in \mathcal{U}(\mathbb{N}) \setminus \mathbb{Z} \\ \mathcal{P}(101,011) & \text{if } \chi \in \mathcal{U}(\mathbb{Z}_-) \setminus \mathbb{Z}. \end{cases}$$

Moreover, they are attainable.

In fact, take any $\chi \in \mathcal{U}(\mathbb{N})\setminus\mathbb{Z}$ and let $\varphi : \mathbb{N} \to \mathbb{Z}$ be the injection such that $\varphi(n) = n$ $(\forall n \in \mathbb{N})$. Then, we have $\varphi(\mathbb{N}) = \mathbb{N} \in \chi$ and $\Omega \circ \varphi = \mathcal{P}(101,110)$. Hence, $\mathcal{P}(101,110)$ is an attainable strong superstationary factor of Ω at χ.

On the other hand, take any $\chi \in \mathcal{U}(\mathbb{Z}_-) \setminus \mathbb{Z}$ and let $\varphi : \mathbb{N} \to \mathbb{Z}$ be the injection such that $\varphi(n) = -n - 1$ $(\forall n \in \mathbb{N})$. Then, we have $\varphi(\mathbb{N}) = \mathbb{Z}_- \in \chi$ and $\Omega \circ \varphi = \mathcal{P}(101,011)$. Hence, $\mathcal{P}(101,011)$ is an attainable strong superstationary factor of Ω at χ.

These 2 superstationary sets are not isomorphic in the sense of [4].

Example 2. ([7]) Let $X = \mathbb{R}/\mathbb{Z}$ and $f : X \to X$ be $f(x) = x + \alpha \ (mod\ 1)$, where α is an irrational number. Let $d \geq 2$ be an integer and $\mathbb{A} = \{0, 1, \ldots, d-1\}$. Let $a_0 < a_1 < \ldots < a_{d-1} < a_d$ be real numbers such that $a_d = a_0 + 1$. Define $\kappa : X \to \mathbb{A}$ by

$$\kappa(x) = i \text{ if } x \in [a_i, a_{i+1}) \ (mod\ 1) \ (i \in \mathbb{A}).$$

Define an imbedding $F : \mathbb{N} \to X$ by $F(n) = n\alpha \ (mod\ 1)$. For $x \in X$, define $\omega_x \in \mathbb{A}^{\mathbb{N}}$ by $\omega_x(n) = \kappa(x + n\alpha) \ (\forall n \in \mathbb{N})$. Let Ω be the closure of $\{T^n \omega_0;\ n \in \mathbb{N}\}$. Then, it is clear that Ω is the closure of $\{\omega_x;\ x \in X\}$.

For $d \geq 3$, define $\Xi_+,\ \Xi_- \subset \mathbb{A}^+$ by

$$\Xi_+ = \{ij \in \mathbb{A}^2;\ j \neq i \text{ and } j \neq i+1 \ (mod\ d)\}$$

and

$$\Xi_- = \{ij \in \mathbb{A}^2;\ j \neq i \text{ and } j \neq i-1 \ (mod\ d)\}.$$

For $d = 2$, define

$$\Xi_+ = \Xi_- = \{101, 010\}.$$

Then, it holds that

$$\{\Omega[\chi^\infty];\ \chi \in \beta\mathbb{N} \setminus \mathbb{N}\} = \{\Omega((\chi));\ \chi \in DS(\Omega)\} = \{\mathcal{P}(\Xi_-), \mathcal{P}(\Xi_+)\}.$$

Moreover, they are attainable and $DS(\Omega) \neq \beta\mathbb{N} \setminus \mathbb{N}$.

In fact, there exist increasing injections $\varphi_i : \mathbb{N} \to \mathbb{N}$ $(i = 1, 2)$ and $x_0 \in X$ such that
(1) $x_0 - \delta < x_0 + \varphi_1(0)\alpha < x_0 + \varphi_1(1)\alpha < x_0 + \varphi_1(2)\alpha < \ldots < x_0 \ (mod\ 1)$,
or
(2) $x_0 + \delta > x_0 + \varphi_2(0)\alpha > x_0 + \varphi_2(1)\alpha > x_0 + \varphi_2(2)\alpha > \ldots > x_0 \ (mod\ 1)$,
where $\delta = \min\{a_{i+1} - a_i;\ i = 0, 1, \ldots, d-1\}$.

Then, it is easy to see that

$$\Omega \circ \varphi_1 = \mathcal{P}(\Xi_+) \ \text{ and } \ \Omega \circ \varphi_2 = \mathcal{P}(\Xi_-).$$

Hence, $\mathcal{P}(\Xi_+)$ and $\mathcal{P}(\Xi_-)$ are attainable strong superstationary factors of Ω. This implies that

$$\{\Omega((\chi));\ \chi \in DS(\Omega)\} \supset \{\mathcal{P}(\Xi_-), \mathcal{P}(\Xi_+)\}. \tag{5.1}$$

Let $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ and $\tilde{\chi} = F(\chi)$. Let $x_0 = id_X(\tilde{\chi})$ as in Section 4. We call $\tilde{\chi} \in \beta X \setminus X$ *increasing* if $(id_X(\tilde{\chi}) - \varepsilon,\ id_X(\tilde{\chi})) \in \tilde{\chi}$ for any $\varepsilon > 0$, and *decreasing* if $(id_X(\tilde{\chi}),\ id_X(\tilde{\chi}) + \varepsilon) \in \tilde{\chi}$ for any $\varepsilon > 0$. For any $k = 1, 2, \ldots$, let

$$U_k^1 = \{(n_1 < \ldots < n_k) \in \mathbb{N}^k;\ x_0 - \delta_0 < n_1\alpha < \ldots < n_k\alpha < x_0 \ (mod\ 1)\},$$

and

$$U_k^2 = \{(n_1 < \ldots < n_k) \in \mathbb{N}^k;\ x_0 + \delta > n_1\alpha > \ldots > n_k\alpha > x_0\ (mod\ 1)\}.$$

Then, by the same argument as in the proof of (1) of Theorem 5, we can prove that $U_k^1 \in \chi^k$ if $\tilde{\chi}$ is increasing and $U_k^2 \in \chi^k$ if $\tilde{\chi}$ is decreasing. Hence, there exists $(n_1 < \ldots < n_k) \in U_k^i$ $(i = 1, 2)$ satisfying that

$$\Omega[(n_1 < \ldots < n_k)] = \Omega[\chi^k].$$

Moreover, the above (1) or (2) with $x_0 = id_X(\tilde{\chi})$ and $\varphi_i(j) = n_{j+1}$ $(j = 0, 1, \ldots, k-1)$ is satisfied. Thus, we have

$$\Omega[\chi^k] = \Omega \circ \varphi_1 = \mathcal{P}(\Xi_+)[\{0, 1, \ldots, k-1\}]$$

if $\tilde{\chi}$ is increasing, and

$$\Omega[\chi^k] = \Omega \circ \varphi_2 = \mathcal{P}(\Xi_-)[\{0, 1, \ldots, k-1\}]$$

if $\tilde{\chi}$ is decreasing. Since $k = 1, 2, \ldots$ is arbitrary, we have

$$\Omega[\chi^\infty] = \begin{cases} \mathcal{P}(\Xi_+) & \text{if } \tilde{\chi} \text{ is increasing} \\ \mathcal{P}(\Xi_-) & \text{if } \tilde{\chi} \text{ is decreasing} \end{cases}$$

Thus, together with (5.1), we have

$$\{\Omega[\chi^\infty];\ \chi \in \beta\mathbb{N} \setminus \mathbb{N}\} = \{\Omega((\chi));\ \chi \in DS(\Omega)\} = \{\mathcal{P}(\Xi_-), \mathcal{P}(\Xi_+)\}.$$

