Hausdorff dimension of the level sets of self-affine functions

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Li PENG* and Teturo KAMAE†

Abstract

Given a partition of $[0,1]$ by closed intervals $I_1 \cup \cdots \cup I_k = [0,1]$ with $k \geq 2$. Given $0 < \alpha < 1$ and closed intervals J_i $(i = 1, \cdots, k)$ with $|I_j| = |J_j|^\alpha$. Given $\tau_1, \cdots, \tau_k \in \{0,1\}$. Let $\Omega \subset [0,1] \times [0,1]$ be the compact set satisfying that

$$\Omega = \cup_{i=1}^k (\varphi_{I_i,1} \times \varphi_{J_i,\tau_i})(\Omega),$$

where for an interval $I = [a,b] \subset [0,1]$ and $\tau \in \{-1,1\}$, $\varphi_{I,\tau} : [0,1] \to I$ is the linear map such that $\varphi_{I,1}(0) = a$, $\varphi_{I,1}(1) = b$ and $\varphi_{I,-1}(0) = b$, $\varphi_{I,-1}(1) = a$. Such Ω is a graph of a Borel function f_Ω almost surely and is called a self-affine set of α-function type. We obtain the Hausdorff dimension of the level set $\Omega^y = \{x;\ (x,y) \in \Omega\}$ in the case that $\lambda \circ f_\Omega^{-1}$ has a bounded density with respect to λ, where λ is the Lebesgue measure on $[0,1]$.

1 Introduction

Let $0 < \alpha < 1$. Assume that

$$(\#1) \qquad \begin{array}{l} k \text{ is an integer with } k \geq 2, \\ 0 = s_0 < s_1 < \cdots < s_k = 1, \text{ and} \\ I_i = [s_{i-1}, s_i] \ (i = 1, \cdots, k) \text{ are intervals.} \end{array}$$

Assume further that

$$(\#2^\alpha) \qquad \begin{array}{l} J_i \ (i = 1, \cdots, k) \text{ are closed intervals in } [0,1] \\ \text{such that } |J_i| = |I_i|^\alpha \text{ and } \tau_i \in \{-1, 1\} \\ (i = 1, \cdots, k), \end{array}$$

*Department of Mathematics Huazhong University of Science and Technology, Wuhan 430074, PR. China (lpeng@hust.edu.cn)

†Advanced Mathematical Institute, Osaka City University, 558-8585 Japan (kamae@apost.plala.or.jp)

where $|I|$ denotes the length $b-a$ of the interval $I=[a,b]$. Let

$$C_0 = \min_{i=1,\cdots,k} |s_i - s_{i-1}| \text{ and } C_1 = \max_{i=1,\cdots,k} |s_i - s_{i-1}|.$$

For an interval $I=[a,b]$ with $0 \le a < b \le 1$ and an *orientation* $\tau \in \{-1,1\}$, define a linear map $\varphi_{I,\tau} : [0,1] \to I$ by

$$\varphi_{I,\tau}(x) = \begin{cases} a(1-x)+bx & (\tau = 1) \\ b(1-x)+ax & (\tau = -1). \end{cases}$$

Therefore, $\varphi_{I,1} \times \varphi_{J,1}$ with intervals $I=[a,b]$, $J=[c,d]$ in $[0,1]$ is the orientation preserving linear bijection $[0,1]^2 \to I \times J$, and $\varphi_{I,1} \times \varphi_{J,-1}$ is the vertically orientation reversing linear bijection $[0,1]^2 \to I \times J$. The former image is denoted with the upward arrow and the latter image is denoted with the downwardward arrow (Figure 1).

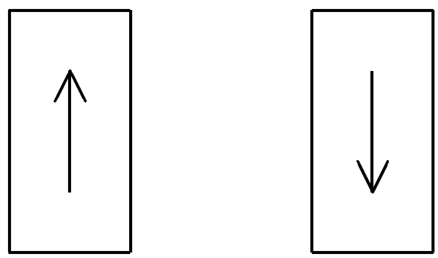

Figure 1: Images of the mappings $\varphi_{I,1} \times \varphi_{J,1}$ (left) and $\varphi_{I,1} \times \varphi_{J,-1}$ (right)

Definition 1. Let $\Omega = \Omega(I_1,\cdots,I_k; J_1,\cdots,J_k; \tau_1,\cdots,\tau_k)$ be the compact set in $[0,1]\times[0,1]$ satisfying the set equation that

$$\Omega = \bigcup_{i=1}^{k} (\varphi_{I_i,1} \times \varphi_{J_i,\tau_i})(\Omega).$$

We call Ω a self-affine set of *α-function type*. Figure 2 gives an example of Ω of $(1/2)$-function type. We fix Ω throughout this paper.

