Multifractal analysis for a class of homogeneous Moran constructions

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Abstract

We consider the homogeneous Moran sets $\mathcal{M}([0,1],\ \{n_k\},\ \{c_k\})$ with increasing spacing such that $\lim_{k\to\infty} \log n_{k+1}/\log n_k = t > 1$. We discuss the multifractal properties of them.

1 Introduction

Let $n_k \geq 2$ be integers and c_k be positive numbers satisfying that $0 < c_k n_k < 1$ $(k = 1, 2, \cdots)$. Let d_k^i $(i = 1, 2, \cdots, n_k;\ k = 1, 2, \cdots)$ be nonnegative numbers such that $d_k^i \geq c_k$ $(i = 0, 1, \cdots, n_k - 1)$ and

$$d_k^1 + d_k^2 + \cdots + d_k^{n_k} \leq 1.$$

Let $D_k = \prod_{i=1}^k \{1, 2, \cdots n_i\}$ and $D = \cup_{k=0}^\infty D_k$, where an element in D_k is denoted by a finite sequence $\sigma_1\sigma_2\cdots\sigma_k$ of $\sigma_i \in \{1, 2, \cdots, n_i\}$ $(i = 1, 2, \cdots, k)$ and D_0 consists of the empty sequence $\emptyset$.

Let $\mathbb{J}_\emptyset$ be a nondegenerate bounded closed interval in $\mathbb{R}$ and define closed intervals $\mathbb{J}_\sigma \subset \mathbb{J}_\emptyset$ for $\sigma \in D$ inductively. Let $\sigma = \sigma' i \in D_k$ with $\sigma' \in D_{k-1}$ and $i \in \{1, 2, \cdots, n_k\}$. Assume that $\mathbb{J}_{\sigma'} = [u, v]$ with $[u, v] \subset \mathbb{J}_\emptyset$ and $v - u = |\mathbb{J}_\emptyset| c_1 \cdots c_{k-1}$ is already defined. Then, define $\mathbb{J}_{\sigma' i}$ as

$$\begin{gathered}[u + (d_k^1 + \cdots + d_k^i - c_k)(v - u)\ ,\ u + (d_k^1 + \cdots + d_k^i)(v - u)] \\ (i = 1, 2, \cdots, n_k) \qquad (1.1) \\ \text{(see Figure 1).}\end{gathered}$$

Let

$$E = \bigcap_{k=0}^\infty \bigcup_{\sigma \in D_k} \mathbb{J}_\sigma,$$

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which we call the *homogeneous Moran set with structure* $(\mathbb{J}_\emptyset, \{n_k\}, \{c_k\}, \{d_k^i\})$ and is denoted by $\mathcal{C}(\mathbb{J}_\emptyset, \{n_k\}, \{c_k\}, \{d_k^i\})$. Each interval $\mathbb{J}_\sigma$ for $\sigma \in D_k$ is called a *basic interval* of level k. Most case, we take $\mathbb{J}_\emptyset = [0,1]$ and denote

$$N_k = n_1 n_2 \cdots n_k \ , \ \delta_k = c_1 c_2 \cdots c_k \quad (k = 1, 2, \cdots).$$

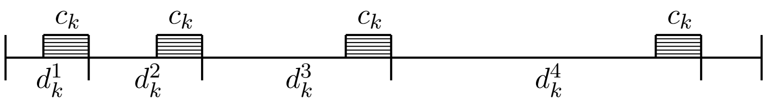

Figure 1: subintervals $\mathbb{J}_{\sigma' i}$ for $\sigma' \in D_{k-1}$ and $i = 1, 2, 3, 4$