Let us show that there exists $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ with $\chi \notin DS(\Omega)$. It is easy to check that there exists $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ such that $\chi \subset \{U \subset \mathbb{N};\ \mathrm{acdeg}\, F(U) \geq 2\}$. Then, we prove that $\chi \notin DS(\Omega)$. Suppose to the contrary that $\chi \in DS(\Omega)$. Then, we must have $\Omega((\chi)) = \mathcal{P}(\Xi_+)$ or $\mathcal{P}(\Xi_-)$. This implies that for any $k = 1, 2, \ldots$, there exists an injection $\varphi_k : \mathbb{N} \to \mathbb{N}$ with $\mathrm{acdeg}\, F(\varphi_k(\mathbb{N})) \geq 2$ such that $\Omega \circ \varphi_k$ is k-superstationary and

$$\Omega \circ \varphi_k[\{0, 1, \ldots, k-1\}] = \mathcal{P}(\Xi_+)[\{0, 1, \ldots, k-1\}] \text{ or } \mathcal{P}(\Xi_-)[\{0, 1, \ldots, k-1\}].$$

We take $k = 3$. Since there exist infinitely many accumulating points of $F(\varphi_3(\mathbb{N}))$, take two accumulating points y_0, y_1 of $F(\varphi_3(\mathbb{N}))$ with $y_0 \neq y_1$. Then, there exist $x_0 \in \mathbb{R}$ and ε with $0 < \varepsilon < 1/2$ such that

$$\pi((x_0 - \varepsilon, x_0 + \varepsilon)) + y_0 \subset \pi([a_0, a_1))$$

and

$$\pi((x_0 - \varepsilon, x_0 + \varepsilon)) + y_1 \subset \pi([a_i, a_{i+1})) \text{ for some } i \in \mathbb{A} \text{ with } i \neq 0,$$

where π is the natural projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. Hence, $\kappa(x_0 + n\alpha) = 0$ if $F(n) = \pi(n\alpha) \in (y_0 - \varepsilon, y_0 + \varepsilon)$ while $\kappa(x_0 + n\alpha) = i \neq 0$ if $F(n) = \pi(n\alpha) \in (y_1 - \varepsilon, y_1 + \varepsilon)$.

Since y_0 and y_1 are accumulating points of $F(\varphi_3(\mathbb{N}))$, there exist nonnegative integers $n_0 < n_1 < n_2$ such that $F(\varphi_3(n_i)) \in (y_0 - \varepsilon, y_0 + \varepsilon)$ for $i = 0, 2$ and $F(\varphi_3(n_1)) \in (y_1 - \varepsilon, y_1 + \varepsilon)$. This implies that

$$\kappa(x_0 + \varphi_3(n_0)\alpha)\kappa(x_0 + \varphi_3(n_1)\alpha)\kappa(x_0 + \varphi_3(n_2)\alpha) = 0i0.$$

Thus, we have a contradiction that

$$0i0 \in \Omega \circ \varphi_3[\{n_0, n_1, n_2\}] = \Omega \circ \varphi_3[\{0, 1, 2\}] \subset (\mathcal{P}(\Xi_+) \cup \mathcal{P}(\Xi_-))[\{0, 1, 2\}].$$

Example 3. ([7]) Let $X = \mathbb{Z}_2 = \{0, 1\}^{\mathbb{N}}$ be the 2-adic compactification of $\mathbb{Z}$ and $F : \mathbb{Z} \to X$ be the canonical imbedding as in Section 4. We identify $n \in \mathbb{N}$ with $F(n) \in \{0, 1\}^{\mathbb{N}}$, sometimes denoting $F(n)$ by n. Let $f : X \to X$ be the addition by 1, that is, $f(x) = x + 1$. Let $\tau : X \to \mathbb{N} \cup \{\infty\}$ be as in Section 4 the maximum n such that 2^n divides x.

Define $\kappa : X \to \{0, 1\}$ by

$$\kappa(x) = \begin{cases} 0 & \text{if } \tau(x+1) \text{ is even} \\ 1 & \text{if } \tau(x+1) \text{ is odd} \end{cases}$$

We define $\kappa(-1)$ to be 0 or 1 arbitrary. The unique discontinuous point of the function $\kappa(x)$ is $x = -1$. For $x \in X$ let $\omega_x \in \{0, 1\}^{\mathbb{N}}$ be such that $\omega_x(n) = \kappa(f^n(x))$ $(\forall n \in \mathbb{N})$. Then, $\omega_0 = 010001010100\ldots$ is called the *Toeplitz word*. Let $T : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ be the shift and $\Omega \subset \{0, 1\}^N$ be the closure of $\{T^n\omega_0;\ n \in \mathbb{N}\}$. Thus, our system (Ω, T) comes from (X, f) through κ and 0.

Then, we have

$$\{\Omega[\chi^\infty];\ \chi \in \beta\mathbb{N}\backslash\mathbb{N}\} = \{\Omega((\chi));\ \chi \in DS(\Omega)\} = \{\mathcal{P}(101, 110),\ \mathcal{P}(010, 001)\}.$$

Moreover, they are attainable and $DS(\Omega) \neq \beta\mathbb{N} \setminus \mathbb{N}$.

In fact, let $\varphi : \mathbb{N} \to \mathbb{N}$ be an injection such that $\varphi(n) = 2^{2n+2}$ $(\forall n \in \mathbb{N})$. Then, for any $N \in \mathbb{N}$ with $\tau(N + 1) = 2k + 1$, where $k \in \mathbb{N}$, we have

$$\tau(N + 1 + \varphi(n)) = \begin{cases} 2n + 2 & \text{if } n < k \\ 2k + 1 & \text{if } n \geq k \end{cases} \quad (\forall n \in \mathbb{N}),$$

and for any $N \in \mathbb{N}$ with $\tau(N + 1) = 2k$, where $k \in \mathbb{N}$, we have

$$\tau(N + 1 + \varphi(n)) \begin{cases} = 2n + 2 & \text{if } n < k - 1 \\ > 2k & \text{if } n = k - 1 \\ = 2k & \text{if } n > k - 1 \end{cases} \quad (\forall n \in \mathbb{N}),$$

where in the middle case, any value larger than $2k$ is possible.

Hence, $(T^N\omega_0) \circ \varphi$ for $N \in \mathbb{N}$ is either 0^k1^∞, 0^∞ or 0^k10^∞ for some $k \in \mathbb{N}$. Therefore, $\Omega \circ \varphi = \mathcal{P}(101, 110)$.

Now, let $\varphi : \mathbb{N} \to \mathbb{N}$ be an injection such that $\varphi(n) = 2^{2n+1}$ $(\forall n \in \mathbb{N})$. Then, for any $N \in \mathbb{N}$ with $\tau(N+1) = 2k$, where $k \in \mathbb{N}$, we have

$$\tau(N+1+\varphi(n)) = \begin{cases} 2n+1 & \text{if } n < k \\ 2k & \text{if } n \geq k \end{cases} \quad (\forall n \in \mathbb{N}),$$

and for any $N \in \mathbb{N}$ with $\tau(N+1) = 2k+1$, where $k \in \mathbb{N}$, we have

$$\tau(N+1+\varphi(n)) \begin{cases} = 2n+1 & \text{if } n < k \\ > 2k+1 & \text{if } n = k \\ = 2k+1 & \text{if } n > k \end{cases} \quad (\forall n \in \mathbb{N}),$$

where in the middle case, any value larger than $2k+1$ is possible. Hence, $(T^N\omega_0) \circ \varphi$ for $N \in \mathbb{N}$ is either $1^k 0^\infty$, 1^∞ or $1^k 01^\infty$ for some $k \in \mathbb{N}$. Therefore, $\Omega \circ \varphi = \mathcal{P}(010, 001)$.

Thus, we have

$$\{\Omega((\chi));\ \chi \in DS(\Omega)\} \supset \{\mathcal{P}(101, 110),\ \mathcal{P}(010, 001)\}, \tag{5.2}$$

both factors being attainable.