If $\alpha = 1/2$, such Ω is called a self-affine set of Brownian motion type and studied by Xue and Kamae [13]. It is proved in [13] (in the case $\alpha = 1/2$, but the proof is same for the general α) that there exists a Borel function $f_\Omega : [0,1] \to [0,1]$ such that $\Omega_x = \{f_\Omega(x)\}$ except for a countably many $x \in [0,1]$, where $\Omega_x = \{y;\ (x,y) \in \Omega\}$. Let λ be the Lebesgue measure on $[0,1]$. Let λ_Ω be the probability measure on $[0,1]\times[0,1]$ such that $\lambda_\Omega(S) = \lambda(\{x;\ (x,y) \in S \cap \Omega\})$. Then, the marginal measure of λ_Ω to the first coordinate is λ. Let μ_Ω be the marginal measure of λ_Ω to the second coordinate. Then, we have $\mu_\Omega = \lambda \circ f_\Omega^{-1}$ and μ_Ω becomes a self-similar measure satisfying that

$$\mu = \sum_{i=1}^{k} |J_i|^{1/\alpha} \mu \circ \varphi_{J_i,\tau_i}^{-1}.$$

If μ_Ω is absolutely continuous with respect to λ, the density function is denoted by ρ_Ω and is called the *local time* of Ω. In [13], the local time is obtained for some self-affine sets of Brownian motion type by solving the *jigsaw puzzle*.

It is proved by Feng and Hu [5] that if

$$\dim_H \mu_\Omega := \min\{\dim_H S;\ \mu_\Omega(S) = 1\} = \beta,$$

then $\dim_H \lambda_\Omega = 1 + \beta - \alpha\beta$. Hence, we have $\dim_H \Omega \geq \dim_H \lambda_\Omega = 2 - \alpha$ if $\beta = 1$. Moreover, since it is well known ([4], for example) that $\dim_H \Omega \leq 2 - \alpha$, we have $\dim_H \Omega = 2 - \alpha$ if $\beta = 1$, and hence, if Ω has a local time. This result is also obtaine by Przytycki and Urbański [12] in the studies of continuous nowhere differentiable fractal functions in general.

In this paper, we prove that the Hausdorff dimension of the level set Ω^y is $1 - \alpha$ for almost all $y \in [0,1]$ such that $\rho_\Omega(y) > 0$ if Ω has a bounded local time (Theorem 1), where $\Omega^y = \{x;\ (x,y) \in \Omega\}$. Furthermore, we can replace "almost all" by "all" under a mild additional condition (Theorem 2).

If the local time of Ω exists, we always take the version $\rho_\Omega(y)$ such that

$$\rho_\Omega(y) = \liminf_{\epsilon \downarrow 0} \frac{\mu_\Omega([y-\epsilon, y+\epsilon])}{2\epsilon} \tag{1.1}$$

for all $y \in [0,1]$.

For $l = 1, 2, \cdots$ and $i_1 \cdots i_l \in \{1, 2, \cdots, k\}^l$, let

$$\psi_{i_1\cdots i_l} = (\varphi_{I_{i_1},1} \times \varphi_{J_{i_1},\tau_{i_1}}) \circ \cdots \circ (\varphi_{I_{i_l},1} \times \varphi_{J_{i_l},\tau_{i_l}})$$

be the linear mapping from $[0,1] \times [0,1]$ to a rectangle, say $I_{i_1\cdots i_l} \times J_{i_1\cdots i_l}$, which is upward or downward corresponding to whether $\tau_{i_1} \cdots \tau_{i_l}$ is 1 or -1. It holds that

$$\begin{aligned}
\Omega &= \bigcap_{l=1}^{\infty} \bigcup_{i_1\cdots i_l \in \{1,2,\cdots,k\}^l} \psi_{i_1\cdots i_l}(([0,1] \times [0,1]) \\
&= \bigcap_{l=1}^{\infty} \bigcup_{i_1\cdots i_l \in \{1,2,\cdots,k\}^l} I_{i_1\cdots i_l} \times J_{i_1\cdots i_l} \\
&= \bigcap_{l=1}^{\infty} \bigcup_{i_1\cdots i_l \in \{1,2,\cdots,k\}^l} \psi_{i_1\cdots i_l}(\Omega).
\end{aligned} \tag{1.2}$$