If

$$d_k^i = (i-1)d_k' + c_k \quad (i = 1, 2, \cdots, n_k; \ k = 1, 2, \cdots)$$

with $d_k' = \dfrac{1 - c_k}{n_k - 1}$, then $\mathcal{C}(\mathbb{J}_\emptyset, \{n_k\}, \{c_k\}, \{d_k^i\})$ is called a *homogeneous Cantor set* which is denoted by $\mathcal{C}^*(\mathbb{J}_\emptyset, \{n_k\}, \{c_k\})$. On the other hand, if

$$d_k^i = i c_k \quad (i = 1, 2, \cdots, n_k; \ k = 1, 2, \cdots),$$

then $\mathcal{C}(\mathbb{J}_\emptyset, \{n_k\}, \{c_k\}, \{d_k^i\})$ is called a *partial homogeneous Cantor set* which is denoted by $\mathcal{C}_*(\mathbb{J}_\emptyset, \{n_k\}, \{c_k\})$.

It is known by Dejun Feng, Zhiying Wen and Jun Wu [3] that

Theorem 1. [3] *The Hausdorff dimensions of a homogeneous Cantor set and a partial homogeneous Cantor set are obtained as follows:*

$$\dim_H \mathcal{C}^*([0,1], \{n_k\}, \{c_k\}) = \liminf_{k \to \infty} \frac{\log N_k}{-\log \delta_k} =: s^*$$

$$\dim_H \mathcal{C}_*([0,1], \{n_k\}, \{c_k\}) = \liminf_{k \to \infty} \frac{\log N_k}{-\log(\delta_k c_{k+1} n_{k+1})} =: s_* \ .$$

For a general homogeneous Moran set $E = \mathcal{C}([0,1], \{n_k\}, \{c_k\}, \{d_k^i\})$, *we have*

$$s_* \le \dim_H E \le s^*.$$

Moreover, for any s *with* $s_* \le s \le s^*$, *there exists* $E = \mathcal{C}([0,1], \{n_k\}, \{c_k\}, \{d_k^i\})$ *such that* $\dim_H E = s$.

Definition 1. Let $E = \mathcal{C}([0,1], \{n_k\}, \{c_k\}, \{d_k^i\})$ and I_E be the minimal closed interval containing E. Let $\mathcal{C}^* = \mathcal{C}^*([0,1], \{n_k\}, \{c_k\})$ be the homogeneous Cantor set. For $\sigma \in D$, let $\mathbb{J}_\sigma^E$ and $\mathbb{J}_\sigma^{\mathcal{C}^*}$ be the intervals defined in (1.1) for E and $\mathcal{C}^*$, respectively. A continuous increasing mapping $G : I_E \to [0,1]$ satisfying
(1) for any $\sigma \in D$, $G(\mathbb{J}_\sigma^E \cap E) = \mathbb{J}_\sigma^{\mathcal{C}^*} \cap \mathcal{C}^*$, and
(2) G restricted to any connected component of $I_E \setminus E$ is linear,
is determined, which we call the *canonical mapping* of E and is denoted by G_E.

The above function G_E was introduced by Hao Li, Qin Wang and Lifeng Xi [5]. It is also known in [5] that

Theorem 2. [5] *If $\{n_k\}$ is bounded in k, and $s^* = \lim_{k\to\infty} \frac{\log N_k}{-\log \delta_k}$ exists and satisfies $0 < s^* < 1$, then for any $\{d_k^i\}$ $(i = 1, 2, \cdots, n_k;\ k = 1, 2, \cdots)$, $E = \mathcal{C}([0,1], \{n_k\}, \{c_k\}, \{d_k^i\})$ satisfies that $dim_H E = s^*$ and the canonical mapping of E is quasi-Lipschitz. That is,*

$$C_1(y-x)^{1+\epsilon} < G_E(y) - G_E(x) < C_2(y-x)^{1-\epsilon}$$

holds for any $x < y$ in I_E, where $\epsilon > 0$ is arbitrary and $C_1,\ C_2$ are positive constants depending on ϵ.

In this paper, we are interested in multifractal structures attached to Moran sets. To get nontrivial multifractal structures, $\{n_k\}$ should increase very fast and the values in $\{d_k^i;\ i = 1, \cdots, n_k\}$ should have big deviations. Under these assumptions together with the monotonicity of d_k^i in i, we obtained the local dimension of the Moran set E and the local Hölder exponent of the function G_E.