Let $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ and $\tilde{\chi} = F(\chi)$. Let $x_0 = id_X(\tilde{\chi})$. We call χ *of even type* if $\tau(\tilde{\chi} - x_0) \in \{0, 2, 4, \ldots, \infty\}$, and *of odd type* if $\tau(\tilde{\chi} - x_0) \in \{1, 3, 5, \ldots\}$. That is, χ is of even type if $\{n \in \mathbb{N};\ \tau(n - x_0) \in \{0, 2, 4, \ldots, \infty\}\} \in \chi$, and of odd type if $\{n \in \mathbb{N};\ \tau(n - x_0) \in \{1, 3, 5, \ldots\}\} \in \chi$ since $\mathbb{N} \in \tilde{\chi}$ and $\tilde{\chi}$ restricted to $\mathbb{N}$ coincides with χ.

Without loss of generality, we assume that χ is of even type. By the same reason as in the proof of (1) of Theorem 5, for any $k = 1, 2, \ldots$,

$$\begin{aligned} U_k := \{(n_1 < \ldots < n_k) \in \mathbb{N}^k;\ & \tau(n_i - x_0) \text{ is even for any} \\ & i = 1, \ldots, k \text{ and } \tau(n_1 - x_0) < \ldots < \tau(n_k - x_0)\} \in \chi^k. \end{aligned}$$

Hence, there exists $(n_1, \ldots, n_k) \in U_k$ such that $\Omega[\chi^k] = \Omega[(n_1, \ldots, n_k)]$. Let $\varphi : \{0, 1, \ldots, k-1\} \to \mathbb{N}$ be such that $\varphi(i) = n_{i+1}$ $(i = 0, 1, \ldots, k-1)$.

Then, for any $N \in \mathbb{N}$ with $\tau(N+1+x_0) = 2l+1$, where $l \in \mathbb{N}$, we have

$$\tau(N+1+\varphi(i)) = \begin{cases} \tau(\varphi(i) - x_0) & \text{if } \tau(\varphi(i) - x_0) < 2l+1 \\ 2l+1 & \text{if } \tau(\varphi(n) - x_0) > 2l+1 \end{cases}$$
$$(i = 0, 1, \ldots, k-1),$$

and for any $N \in \mathbb{N}$ with $\tau(N+1+x_0) = 2l$, where $l \in \mathbb{N}$, we have

$$\tau(N+1+\varphi(i)) \begin{cases} = \tau(\varphi(n) - x_0) & \text{if } \tau(\varphi(i) - x_0) < 2l \\ > 2l & \text{if } \tau(\varphi(i) - x_0) = 2l \\ = 2l & \text{if } \tau(\varphi(i) - x_0) > 2l \end{cases}$$
$$(i = 0, 1, \ldots, k-1),$$

where in the middle case, any value larger than $2l$ is possible.

Hence, $(T^N\omega_0)\circ\varphi$ for $N\in\mathbb{N}$ is either 0^i1^{k-i} $(i=0,1,\dots,k-1)$ or 0^i10^{k-1-i} $(i=0,1,\dots,k-1)$. Then,

$$\Omega[\chi^k]=\Omega\circ\varphi=\mathcal{P}(101,110)[\{0,1,\dots,k-1\}].$$

Therefore, we have $\Omega[\chi^\infty]=\mathcal{P}(101,110)$. In the other case, we have $\Omega[\chi^\infty]=\mathcal{P}(010,001)$.

Hence, it holds for any $\chi\in\beta\mathbb{N}\setminus\mathbb{N}$ that

$$\Omega[\chi^\infty]=\begin{cases}\mathcal{P}(101,110) & \text{if } \chi \text{ is of even type}\\ \mathcal{P}(010,001) & \text{if } \chi \text{ is of odd type.}\end{cases}$$

Thus, together with (5.2), we have

$$\{\Omega[\chi^\infty];\ \chi\in\beta\mathbb{N}\backslash\mathbb{N}\}=\{\Omega((\chi));\ \chi\in DS(\Omega)\}=\{\mathcal{P}(101,110),\mathcal{P}(010,001)\}.$$

There exists $\chi\in\beta\mathbb{N}\setminus\mathbb{N}$ satisfying $\chi\subset\{U\subset\mathbb{N};\ \operatorname{acdeg}F(U)\geq 2\}$, where $F:\mathbb{Z}\to\{0,1\}^{\mathbb{N}}$ is the canonical imbedding. As before, we identify $F(n)$ with n. Then, we prove that $\chi\notin DS(\Omega)$. Suppose to the contrary that $\chi\in DS(\Omega)$. Then, we must have $\Omega((\chi))\in\{\mathcal{P}(101,110),\ \mathcal{P}(010,001)\}$. This implies that for any $k=1,2,\dots$, there exists an injection $\varphi_k:\mathbb{N}\to\mathbb{N}$ with $\operatorname{acdeg}\varphi_k(\mathbb{N})\geq 2$ such that $\Omega\circ\varphi_k$ is a k-superstationary set satisfying either

$$\Omega\circ\varphi_k[\{0,1,\dots,k-1\}]=\mathcal{P}(101,110)[\{0,1,\dots,k-1\}]$$

or

$$\Omega\circ\varphi_k[\{0,1,\dots,k-1\}]=\mathcal{P}(010,001)[\{0,1,\dots,k-1\}].$$

We take $k=4$. Since there exist infinitely many accumulating points of $\varphi_4(\mathbb{N})$, take two accumulating points $\omega^1,\ \omega^2\in\{0,1\}^{\mathbb{N}}$ of $\varphi_4(\mathbb{N})$ with $\omega^1\neq\omega^2$. Let $\tau(\omega^2-\omega^1)=r$.

Take $n_1<n_2<n_3<n_4$ in $\mathbb{N}$ such that $\tau(\varphi_4(n_i)-\omega^1)\geq r+2$ for $i=1,3$ and $\tau(\varphi_4(n_i)-\omega^2)\geq r+1$ for $i=2,4$. Take $N\in\mathbb{N}$ such that $\tau(N+1+\omega^1)=r+1$. Then, we have

$$\tau(N+1+\varphi_4(n_i))=\tau(N+1+\omega^1+\varphi_4(n_i)-\omega^1)=\tau(N+1+\omega^1)=r+1$$

for $i=1,3$, and

$$\tau(N+1+\varphi_4(n_i))=\tau(N+1+\omega^1+\omega^2-\omega^1+\varphi_4(n_i)-\omega^2)=\tau(\omega^2-\omega^1)=r$$

for $i=2,4$. Hence, we have

$$\omega_0(N+\varphi_4(n_1))\omega_0(N+\varphi_4(n_2))\omega_0(N+\varphi_4(n_3))\omega_0(N+\varphi_4(n_4))=0101 \text{ or } 1010.$$

This implies that

$$\begin{aligned} & 0101 \text{ or } 1010\in\Omega\circ\varphi_4[\{n_0,n_1,n_2,n_3\}]=\Omega\circ\varphi_4[\{0,1,2,3\}]\\ = \ & \mathcal{P}(101,110)[\{0,1,2,3\}] \text{ or } \mathcal{P}(010,001)[\{0,1,2,3\}],\end{aligned}$$

which is impossible. Thus, $\chi\notin DS(\Omega)$.

Example 4. Let $0 = L_0 < L_1 < L_2 < \dots$ be sequence of positive integers such that $L_{i+1} - L_i \to \infty$ as $i \to \infty$. For $i = 0, 1, 2, \dots$, let $\Omega_i = 0^{L_i}\{0,1\}^{L_{i+1}-L_i}0^\infty$ and $\Omega = \cup_{i=0}^\infty \Omega_i$. Then, Ω is clearly a closed set with the full maximal pattern complexity [8]. On the other hand, it holds that $DS(\Omega) \neq \beta\mathbb{N} \setminus \mathbb{N}$ and

$$\{\Omega[\chi^\infty];\ \chi \in \beta\mathbb{N} \setminus \mathbb{N}\} = \{\Omega((\chi));\ \chi \in DS(\Omega)\} = \{\mathcal{P}(11)\}.$$

Moreover, $\mathcal{P}(11)$ is attainable.