We call $\psi_{i_1\cdots i_l}$ a *fundamental mapping* of level l. Moreover, it is easy to see that $|J_{i_1\cdots i_l}| = |I_{i_1\cdots i_l}|^\alpha$. We call the rectangle $I_{i_1\cdots i_l} \times J_{i_1\cdots i_l}$ a *fundamental rectangle* of level l. We denote the orientation $\tau_{i_1}\cdots\tau_{i_l}$ of $J = J_{i_1\cdots i_l}$ by $\tau(J)$. It is clear that there exist k^l+1 points

$$0 = s_0^l < s_1^l < \cdots < s_{k^l}^l = 1$$

such that $I_{i_1\cdots i_l} = [s_{j-1}^l, s_j^l]$, where j is such that $i_1\cdots i_l$ is the j-th element from below in the lexicographical order in $\{1,2,\cdots,k\}^l$. The interval $[s_{j-1}^l, s_j^l]$ for $j = 1,2,\cdots,k^l$ is called the j-th *fundamental interval* of level l.

Let $\mathcal{F}_l$ be the set of fundamental rectangles of Ω of level l and let

$$\mathcal{F}_l^y = \{[a,b]\times[c,d] \in \mathcal{F}_l \ ; \ y \in [c,d]\}.$$

For $S \subset [0,1]$, $0 \le d \le 1$ and $\delta > 0$, let

$$\mathcal{H}_\delta^d(S) = \inf\{ \sum_{i=1}^N |U_i|^d;\ U_i \text{ is an interval with } |U_i| < \delta\ (i = 1,\cdots,N) \\ \text{such that } S \subset \cup_{i=1}^N U_i,\ N \text{ is arbitrary}\}.$$

The above family $\{U_i;\ i = 1,2,\cdots\}$ is called a δ-cover of S. Let $\mathcal{H}^d(S) = \lim_{\delta\to 0}\mathcal{H}_\delta^d(S)$. It is known that $\mathcal{H}^d$ restricted to the Borel field is a measure called the d-dimensional Hausdorff measure. The Hausdorff dimension (Hutchinson [7]) of S is defined as

$$\dim_H S = \inf\{s;\ \mathcal{H}^d(S) < \infty\} = \sup\{d;\ \mathcal{H}^d(S) > 0\}.$$

In this paper, we obtained the Hausdorff dimension of the level set Ω^y. In general, the relation between the Hausdorff dimensions of a self-affine set $E \subset \mathbf{R}^n$ and of its intersection with a hyperplane S has been studied by many authors (Bárány, Ferguson and Simon [1], Manning and Simon [10], for example). In a recent study by Kempton [9], it is proved that if E is a self-similar set with $\dim_H E = s$ such that the similarities do not include rotations and that the orthogonal projection of the s-dimensional Hausdorff measure on E to a line θ is absolutely continuous with a bounded density, then $\mathcal{H}^{s-1}(E\cap S) > 0$ and hence, $\dim_H(E\cap S) \ge s-1$ holds for almost all S perpendicular to θ.

2 Almost all level sets

The following lemma is well known (Falconer [4], for example), but we give the proof just for the self-containedness.

Lemma 1. *$\mathcal{H}^{1-\alpha}(\Omega^y) < \infty$ holds for almost all $y \in [0,1]$.*

Proof Take any $y \in [0,1]$. For an arbitrary $\delta > 0$, take a sufficiently large l such that $C_1^l < \delta$. Since the intervals in $\mathcal{F}_l^y$ cover Ω^y, we have

$$\mathcal{H}_\delta^{1-\alpha}(\Omega^y) \leq \sum_{[a,b]\times[c,d]\in\mathcal{F}_l^y} |b-a|^{1-\alpha}$$

Therefore,

$$\int_0^1 \mathcal{H}_\delta^{1-\alpha}(\Omega^y)dy \leq \int_0^1 \sum_{[a,b]\times[c,d]\in\mathcal{F}_l^y} |b-a|^{1-\alpha}dy$$
$$= \sum_{[a,b]\times[c,d]\in\mathcal{F}_l} |b-a|^{1-\alpha}|d-c| = \sum_{[a,b]\times[c,d]\in\mathcal{F}_l} |b-a| = 1.$$

Letting $\delta \to 0$, we have

$$\int \mathcal{H}^{1-\alpha}(\Omega^y)dy \leq 1,$$

and hence, $\mathcal{H}^{1-\alpha}(\Omega^y) < \infty$ for almost all $y \in [0,1]$. □

Lemma 2. *If Ω has a bounded local time, then $\mathcal{H}^{1-\alpha}(\Omega^y) > 0$ holds for all $y \in [0,1]$ such that $\rho_\Omega(y) > 0$.*