For any function $f : \mathbb{N} \to \mathbb{N}$ and $k = 1, 2, \dots$, define

$$N_f = \{(n_1, \dots, n_k) \in \mathbb{N}^k;\ n_{i+1} > n_i + f(n_i)\ (i = 1, \dots, k-1)\}.$$

Then, for any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$, it is easy to check that $N_f \in \chi^k$.

Define $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = L_{i+1} - L_i$ if $n \in [L_i, L_{i+1})$ $(i = 0, 1, 2, \dots)$. Then, there exists $S = (n_1, \dots, n_k) \in N_f$ such that $\Omega[S] = \Omega[\chi^k]$. For any $\omega \in \Omega$ and $j = 1, \dots, k-1$, if $\omega(n_j) = 1$, then $\omega \in \Omega_i$ with $n_j \in [L_i, L_{i+1})$. Hence, for any $h = i+1, \dots, k$, $\omega(n_h) = 0$ holds since $n_h > n_j + L_{i+1} - L_i \geq L_{i+1}$. Therefore, we have $\Omega[\chi^k] = \mathcal{P}(11)[\{0, 1, \dots, k-1\}]$. Since $k = 1, 2, \dots$ and $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ are arbitrary, we have

$$\{\Omega[\chi^\infty];\ \chi \in \beta\mathbb{N} \setminus \mathbb{N}\} = \{\Omega((\chi));\ \chi \in DS(\Omega)\} = \{\mathcal{P}(11)\}.$$

Let an injection $\varphi : \mathbb{N} \to \mathbb{N}$ be such that $\#(\varphi(\mathbb{N}) \cap [L_i, L_{i+1})) \leq 1$ for any $i = 0, 1, 2, \dots$. Then, it is clear that $\Omega \circ \varphi = \mathcal{P}(11)$. Take any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ with $\varphi(\mathbb{N}) \in \chi$. Then, $\mathcal{P}(11)$ coincides with $\Omega((\chi))$ and is attainable.

For $U \subset \mathbb{N}$, let

$$\rho(U) = \limsup_{n \to \infty}\ (1/N)\#\{n < N;\ n \in U\}.$$

Then, there exists $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ such that $\chi \subset \{U \subset \mathbb{N};\ \rho(U) > 0\}$. We prove that $\chi \notin DS(\Omega)$. To the contrary suppose that $\chi \in DS(\Omega)$. Then, we must have $\Omega((\chi)) = \mathcal{P}(11)$. This implies that for any $k = 1, 2, \dots$, there exists an injection $\varphi_k : \mathbb{N} \to \mathbb{N}$ with $\rho(\varphi_k(\mathbb{N})) > 0$ such that (2), (3) of Definition 5 hold for $\Theta = \mathcal{P}(11)$. Here we take $k = 3$.

Take $K \in \mathbb{N}$ such that $2/K < \rho(\varphi_3(\mathbb{N}))$. Then, there exist infinitely many $N \in \mathbb{N}$ such that $\#(\varphi_3(\mathbb{N}) \cap [NK, (N+1)K)) \geq 3$. Hence, there exist $u, v, w \in \mathbb{N}$ with $0 \leq u < v < w < K$ and an infinitely many $N \in \mathbb{N}$ such that $N + \{u, v, w\} \in \varphi_3(\mathbb{N})$. Take sufficiently large N as this. Then, there exists $i \in \mathbb{N}$ such that either $\{N+u, N+v\} \subset [L_i, L_{i+1})$ or $\{N+v, N+w\} \subset [L_{i+1}, L_{i+2})$. In the former case, we have $110 \in \Omega \circ \varphi_3[\{0,1,2\}]$, while in the latter case, we have $011 \in \Omega \circ \varphi_3[\{0,1,2\}]$, contradicting that $\Omega \circ \varphi_3[\{0,1,2\}] = \mathcal{P}(11)[\{0,1,2\}]$

Thus, $\chi \notin DS(\Omega)$ and $DS(\Omega) \neq \beta\mathbb{N} \setminus \mathbb{N}$.

Example 5. For $i \geq 0$, let

$$\Sigma_i = \{\omega \in \{0,1\}^{\mathbb{N}};\ \sum_{n \in \mathbb{N}} \omega(n) = i\}.$$

For $i \geq 2$, let

$$\Delta_i = \{\omega \in \{0,1\}^{\mathbb{N}} \quad ; \quad |n-m| \geq 2^i \text{ for any } n, m \in \mathbb{N} \\ \text{such that } n \neq m \text{ and } \omega(n) = \omega(m) = 1\}.$$

Let $\Omega = \Sigma_0 \cup \Sigma_1 \cup \bigcup_{i \geq 2}(\Sigma_i \cap \Delta_i)$. Then, Ω is a nonempty closed set such that

$$\{\Omega[\chi^\infty];\ \chi \in \beta\mathbb{N} \setminus \mathbb{N}\} = \{\Omega((\chi));\ \chi \in DS(\Omega)\} = \{\mathcal{P}(\emptyset)\}\ (\mathcal{P}(\emptyset) = \{0,1\}^{\mathbb{N}}),$$

and $DS(\Omega) = \beta\mathbb{N} \setminus \mathbb{N}$. Moreover, $\mathcal{P}(\emptyset)$ is not attainable.

In fact, take any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$. Then for any $k = 1, 2, \ldots$, there exists a unique $i = 0, 1, \ldots, k-1$ such that $\{i + n2^k;\ n \in \mathbb{N}\} \in \chi$. Let $\varphi_k(n) = i + n2^k$ ($\forall n \in \mathbb{N}$) with this i. Then, $\varphi_k : \mathbb{N} \to \mathbb{N}$ ($k = 1, 2, \ldots$) are injections satisfying (1), (2) of Definition 5 with $\Theta = \mathcal{P}(\emptyset)$. Moreover, $\varphi_k(\mathbb{N}) \in \chi$ ($k = 1, 2, \ldots$). Thus, $\Omega((\chi)) = \mathcal{P}(\emptyset)$ for any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$.

Let us prove that $\mathcal{P}(\emptyset)$ is not attainable. Take any injection $\varphi : \mathbb{N} \to \mathbb{N}$. Let $|\varphi(0) - \varphi(1)| < 2^k$. Then, $1^k \notin \Omega \circ \varphi[\{0, 1, \ldots, k-1\}]$. Hence, $\Omega \circ \varphi$ cannot be $\mathcal{P}(\emptyset)$. Thus, $\mathcal{P}(\emptyset)$ is not attainable.

Example 6. ([2, 3]) Let $\omega_0 = 0110100110010110\ldots \in \{0,1\}^{\mathbb{N}}$ be the Thue-Morse word. That is, $\omega_0(n) = 0$ if and only if the number of 1 in the 2-adic representation of n is even. Let $T : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ be the shift and $\Omega \subset \{0,1\}^{\mathbb{N}}$ be the closure of $\{T^n\omega_0;\ n \in \mathbb{N}\}$.

Let $X = \mathbb{Z}_2 = \{0,1\}^{\mathbb{N}}$ be the 2-adic compactification of $\mathbb{Z}$ and $F : \mathbb{Z} \to X$ be the canonical imbedding as in Section 4. We identify $n \in \mathbb{N}$ with $F(n) \in \{0,1\}^{\mathbb{N}}$, sometimes denoting $F(n)$ by n. Let $f : X \to X$ be the addition by 1, that is, $f(x) = x + 1$. Let γ be any nonprincipal ultrafilter on $\mathbb{N}$. Define $\kappa : X \to \{0,1\}$ by

$$\kappa(x) = \begin{cases} 0 & \text{if } \{n \in \mathbb{N};\ \sum_{i=0}^{n-1} x(i) \text{ is even}\} \in \gamma \\ 1 & \text{if } \{n \in \mathbb{N};\ \sum_{i=0}^{n-1} x(i) \text{ is odd}\} \in \gamma. \end{cases}$$

Then, we have $\omega_0(n) = \kappa(n) = \kappa(f^n(0))$ ($n \in \mathbb{N}$). Thus, our system (Ω, T) comes from (X, f) through κ and 0. Different from Example 3, our κ is discontinuous everywhere.