Proof For any $y \in [0,1]$ with $\rho_\Omega(y) > 0$, there exists $\eta_1 > 0$ such that $\mu_\Omega([y-\epsilon, y+\epsilon]) > \epsilon\eta_1$ for any ϵ with $0 < \epsilon < \eta_1$ by (1.1). Take an arbitrary y as this. For any $\delta > 0$ and $\sigma > 0$, take a finite δ-cover $\{H_j \times \{y\};\ j = 1, \cdots, M\}$ of Ω^y such that

$$\sum_{j=1}^M |H_j|^{1-\alpha} < \mathcal{H}_\delta^{1-\alpha}(\Omega^y) + \sigma.$$

We may assume that each H_j is an interval in $[0,1]$. For each H_j, there exist 2 neighboring fundamental intervals, say K_{2j-1}, K_{2j} (one of them may be the empty set) of certain levels such that $K_{2j-1} \cup K_{2j} \supset H_j$ and $|K_{2j-p}| \leq C_0^{-1}|H_j|$ $(p = 0, 1)$. Then, $\{K_j;\ j = 1, \cdots, 2M\}$ is a $C_0^{-1}\delta$-cover of Ω^y such that

$$\sum_{j=1}^{2M} |K_j|^{1-\alpha} \leq 2C_0^{-(1-\alpha)} \sum_{j=1}^M |H_j|^{1-\alpha} < 2C_0^{-(1-\alpha)}(\mathcal{H}_\delta^{1-\alpha}(\Omega^y) + \sigma).$$

If $x \notin \cup_{j=1}^{2M} K_j$, then $x \notin \Omega^y$ since $\Omega^y \subset \cup_{j=1}^{2M} K_j$. Therefore, if $x \in \overline{[0,1] \setminus \cup_{j=1}^{2M} K_j} \cap \Omega^y$, x must be an endpoint of some of K_j. In this case, take a small fundamental interval $K_j{}'$ neighboring to K_j containing x such that $|K_j{}'| \leq |K_j|$. If otherwise, let $K_j{}' = \emptyset$. Then,

$$\sum_{j=1}^{2M} (|K_j|^{1-\alpha} + |K_j{}'|^{1-\alpha}) < 4C_0^{-(1-\alpha)}(\mathcal{H}_\delta^{1-\alpha}(\Omega^y) + \sigma)$$

and

$$\overline{[0,1]\setminus\cup_{j=1}^{2M}(K_j\cup K_j{}')}\cap\Omega^y=\emptyset.$$

Since

$$\overline{[0,1]\setminus\cup_{j=1}^{2M}(K_j\cup K_j{}')}\times\{y\}$$

and Ω are compact sets which are disjoint each other, there exists $\eta_2>0$ such that

$$\overline{[0,1]\setminus\cup_{j=1}^{2M}(K_j\cup K_j{}')}\times[y-\eta_2,y+\eta_2]\cap\Omega=\emptyset.$$

Hence, we have

$$\mu_\Omega([y-\epsilon,y+\epsilon])=\lambda_\Omega(\cup_{j=1}^{2M}(K_j\cup K_j{}')\times[y-\epsilon,y+\epsilon])$$

for any ϵ with $0<\epsilon<\eta_2$. For any $j=1,\cdots,2M$, let $K_j\times L_j$ be the fundamental rectangle corresponding to K_j. Then, $|L_j|=|K_j|^\alpha$ holds. For any $j=1,\cdots,2M$ and any sufficiently small ϵ with compact sets $0<\epsilon<\min\{\eta_1,\eta_2\}$, by (1.2) we have

$$\begin{aligned}&\lambda_\Omega(K_j\times[y-\epsilon,y+\epsilon])=\lambda_\Omega(K_j\times(L_j\cap[y-\epsilon,y+\epsilon]))\\&=|K_j|\lambda_\Omega([0,1]\times\varphi_{L_j,\tau}^{-1}(L_j\cap[y-\epsilon,y+\epsilon])\le|K_j|C_2(2\epsilon/|L_j|)\\&=|K_j|^{1-\alpha}2\epsilon C_2\quad(\tau\in\{1,-1\})\end{aligned}$$

for $j=1,\cdots,2M$, where C_2 is an upper bound of ρ_Ω. The same inequality holds for $K_j{}'$. Hence,

$$\begin{aligned}&4C_0^{-(1-\alpha)}(\mathcal{H}_\delta^{1-\alpha}(\Omega^y)+\sigma)>\sum_{j=1}^{2M}(|K_j|^{1-\alpha}+|K_j{}'|^{1-\alpha})\\&\ge\frac{1}{2\epsilon C_2}\sum_{j=1}^{2M}(\lambda_\Omega(K_j\times[y-\epsilon,y+\epsilon])+\lambda_\Omega(K_j{}'\times[y-\epsilon,y+\epsilon]))\\&\ge\frac{1}{2\epsilon C_2}\lambda_\Omega(\cup_{j=1}^{2M}(K_j\cup K_j{}')\times[y-\epsilon,y+\epsilon])\\&=\frac{1}{2\epsilon C_2}\mu_\Omega([y-\epsilon,y+\epsilon])>\frac{\eta_1}{2C_2}.\end{aligned}$$

Letting $\sigma\to0$, we have

$$4C_0^{-(1-\alpha)}\mathcal{H}_\delta^{1-\alpha}(\Omega^y)\ge\frac{\eta_1}{2C_2}.$$

Thus,

$$\mathcal{H}^{1-\alpha}(\Omega^y)=\lim_{\delta\to0}\mathcal{H}_\delta^{1-\alpha}(\Omega^y)\ge\frac{\eta_1C_0^{1-\alpha}}{8C_2}>0.$$

□

By Lemmas 1 and 2, we have the following Theorem 1.

Theorem 1. *If Ω has a bounded local time, then $\dim_H\Omega^y=1-\alpha$ holds for almost all $y\in[0,1]$ such that $\rho_\Omega(y)>0$. In fact, under these conditions, we have $0<\mathcal{H}^{1-\alpha}(\Omega^y)<\infty$.*

3 All the level sets

Definition 2. We say that Ω is *admissible* if it has a bounded local time ρ_Ω with the property that that there exists $\delta_0 > 0$ such that for any $y \in [0,1]$ with $\#\mathcal{F}_1^y \geq 2$, $\rho_\Omega(y) \geq \delta_0$.

Lemma 3. *Assume that Ω is admissible with the constant $\delta_0 > 0$. Then, for any $y \in [0,1]$, $l = 1, 2, \cdots$ and $[a,b] \times [c,d] \in \mathcal{F}_l^y$ such that $\#\{U \in \mathcal{F}_{l+1}^y;\ U \subset [a,b] \times [c,d]\} \geq 2$, it holds that*

$$\lambda_\Omega([a,b] \times [y-\epsilon, y+\epsilon]) \geq \epsilon\delta_0(b-a)^{1-\alpha}$$

for any sufficient small $\epsilon > 0$.

Proof Since by (1.2), for any sufficiently small $\epsilon > 0$,

$$\lambda_\Omega([a,b] \times [y-\epsilon, y+\epsilon]) = (b-a)\mu_\Omega(\varphi_{[c,d],\tau}^{-1}([y-\epsilon, y+\epsilon]),$$

where $\tau = \tau([c,d])$, we have

$$\lambda_\Omega([a,b] \times [y-\epsilon, y+\epsilon]) = (b-a)\mu_\Omega([y' - (d-c)^{-1}\epsilon,\ y' + (d-c)^{-1}\epsilon]),$$

where $y' = \varphi_{[c,d],\tau}^{-1}(y)$. Since $\#\{U \in \mathcal{F}_{l+1}^y;\ U \subset [a,b] \times [c,d]\} \geq 2$, $\#\mathcal{F}_1^{y'} \geq 2$ holds and hence, $\rho_\Omega(y') \geq \delta_0$ holds by the admissibility. Thus,

$$\lambda_\Omega([a,b] \times [y-\epsilon, y+\epsilon]) \geq (b-a)(d-c)^{-1}\epsilon\delta_0 = (b-a)^{1-\alpha}\epsilon\delta_0$$

for any sufficiently small $\epsilon > 0$ by (1.2). $\square$

Theorem 2. *If Ω is admissible, then $\dim_H \Omega^y = 1 - \alpha$ holds for all $y \in [0,1]$ such that $\rho_\Omega(y) > 0$. In fact, under these conditions, we have $0 < \mathcal{H}^{1-\alpha}(\Omega^y) < \infty$.*

Proof By Lemma 2, it is sufficient to prove that $\mathcal{H}^{1-\alpha}(\Omega^y) < \infty$ for all $y \in [0,1]$ such that $\rho_\Omega(y) > 0$. Take any $y \in [0,1]$ such that $\rho_\Omega(y) > 0$. For an arbitrary $\delta > 0$, take a sufficiently large l such that $C_1^l < \delta$. Let $\mathcal{F}_l^y = \{[a_i, b_i] \times [c_i, d_i];\ i = 1, \cdots, N\}$. Then, $\mathcal{U} := \{[a_i, b_i];\ i = 1, \cdots, N\}$ is a δ-cover of Ω^y.