Then, we have

$$\{\Omega[\chi^\infty];\ \chi \in \beta\mathbb{N} \setminus \mathbb{N}\} = \{\Omega((\chi));\ \chi \in DS(\Omega)\} = \{\mathcal{P}(\emptyset)\}\ (\mathcal{P}(\emptyset) = \{0,1\}^{\mathbb{N}}),$$

and $DS(\Omega) \neq \beta\mathbb{N} \setminus \mathbb{N}$. Moreover, $\mathcal{P}(\emptyset)$ is attainable.

In fact, let $\varphi(n) = 2^{2n}$ ($\forall n \in \mathbb{N}$). For any $k = 1, 2, \ldots$ and $\xi = \xi_1\xi_2 \ldots \xi_k \in \{0,1\}^k$, let $N_\xi = l2^{2k} + \sum_{i=1}^{k}(1-\xi_i)2^{2(i-1)}$, where $l \in \{0,1\}$ is determined depending on ξ so that $\omega_0(N_\xi) = 0$. Then, we have $\omega_0(N_\xi + \varphi(i-1)) = \xi_i$ for any $i = 1, 2, \ldots, k$. That is, $\xi \in \Omega \circ \varphi[\{0, 1, \ldots, k-1\}]$ for any $\xi \in \{0,1\}^k$. Hence, $\Omega \circ \varphi[\{0, 1, \ldots, k-1\}] = \{0,1\}^k$ ($k = 1, 2, \ldots$), which implies that $\Omega \circ \varphi = \mathcal{P}(\emptyset)$. Thus, $\mathcal{P}(\emptyset)$ is an attainable strong superstationary factor of Ω.

To prove $\Omega[\chi^\infty] = \mathcal{P}(\emptyset)$ for any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$, we use the following lemmas and Theorem 7. Recall the definition of τ in Section 4.

Lemma 7. *For any $n, m \in \mathbb{N}$ such that $n \neq m$ (mod 2), there exists $i \in \{1, 2, 3\}$ such that*

$$\omega_0(n) + \omega_0(n+i) \neq \omega_0(m) + \omega_0(m+i) \ (mod\ 2).$$

Proof Assume without loss of generality that n is odd and m is even.
Case 1: If $\tau(n+1)$ is odd, then $\omega_0(n+1) = \omega_0(n)$ and $\omega_0(m+1) \neq \omega_0(m)$. Hence, $\omega_0(n) + \omega_0(n+1) \neq \omega_0(m) + \omega_0(m+1)$ (*mod* 2).
Case 2: If $\tau(n+1)$ is even and $\tau(m+2)$ is odd, then $\omega_0(n+2) = \omega_0(n)$ and $\omega_0(m+2) \neq \omega_0(m)$. Hence, $\omega_0(n)+\omega_0(n+2) \neq \omega_0(m)+\omega_0(m+2)$ (*mod* 2).
Case 3: If $\tau(n+1)$ is even and $\tau(m+2)$ is even, then $\omega_0(n+3) = \omega_0(n)$ and $\omega_0(m+3) \neq \omega_0(m)$. Hence, $\omega_0(n)+\omega_0(n+3) \neq \omega_0(m)+\omega_0(m+3)$ (*mod* 2).
□

Lemma 8. *For any set $S = \{s_1 < s_2 < \ldots < s_k\} \subset \mathbb{N}$ satisfying that $20s_i \leq s_{i+1}$ ($i = 1, 2, \ldots, k-1$), there exists $m \in \mathbb{N}$, such that $\omega_0(s_i + m) \neq \omega_0(s_{i+1} + m)$ for any $i = 1, 2, \ldots, k-1$.*

Proof Note that $\omega_0(m+n2^e) = \omega_0(m)+\omega_0(n)$ (*mod* 2) for any $m, n, e \in \mathbb{N}$ with $m < 2^e$.

Let $e_2 = \lfloor \log_2 s_2 \rfloor$. Since $\lfloor s_1/2^{e_2} \rfloor = 0$ while $\lfloor s_2/2^{e_2} \rfloor = 1$, by Lemma 7, there exists $i \in \{1, 2, 3\}$ such that

$$\omega_0(s_1) + \omega_0(s_1 + i2^{e_2}) \neq \omega_0(s_2) + \omega_0(s_2 + i2^{e_2}) \ (mod\ 2).$$

Hence, either $\omega_0(s_1) \neq \omega_0(s_2)$ or $\omega_0(s_1 + i2^{e_2}) \neq \omega_0(s_2 + i2^{e_2})$ holds. Define $i_2 = 0$ in the former case, and i_2 equal to the above i in the latter case. Then, we have

$$\omega_0(s_1 + i_2 2^{e_2}) \neq \omega_0(s_2 + i_2 2^{e_2})$$

for some $i_2 \in \{0, 1, 2, 3\}$. Let $e_3 = \lfloor \log_2(s_3 + i_2 2^{e_2}) \rfloor$. Then again by Lemma 7, there exists $i_3 \in \{0, 1, 2, 3\}$ such that

$$\omega_0(s_2 + i_2 2^{e_2} + i_3 2^{e_3}) \neq \omega_0(s_3 + i_2 2^{e_2} + i_3 2^{e_3}).$$

On the other hand,

$$\begin{aligned} & \omega_0(s_1 + i_2 2^{e_2} + i_3 2^{e_3}) = \omega_0(s_1 + i_2 2^{e_2}) + \omega_0(i_3) \\ \neq\ & \omega_0(s_2 + i_2 2^{e_2}) + \omega_0(i_3) = \omega_0(s_2 + i_2 2^{e_2} + i_3 2^{e_3}) \ \ (mod\ 2) \end{aligned}$$

In this way, we can prove that $\omega_0(s_i + m) \neq \omega_0(s_{i+1} + m)$ for any $i = 1, 2, \ldots, k-1$ with $m = i_2 2^{e_2} + i_3 2^{e_3} + \ldots + i_k 2^{e_k}$. □

Now, let us prove $\Omega[\chi^\infty] = \mathcal{P}(\emptyset)$ for any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$. By Theorem 7, it is sufficient to prove that for any $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$, there exists a sequence $(U_k \in \chi^k;\ k = 1, 2, \ldots)$ such that for any $S = (s_1, \ldots, s_k) \in U_k \cap \Delta_k$, $\lambda(S, \omega_0) \geq c_k$ and $c_k \to \infty$ as $k \to \infty$. This holds since

$$U_k := \{(s_1, \ldots, s_k) \in \mathbb{N}^k;\ 20 s_i \leq s_{i+1}\ (i = 1, 2, \ldots, k-1)\} \in \chi^k$$

and for any $S \in U_k \cap \Delta_k$, we have $\lambda(S, \omega_0) = k - 1$ by Lemma 8.

Thus, we have

$$\{\Omega[\chi^\infty];\ \chi \in \beta\mathbb{N} \setminus \mathbb{N}\} = \{\Omega((\chi));\ \chi \in DS(\Omega)\} = \{\mathcal{P}(\emptyset)\}.$$

Take $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ such that $\chi \subset \{U \subset \mathbb{N};\ \rho(U) > 0\}$ (see Example 4). We prove that $\chi \notin DS(\Omega)$. To the contrary suppose that $\chi \in DS(\Omega)$. Then, we must have $\Omega((\chi)) = \mathcal{P}(\emptyset)$. Then for any $k = 1, 2, \ldots$, there exists an injection $\varphi_k : \mathbb{N} \to \mathbb{N}$ with $\rho(\varphi_k(\mathbb{N})) > 0$ such that (1), (2) of Definition 5 hold for $\Theta = \mathcal{P}(\emptyset)$ and that $\varphi_k(\mathbb{N}) \in \chi$ $(k = 1, 2, \ldots)$. Here we take $k = 14$.