If $[a,b] \times [c,d] \in \mathcal{F}_l^y$ satisfies that $\{U \in \mathcal{F}_{l+1}^y;\ U \subset [a,b] \times [c,d]\} = \emptyset$, then $\mathcal{U} \setminus \{[a,b]\}$ is still a cover of Ω^y. If $[a,b] \times [c,d] \in \mathcal{F}_l^y$ satisfies that

$$\{U \in \mathcal{F}_{l+1}^y;\ U \subset [a,b] \times [c,d]\} = \{[a',b'] \times [c',d']\},$$

then $(\mathcal{U} \setminus \{[a,b]\}) \cup \{[a',b']\}$ is still a cover of Ω^y. Note that $b' - a' \leq C_1(b-a)$. If again

$$\{U \in \mathcal{F}_{l+2}^y;\ U \subset [a',b'] \times [c',d']\} = \{[a'',b''] \times [c'',d'']\},$$

then $(\mathcal{U} \setminus \{[a,b]\}) \cup \{[a'',b'']\}$ is still a cover of Ω^y. In this case, we have $b'' - a'' \leq C_1^2(b-a)$.

In this way, take the smallest $l' > l$ such that

$$\#\{U \in \mathcal{F}^y_{l'};\ U \subset [a,b] \times [c,d]\} \geq 2.$$

If such l' does not exist, we can remove $[a,b]$ from $\mathcal{U}$ and add possibly a one point set to be still a cover of Ω^y.

Let the element in $\mathcal{F}^y_{l'}$ as above corresponding to $[a_i,b_i]\times[c_i,d_i]$ be $[a'_i,b'_i]\times[c'_i,d'_i]$. Then, $\{[a'_i,b'_i]\times[c'_i,d'_i] \in \mathcal{F}^y_{l_i};\ i=1,\cdots,N\}$ satisfies that
(1) $l_i \geq l$ $(i=1,\cdots,N)$,
(2) $\#\{U \in \mathcal{F}^y_{l_{i+1}};\ U \subset [a'_i,b'_i]\times[c'_i,d'_i]\} \geq 2$ $(i=1,\cdots,N)$, and
(3) $\{[a'_i,b'_i];\ i=1,\cdots,N\}$ covers Ω^y except possibly for a finite set.
Hence, we have

$$\mathcal{H}^{1-\alpha}_\delta(\Omega^y) \leq \sum_{i=1}^N (b'_i - a'_i)^{1-\alpha}. \tag{3.1}$$

Since Ω is admissible with the constant $\delta_0 > 0$, by the above (2),

$$\lambda_\Omega([a'_i,b'_i]\times[y-\epsilon,y+\epsilon]) \geq \epsilon\delta_0(b'_i-a'_i)^{1-\alpha} \quad (i=1,\cdots,N)$$

holds for any sufficiently small $\epsilon > 0$ by Lemma 3, which is common for finitely many $i=1,\cdots,N$. Hence, we have

$$\begin{aligned}\mathcal{H}^{1-\alpha}_\delta(\Omega^y) &\leq \sum_{i=1}^N (b'_i-a'_i)^{1-\alpha} \leq \sum_{i=1}^N \lambda_\Omega([a'_i,b'_i]\times[y-\epsilon,y+\epsilon])/(\epsilon\delta_0) \\ &\leq \mu_\Omega([y-\epsilon,y+\epsilon])/(\epsilon\delta_0) \leq 4\rho_\Omega(y)/\delta_0 \leq 4C_2/\delta_0,\end{aligned}$$

where C_2 is an upper bound of ρ_Ω. Letting $\delta \to 0$, we have $\mathcal{H}^{1-\alpha}(\Omega^y) < \infty$, which completes the proof. □

Corollary 1. *If Ω has a bounded local time such that $\inf_{y\in[0,1]}\rho_\Omega(y) > 0$, then $\dim_H \Omega^y = 1-\alpha$ holds for all $y \in [0,1]$.*

Proof It is clear that Ω is admissible, if Ω has a bounded local time such that $\inf_{y\in[0,1]}\rho_\Omega(y) > 0$. Hence, by Theorem 2, we have Corollary 1. □