Take $K \in \mathbb{N}$ such that $7/2^K \leq \rho(\varphi_{14}(\mathbb{N}))$. Then, there exist infinitely many $N \in \mathbb{N}$ such that $\#(\varphi_{14}(\mathbb{N}) \cap [N2^K, (N+1)2^K)) \geq 7$. Hence, we can take seven elements $u_1, \ldots, u_7 \in \{0, 1, \ldots, 2^K - 1\}$ with $u_1 < \ldots < u_7$ such that $N2^K + \{u_1, \ldots, u_7\} \subset \varphi_{14}(\mathbb{N})$ for infinitely many $N \in \mathbb{N}$. Take U_1 and U_2 with $U_1 < U_2$ as this N. Let

$$n_{ij} = u_i + U_j 2^K\ (i = 1, \ldots, 7 : j = 1, 2).$$

Note that they are distinct and all of them are in $\varphi_{14}(\mathbb{N})$, and hence, we must have

$$\Omega[\{n_{ij};\ i = 1, \ldots, 7 : j = 1, 2\}] = \{0, 1\}^{14} \tag{5.3}$$

by (2) of Definition 5. We prove that this is not true.

We represent $N \in \mathbb{N}$ as $N = N_1 + N_2 2^K$, where $N_1, N_2 \in \mathbb{N}$ and $N_1 < 2^K$. In this case, we denote $N_1 = (N)_1$ and $N_2 = (N)_2$. Then, we have

$$\omega_0(N) = \omega_0((N)_1) + \omega_0((N)_2)\ (mod\ 2).$$

If $(N)_1 + u_4 < 2^K$, then we have

$$\omega_0(N + n_{ij}) = \omega_0((N)_1 + u_i) + \omega_0((N)_2 + U_j)\ (mod\ 2)$$

for any $i = 1, 2, 3, 4$ and $j = 1, 2$. Let

$$\begin{aligned} p(N) &= (\omega_0(N + n_{ij});\ i = 1, 2, 3, 4 : j = 1, 2) \\ q(N) &= (\omega_0((N)_1 + u_i);\ i = 1, 2, 3, 4 : j = 1, 2) \\ r(N) &= (\omega_0((N)_2 + U_j);\ i = 1, 2, 3, 4 : j = 1, 2) \end{aligned}$$

be the 8-dimensional vectors on $\mathbb{Z}/2\mathbb{Z}$. Then,

$$\{q(N);\ N \in \mathbb{N}\} \subset \text{linear space spanned by } \{\mathbf{e}_{i1} + \mathbf{e}_{i2};\ i = 1, 2, 3, 4\},$$

where $\mathbf{e}_{ij}$ is the unit vector corresponding to the suffix (i, j) in the above. Hence, $\dim\{q(N);\ N \in \mathbb{N}\} \leq 4$. Similarly, $\dim\{r(N);\ N \in \mathbb{N}\} \leq 2$. Since $p(N) = q(N) + r(N)$ holds if $(N)_1 + u_4 < 2^K$, we have

$$\begin{aligned}
&\dim\{p(N);\ (N)_1 + u_4 < 2^K\} \\
&\leq \dim\{q(N);\ (N)_1 + u_4 < 2^K\} + \dim\{r(N);\ (N)_1 + u_4 < 2^K\} \\
&\leq 4 + 2 = 6.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\dim\{(\omega_0(N + n_{ij});\ i = 1, 2, \ldots, 7 : j = 1, 2);\ (N)_1 + u_4 < 2^K\} \\
&\leq \dim\{p(N);\ (N)_1 + u_4 < 2^K\} \\
&\qquad + \dim\{(\omega_0(N + n_{ij});\ i = 5, 6, 7 : j = 1, 2);\ (N)_1 + u_4 < 2^K\} \\
&\leq 6 + 6 = 12
\end{aligned}$$

Thus, the cardinality of the set of the vectors

$$\{(\omega_0(N + n_{ij});\ i = 1, 2, \ldots, 7 : j = 1, 2);\ (N)_1 + u_4 < 2^K\}$$

is at most 2^{12}.

If $(N)_1 + u_4 \geq 2^K$, then we have

$$\omega_0(N + n_{ij}) = \omega_0((N)_1 + u_i - 2^K) + \omega_0((N)_2 + U_j + 1)\ (mod\ 2)$$

for any $i = 4, 5, 6, 7$ and $j = 1, 2$. By the same argument as above, we can conclude that the number of vectors $(\omega_0(N + n_{ij});\ i = 1, \ldots, 7 : j = 1, 2)$ corresponding to $N \in \mathbb{N}$ satisfying $(N)_1 + u_4 \geq 2^K$ is at most 2^{12}.

Therefore, the total number of vectors $(\omega_0(N+n_{ij});\ i = 1, \ldots, 7 : j = 1, 2)$ for $N \in \mathbb{N}$ is at most $2^{12} + 2^{12} = 2^{13}$, which contradicts (5.3). Thus, $DS(\Omega) \neq \beta\mathbb{N} \setminus \mathbb{N}$.

Example 7. ([10]) Let $f = (f_1, f_2) \in \mathbb{R}^2$ be an irrational vector, that is, $1, f_1, f_2$ are linearly independent over the rational field. Let $x \mapsto x + f$ be the rotation in $(\mathbb{R}/\mathbb{Z})^2$ by f. For $0 < \delta < 1/4$, let $\mathcal{D}$ be the closed disc with radius δ and center at the origin. For $x \in (\mathbb{R}/\mathbb{Z})^2$, define $\omega_x \in \{0, 1\}^{\mathbb{N}}$ by $\omega_x(n) = 1$ if and only if $x + nf \in \mathcal{D}$. Specially, ω_x for $x = (0, 0)$ is denoted by ω_0. Let $\Omega \subset \{0, 1\}^{\mathbb{N}}$ be the closure of $\{T^n\omega_0;\ n \in \mathbb{N}\}$. Then, we have

$$\{\Omega[\chi^\infty];\ \chi \in \beta\mathbb{N} \setminus \mathbb{N}\} = \{\Omega((\chi));\ \chi \in DS(\Omega)\} = \{\mathcal{P}(101), \mathcal{P}(0101, 1010)\}.$$

These factors are attainable. Moreover, $DS(\Omega) \neq \beta\mathbb{N} \setminus \mathbb{N}$.

Let $X = (\mathbb{R}/\mathbb{Z})^2$ and define an imbedding map $F : \mathbb{N} \to X$ by $F(n) = nf$. Let $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ and $\tilde{\chi} = F(\chi) \in \beta X \setminus X$. Recall the notion of radius $R(\tilde{\chi}^3)$ in Section 4. We prove that

Lemma 9.
(1) *If* $R(\tilde{\chi}^3) \geq \delta$, *then* $\Omega[\chi^\infty] = \mathcal{P}(101)$.
(2) *If* $R(\tilde{\chi}^3) < \delta$, *then* $\Omega[\chi^\infty] = \mathcal{P}(0101, 1010)$.

Proof Let $id_X(\tilde{\chi}) = x_0$ and W be the γ-neighborhood of x_0 for a sufficiently small $\gamma > 0$ (particularly, $\gamma < \delta/2$). Let $\tilde{v}$ be the tangent vector of $\tilde{\chi}$ and for $(n_1 < n_2) \in \mathbb{N}^2$ with $\{F(n_1), F(n_2)\} \subset W$, let $v(n_1, n_2) = \vec{F(n_2)F(n_1)}/||\vec{F(n_2)F(n_1)}||$. Then, either

$$V_2 := \{(n_1 < n_2) \in \mathbb{N}^2;\ \{F(n_1), F(n_2)\} \subset W \text{ and } \det(v(n_1, n_2), \tilde{v}) \leq 0\} \in \chi^2$$

or

$$\{(n_1 < n_2) \in \mathbb{N}^2;\ \{F(n_1), F(n_2)\} \subset W \text{ and } \det(v(n_1, n_2), \tilde{v}) > 0\} \in \chi^2$$

holds. Without loss of generality, we assume the former. That is, the set of $(n_1 < n_2) \in \mathbb{N}^2$ such that the vector $v(n_1, n_2)$ is rotated from the vector $\tilde{v}$ within angle 0 to π in the positive direction is in χ^2. For $k = 4, 5, \ldots$, let

$$V_k = \{(n_1 < n_2 < \ldots < n_k) \in \mathbb{N}^k;\ ||F(n_k) - x_0|| < \ldots < ||F(n_1) - x_0|| < \gamma, \\ \text{and } ||v(n_{k-1}, n_k) - \tilde{v}|| \leq \ldots \leq ||v(n_1, n_2) - \tilde{v}|| < \gamma\}.$$

Then, we have $V_k \in \chi^k$. Let

$$V_3 = \{(n_1 < n_2 < n_3) \in \mathbb{N}^3;\ R(F(n_1), F(n_2), F(n_3)) \geq \delta\}$$
$$V_3' = \{(n_1 < n_2 < n_3) \in \mathbb{N}^3;\ R(F(n_1), F(n_2), F(n_3)) < \delta\}$$

Then, $V_3 \in \chi^3$ if $R(\tilde{\chi}) \geq \delta$ and $V_3' \in \chi^3$ if $R(\tilde{\chi}) < \delta$.

By Lemma 5, for any $k = 4, 5, \ldots$, there exists $U \in \chi^k$ satisfying that $U[S] \subset V_2$ for any $S \subset \{1, 2, \ldots, l\}$ with $\#S = 2$, $U[S] \subset V_3$ for any $S \subset \{1, 2, \ldots, l\}$ with $\#S = 3$ and $U \subset V_k$, if $R(\tilde{\chi}) \geq \delta$. Similarly, there exists $U' \in \chi^k$ satisfying that $U'[S] \subset V_2$ for any $S \subset \{1, 2, \ldots, l\}$ with $\#S = 2$, $U'[S] \subset V_3'$ for any $S \subset \{1, 2, \ldots, l\}$ with $\#S = 3$ and $U' \subset V_k$, if $R(\tilde{\chi}) < \delta$.

Then, the points $F(n_1), F(n_2), \ldots, F(n_k)$ for $(n_1, n_2, \ldots, n_k)$ in U or U' look like Fig. 1. In particular, $F(n_1)F(n_2)\ldots F(n_k)$ is a convex polygon (possibly, degenerate). Choosing γ small enough, we may assume that $3\pi/4 < \angle F(n_i)F(n_{i+1})F(n_{i+2}) \leq \pi$ for any $i = 1, 2, \ldots, k-2$.

Assume that $R(\tilde{\chi}^3) \geq \delta$. Since $U \in \chi^k$, there exists $(n_1 < n_2 < \ldots < n_k) \in U$ such that $\Omega[\chi^k] = \Omega[(n_1, n_2, \ldots, n_k)]$. For any $\omega \in \Omega$, there exists $x \in X$ such that $\omega(n) = \omega_x(n)$ for $n = 0, 1, \ldots, n_k$. Hence,

$$\omega(n_i)\omega(n_j)\omega(n_h) = \kappa(x + F(n_i))\kappa(x + F(n_j))\kappa(x + F(n_h)),$$

where $\kappa : X \to \mathbb{A}$ is such that $\kappa(x) = 1$ if $x \in \mathcal{D}$ and $\kappa(x) = 0$ otherwise.

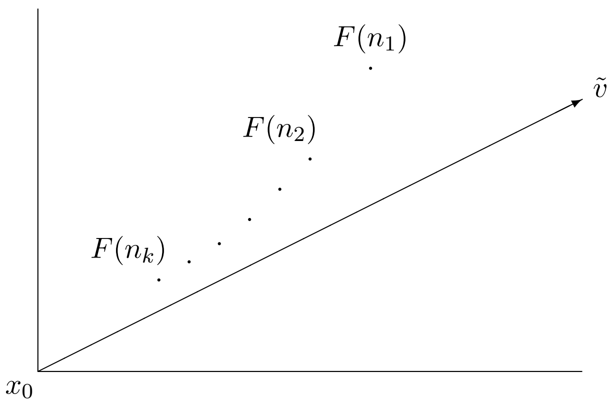

Figure 1: Converging points along $\tilde{\chi}$

We prove that 101 for $\kappa(x+F(n_i))\kappa(x+F(n_j))\kappa(x+F(n_h))$ with $i<j<h$ is impossible. Suppose to the contrary that this holds. Then, we have

$$F(n_i) \in \mathcal{D}-x \ , F(n_j) \notin \mathcal{D}-x,\ F(n_h) \in \mathcal{D}-x$$

together with $3\pi/4 < \angle F(n_i)F(n_j)F(n_h) \leq \pi$. Let the intersection of the straight line passing $F(n_i), F(n_h)$ and the circle $\partial\mathcal{D}-x$ be C and D. Let E be the intersection of the circle $\partial\mathcal{D}-x$ and the line segment $F(n_i)F(n_j)$. Then, we have

$$3\pi/4 < \angle F(n_i)F(n_j)F(n_h) < \angle CED < \pi.$$

Therefore, we have a contradiction that

$$\begin{aligned}\delta = \frac{CD}{2\sin\angle CED} &> \frac{F(n_i)F(n_h)}{2\sin\angle F(n_i)F(n_j)F(n_h)} \\ &= \text{radius of the circle determined by } F(n_i)F(n_j)F(n_h) \geq \delta.\end{aligned}$$

Hence, we have $\Omega[(n_1,n_2,\ldots,n_k)] \subset \mathcal{P}(101)[\{0,1,\ldots,k-1\}]$ for any $k=1,2,\ldots$. Conversely, any $\xi = 0^i1^j0^{k-i-j}$ for some $i,j\in\mathbb{N}$ with $i+j\leq k$ is contained in $\Omega[(n_1,n_2,\ldots,n_k)]$ since $F(n_1)F(n_2)\ldots F(n_k)$ is a convex polygon with diameter less than δ (recall that $\gamma<\delta/2$). Hence,

$$\Omega[\chi^k] = \Omega[(n_1,n_2,\ldots,n_k)] = \mathcal{P}(101)[\{0,1,\ldots,k-1\}]$$

and we have $\Omega[\chi^\infty] = \mathcal{P}(101)$.

Now assume that $R(\tilde{\chi}^3)<\delta$. At first, we prove that there does not exist $\omega\in\Omega$ and $1\leq i<j<h<l\leq k$ such that

$$\omega(n_i)\omega(n_j)\omega(n_h)\omega(n_l) = 0101.$$

Suppose to the contrary that such ω exists. Take $x \in X$ such that $\omega(n) = \omega_x(n)$ for $n = 0, 1, \ldots, n_k$. Hence,

$$\omega(n_i)\omega(n_j)\omega(n_h)\omega(n_l) = \kappa(x+F(n_i))\kappa(x+F(n_j))\kappa(x+F(n_h))\kappa(x+F(n_l)).$$

Then, we have

$$F(n_i) \notin \mathcal{D} - x \ , F(n_j) \in \mathcal{D} - x, \ F(n_h) \notin \mathcal{D} - x, \ F(n_l) \in \mathcal{D} - x.$$