4 Examples

Example 1. Let $\alpha = 1/2$ and

$$\Omega = \Omega([0,4/9],[4/9,5/9],[5/9,1];[0,2/3],[1/3,2/3],[1/3,1];\tau_1,\tau_2,\tau_3),$$

where $\tau_i \in \{-1,1\}$ $(i=1,2,3)$ are arbitrary. By [13], Ω has the following bounded local time.

$$\rho_\Omega(x) = \begin{cases} 4x & 0 \leq x \leq 1/2 \\ 4-4x & 1/2 < x \leq 1 \end{cases}.$$

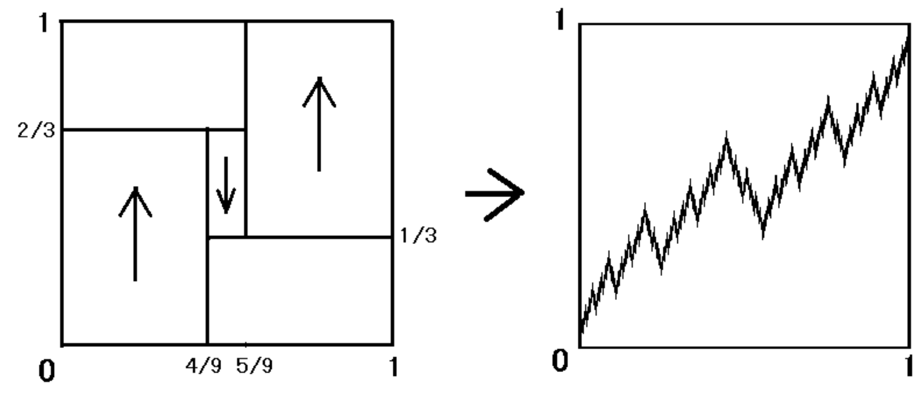

Figure 2: $\Omega([0,4/9],[4/9,5/9],[5/9,1];[0,2/3],[1/3,2/3],[1/3,1];1,-1,1)$

The condition for $y \in [0,1]$ that $\#\mathcal{F}_1^y \geq 2$ is equivalent to $1/3 \leq y \leq 2/3$, and $\rho_\Omega(y) \geq 4/3$ holds for any $1/3 \leq y \leq 2/3$. Hence, Ω is admissible with the constant $4/3$. Thus by Theorem 3, $\dim_H \Omega^y = 1/2$ holds for any $y \in (0,1)$. If $\tau_1 = 1$, $\tau_2 = -1$, $\tau_3 = 1$, then Ω is a graph of a continuous function as is seen in Figure 2. It is a sample path of so called *deterministic Brownian motion* and studied by Kamae [8].

Example 2. Let $\alpha = 1/2$ and

$$\Omega = \Omega([0,1/2],[1/2,1];[0,\sqrt{2}/2],[(2-\sqrt{2})/2,1];\tau_1,\tau_2),$$

where $\tau_i \in \{-1,1\}$ $(i=1,2)$ are arbitrary. Then, it is known ([13], originally by Dai, Feng and Wang [2]) that

$$\rho_\Omega(x) = \begin{cases} \frac{4+3\sqrt{2}}{2}\,x & 0 \leq x \leq \sqrt{2}-1 \\ \frac{2+\sqrt{2}}{2} & \sqrt{2}-1 < x \leq 2-\sqrt{2} \\ \frac{4+3\sqrt{2}}{2}\,(1-x) & 2-\sqrt{2} < x \leq 1 \end{cases}.$$

Hence, Ω is admissible with the constant $\delta_0 = \frac{1+\sqrt{2}}{2}$. Thus by Theorem 3, $\dim_H \Omega^y = 1/2$ holds for any $y \in (0,1)$.

Example 3. Let $\alpha = 1/2$ and

$$J_1 = [0,1/2],\ J_2 = J_3 = J_4 = [1/2,1];\ \tau_1 = \tau_2 = 1,\ \tau_3 = -1,\ \tau_4 = 1.$$

Then, it is known ([13])that Ω has the local time ρ_Ω such that

$$\rho_\Omega(x) = \begin{cases} 2x & (0 \leq x < 1) \\ 1 & (x = 1) \end{cases}$$

Hence, Ω is admissible with the constant $\delta_0 = 1$. Thus by Theorem 3, $\dim_H \Omega^y = 1/2$ holds for any $y \in (0,1]$.