Let A_1, A_2, A_3, A_4 be the intersections of the circle $\partial\mathcal{D} - x$ with the line segments $F(n_i)F(n_j)$, $F(n_j)F(n_h)$, $F(n_h)F(n_l)$ and $F(n_l)F(n_i)$, respectively. Then, $F(n_i)F(n_h) > A_1A_2$. Since A_1, A_2, A_3, A_4 are on a circle, we have $\angle A_1A_2A_3 + \angle A_3A_4A_1 = \pi$. Moreover, since

$$\begin{aligned} \angle A_1A_2A_3 &\geq \pi - (\pi - \angle F(n_i)F(n_j)F(n_h)) - (\pi - \angle F(n_j)F(n_h)F(n_l)) \\ &> \pi - (\pi - 3\pi/4) - (\pi - 3\pi/4) = \pi/2, \end{aligned}$$

we have $\angle A_3A_4A_1 = \pi - \angle A_1A_2A_3 < \pi/2$. On the other hand, since $F(n_j)$ is inside the circle determined by A_1, A_2, A_4, we have $\angle F(n_i)F(n_j)F(n_h) + \angle A_2A_4A_1 \geq \pi$. Hence,

$$0 \leq \pi - \angle F(n_i)F(n_j)F(n_h) \leq \angle A_2A_4A_1 \leq \angle A_3A_4A_1 < \pi/2.$$

Thus, $\sin \angle F(n_i)F(n_j)F(n_h) \leq \sin \angle A_2A_4A_1$, and hence, we have a contradiction that

$$\begin{aligned} \delta = \frac{A_1A_2}{2\sin \angle A_2A_4A_1} &< \frac{F(n_i)F(n_h)}{2\sin \angle F(n_i)F(n_j)F(n_h)} \\ &= \text{radius of the circle determined by } F(n_i), F(n_j), F(n_h) < \delta. \end{aligned}$$

Thus, we have $\omega(n_i)\omega(n_j)\omega(n_h)\omega(n_l) \neq 0101$ for any $\omega \in \Omega$ and $1 \leq i < j < h < l \leq k$. In the same way, we have $\omega(n_i)\omega(n_j)\omega(n_h)\omega(n_l) \neq 1010$ for any $\omega \in \Omega$ and $1 \leq i < j < h < l \leq k$.

Hence, we have $\Omega[(n_1, n_2, \ldots, n_k)] \subset \mathcal{P}(0101, 1010)[\{0, 1, \ldots, k-1\}]$. Note that $\mathcal{P}(0101, 1010)[\{0, 1, \ldots, k-1\}]$ consists of words $\omega \in \{0, 1\}^k$ such that either $\omega = 0^i 1^j 0^{k-i-j}$ for some $i, j \in \mathbb{N}$ with $i + j \leq k$ or $\omega = 1^i 0^j 1^{k-i-j}$ for some $i, j \in \mathbb{N}$ with $i + j \leq k$. We have $0^i 1^j 0^{k-i-j} \in \Omega[(n_1, n_2, \ldots, n_k)]$ since $F(n_1)F(n_2)\ldots F(n_k)$ is a convex polygon with diameter less than δ. Moreover, $1^i 0^j 1^{k-i-j} \in \Omega[(n_1, n_2, \ldots, n_k)]$ holds in our case since if we take $x \in X$ such that the points $F(n_i)$ and $F(n_{i+j+1})$ are on the circle $\partial\mathcal{D} - x$ and the center of the circle is in the same side of $F(n_1)$ or $F(n_k)$ but in the opposite side of $F(n_{i+1})$ separated by the line $F(n_i)F(n_{i+j+1})$, then

$$\{F(n_{i+1}), \ldots, F(n_{i+j})\} \cap (\mathcal{D} - x) = \emptyset$$

and

$$\{F(n_1), \ldots, F(n_i)\} \cup \{F(n_{i+j+1}), \ldots, F(n_k)\} \subset \mathcal{D} - x$$

since $R(\tilde{\chi}^3) < \delta$. Hence, $\Omega[(n_1, n_2, \ldots, n_k)] = \mathcal{P}(0101, 1010)[\{0, 1, \ldots, k-1\}]$. Thus, we have $\Omega[\chi^\infty] = \mathcal{P}(0101, 1010)$, which completes the proof. $\Box$

Thus, we have proved that

$$\{\Omega[\chi^\infty];\ \chi \in \beta\mathbb{N} \setminus \mathbb{N}\} = \{\mathcal{P}(101), \mathcal{P}(0101, 1010)\}.$$

Now, we prove that both factors are attainable.

For $n \in \mathbb{N}$, let $x_n = (c2^{-n},\ d2^{-2n}) \in (\mathbb{R}/\mathbb{Z})^2$ and define an increasing injection $\varphi : \mathbb{N} \to \mathbb{N}$ such that $||\varphi(n)f - x_n|| < \epsilon 2^{-2n}$ $(\forall n \in \mathbb{N})$. Then, it is easy to see that for any $0 < \delta_0 < \delta_1 < \infty$, there exist $c > 0,\ d > 0,\ \epsilon > 0$ such that

$$\delta_0 < R(\varphi(n_1)f, \varphi(n_2)f, \varphi(n_3)f) < \delta_1$$

for any $0 \le n_1 < n_2 < n_3$. Thus, by choosing the constants c, d, ϵ, both $\Omega \circ \varphi = \mathcal{P}(101)$ and $\Omega \circ \varphi = \mathcal{P}(0101, 1010)$ are attainable.

Finally, we prove that $DS(\Omega) \neq \beta\mathbb{N}\setminus\mathbb{N}$. There exists $\chi \in \beta\mathbb{N}\setminus\mathbb{N}$ such that $\chi \subset \{U \subset \mathbb{N};\ \mathrm{acdeg}(Uf) \ge 2\}$. Then, we prove that $\chi \notin DS(\Omega)$. Suppose to the contrary that $\chi \in DS(\Omega)$. Then, we must have $\Omega((\chi)) = \mathcal{P}(101)$ or $\mathcal{P}(0101, 1010)$.

There exist injections $\varphi_k : \mathbb{N} \to \mathbb{N}$ $(k = 1, 2, \ldots)$ with $\mathrm{acdeg}(\varphi_k(\mathbb{N})f) \ge 2$ such that (1), (2) of Definition 5 hold for $\Theta = \mathcal{P}(101)$ or $\mathcal{P}(0101, 1010)$. Here we take $k = 4$. Since there exist infinitely many accumulating points of $\varphi_4(\mathbb{N})f$, take two accumulating points $y, z \in (\mathbb{R}/\mathbb{Z})^2$ of them with $y \neq z$. There exist $x \in (\mathbb{R}/\mathbb{Z})^2$ and $\epsilon > 0$ such that $x + y + U_\epsilon \subset \mathcal{D}$ and $(x + z + U_\epsilon) \cap \mathcal{D} = \emptyset$, where $U_\epsilon = \{u \in (\mathbb{R}/\mathbb{Z})^2;\ ||u|| < \epsilon\}$. Take $n_1 < n_2 < n_3 < n_4$ in $\mathbb{N}$ such that $\varphi_4(n_i)f \in y + U_\epsilon$ for $i = 1, 3$ and $\varphi_4(n_i)f \in z + U_\epsilon$ for $i = 2, 4$. This implies that $\omega_x[\{n_1, n_2, n_3, n_4\}] = 1010$, and

$$\begin{aligned} 1010 &\in \Omega \circ \varphi_4[\{n_1, n_2, n_3, n_4\}] \\ &= \Omega \circ \varphi_4[\{0, 1, 2, 3\}] = \Theta[\{0, 1, 2, 3\}] \\ &= \mathcal{P}(101)[\{0, 1, 2, 3\}] \text{ or } \mathcal{P}(0101, 1010)[\{0, 1, 2, 3\}, \end{aligned}$$

which is a contradiction. Thus, $\chi \notin DS(\Omega)$.

Note that the complexities $p_{\Theta_i}(k)$ for $\Theta_1 = \mathcal{P}(101)$ and $\Theta_2 = \mathcal{P}(0101, 1010)$ satisfy that

$$p_{\Theta_1}(k) = (1/2)k^2 + (1/2)k + 1 \quad (k = 0, 1, 2, \ldots)$$

and

$$p_{\Theta_2}(k) = k^2 - k + 2 \quad (k = 1, 2, \ldots).$$

We called the isomorphic class of Θ_2 a *primitive factor* of Ω in [7] since it attains the maximal pattern complexity.

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(All the above papers except for [1][9] can be downloaded from the site: http://www14.plala.or.jp/kamae)