Example 4. Let $\alpha = 1/2$,

$$J_1 = [0, 1/2],\ J_2 = [1/4, 3/4],\ J_3 = J_4 = [1/2, 1]$$

and $\tau_1 = -1,\ \tau_2 = \tau_3 = \tau_4 = 1$, then it is known ([13]) that

$$\rho_\Omega(x) = \begin{cases} 1/3 & (x = 0) \\ 2/3 & (0 < x < 1/2) \\ 1 & (x = 1/2) \\ 4/3 & (1/2 < x < 1) \\ 2/3 & (x = 1) \end{cases} .$$

Hence, by Corollary 1, $\dim_H \Omega^y = 1/2$ holds for any $y \in [0, 1]$.

Example 5. Let $\alpha = 1/2$ and

$$J_1 = [0, 1/2],\ J_2 = J_3 = J_4 = [1/2, 1];\ \tau_1 = \tau_2 = \tau_3 = \tau_4 = 1.$$

That is, the same as Example 3 except for τ_3. Then, μ_Ω is singular ([13]). In this case, $\dim_H \Omega^y$ $(y \in [0, 1])$ takes all values between 0 and $\log 3/\log 4$ since Ω^y is a Moran set (Feng, Wen and Wu [6]) with $c_1 = c_2 = \cdots = 1/4$ and $n_i = 1$ or 3 according to $y_i = 0$ or $y_i = 1$ $(i = 1, 2, \cdots)$, where $y = \sum_{i=1}^{\infty} y_i 2^{-i}$, and is proved in [6] that

$$\dim_H \Omega^y = \liminf_{k\to\infty} \frac{\log(n_1 n_2 \cdots n_k)}{-\log(c_1 c_2 \cdots c_k)} = \frac{\log 3}{\log 4} \liminf_{k\to\infty} (1/k) \sum_{i=1}^{k} y_i.$$

It is also proved by McMullen [11] that $\dim_H \Omega = \log(1 + \sqrt{3})/\log 2$.

Example 6. Let $\alpha = \log 2/\log k$ with even $k \geq 4$ and

$$J_i = [0, 1/2] \text{ for } k/2 \text{ number of } i, \text{ and}$$
$$J_i = [1/2, 1] \text{ for } k/2 \text{ number of } i.$$

Then, for any τ_i $(i = 1, \cdots, k)$, ρ_Ω exists and satisfies that

$$\rho_\Omega(x) = \begin{cases} 1/2 & (x = 0) \\ 1 & (0 < x \leq 1) \\ 1/2 & (x = 1) \end{cases} .$$

Hence, by Corollary 1, $\dim_H \Omega^y = 1 - (\log 2/\log k)$ holds for any $y \in [0, 1]$. It is proved in [11] that $\dim_H \Omega = 2 - (\log 2/\log k)$.

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References

[1] Balázs Bárány, Andrew Ferguson, and Károly Simon, Slicing the Sierpinski gasket. Nonlin-earity 25-6 (2012), pp. 1753-1770.

[2] Xin-Rong Dai, De-Jun Feng and Yang Wang, Classification of refinable splines, Constr. Approx. 24 (2006), pp. 187-200.

[3] K. J. Falconer, The dimension of self-affine fractals II, Math. Proc. Camb. Phil. Soc. 111 (1992), pp. 169-179.

[4] K. J. Falconer, Fractal geometry, John-Wiley & Sons, Inc., Hoboken, NJ, second edition, 2003

[5] De-Jun Feng and Huyi Hu, Dimension Theory of Iterated Function Systems, Communications on Pure and Applied Mathematics, Vol. LXII (2009), pp. 1435-1500.

[6] Dejun Feng, Zhiying Wen and Jun Wu, Some dimensional results for homogeneous Moran sets, Science in China 40-5 (1997), pp. 475-482.

[7] J. E. Hutchinson, Fractal and self-similarity, Indiana Univ. Math. J. 30 (1981), pp. 713-747.

[8] T. Kamae, Stochastic analysis based on deterministic Brownian motion, Israel J. Math. 125 (2001), pp. 317-346.

[9] Tom Kempton, Sets of beta-expansions and the hausdorff measure of slices through fractals (available at http://arxiv.org/abs/1307.2091).

[10] Anthony Manning and Károly Simon, Dimension of slices through the Sierpinski carpet, Trans. Amer. Math. Soc. 365-1 (2013), pp. 213-250.

[11] C. McMullen, The Hausdorff dimension of general Sierpiński carpets. Nagoya Math. J. 96 (1984), pp. 1-9.

[12] F. Przytycki and M. Urbański, On the Hausdorff dimension of some fractal sets, Studia Math. 93-2 (1989), pp. 155-186.

[13] Yu-Mei Xue and Teturo Kamae, Local time of self-affine sets of Brownian motion type and the jigsaw puzzle problem, J. Mathematical Analysis and Applications 419 (2014), pp. 79-93.