We assume that

$$(*1) \begin{cases} d_k^1 \le d_k^2 \le d_k^2 \le \cdots \le d_k^{n_k} \text{ and } d_k^2 \ge 2c_k\ (k = 1, 2, \cdots), \\ \text{the limit } t := \lim_{k\to\infty} \frac{\log n_{k+1}}{\log n_k} \text{ exists and } t > 1, \text{ and} \\ \text{the limit } s^* := \lim_{k\to\infty} \frac{\log N_k}{-\log \delta_k} \text{ exists and } 0 < s^* < 1. \end{cases}$$

In this setting, we introduce natural parameters a, b to describe the local Hölder continuity degree of G_E and the local Hausdorff dimension of E.

Definition 2. Let $E = \mathcal{C}([0,1], \{n_k\}, \{c_k\}, \{d_k^i\})$ satisfying $(*1)$ be given. For $k = 1, 2, \cdots$ and $i = 1, 2, \cdots, n_k$, define $a \in [0,1]$ and $b \in \mathbb{R}$ by

$$i - 1 = n_k - n_k^{1-a} \quad \text{and} \quad \delta_{k-1}^{t-b} = (d_k^1 + \cdots + d_k^i)\delta_{k-1}. \tag{1.2}$$

We denote a and b related like this with some k and i by $a(i,k)$ and $b(i,k)$, respectively. Since both of $a(i,k)$ and $b(i,k)$ are strictly increasing in i for

any fixed k, there exists a function $f_k : [0,1] \to \mathbb{R}$ such that
(1) $f_k(a(i,k)) = b(i,k)$ for any $i = 1, 2, \cdots, n_k,\ k = 1, 2, \cdots,$
(2) f_k is a strictly increasing continuous function for any $k = 1, 2, \cdots$.
We call such a function f_k a *pre-spacing function.*

We also always assume that

$$(*2) \begin{cases} \text{there exists a strictly increasing continuous function} \\ f : [0,1] \to \mathbb{R} \text{ such that } f_k \text{ converges to } f \text{ as } k \to \infty \text{ on } [0,1]. \end{cases}$$

This function f is determined by the Moran structure $([0,1], \{n_k\}, \{c_k\}, \{d_k^i\})$, which we call the *spacing function.*

Definition 3. For $x \in E$ and $k = 1, 2, \cdots$, let σ be the unique element in D_k such that $x \in \mathbb{J}_\sigma$. We define

$$\Pi_k(x) = \pi_1(x)\pi_2(x)\cdots\pi_k(x) := \sigma.$$

Definition 4. For $x \in E$, let $\underline{a}(x) = \liminf_{k\to\infty} a(\pi_k(x), k)$. We call it the *deviation index* of x in E.

We prove the following theorems.

Theorem 3. *The local Hölder continuity degree of G_E at $x \in E$ is equal to $1/(t - f(\underline{a}(x)))$. That is,*

$$\liminf_{y\to x,\ y\in E} \log|G_E(y) - G_E(x)|/\log|y - x| = 1/(t - f(\underline{a}(x))).$$

Theorem 4. *For any $a_0 \in [0,1]$, we have*

$$\begin{aligned} \dim_H\{x \in E;\ \underline{a}(x) = a_0\} &= \lim_{\epsilon\to 0} \dim_H\{x \in E;\ \underline{a}(x) \in (a_0 - \epsilon,\ a_0 + \epsilon)\} \\ &= \frac{1 - a_0}{t - f(a_0)}\ s^*. \end{aligned}$$

This value is called the local dimension of E at deviation index a_0.

Corollary 1. *It holds that*

$$\dim_H E = \sup_{a\in[0,1]} \frac{1 - a}{t - f(a)}\ s^*.$$

This Corollary generalizes Theorem 1 in [9].

2 Preliminary Lemmas

The following fact is well known since G_E maps E onto $\mathcal{C}^*$.

Fact 1. *If there exists* $\epsilon > 0$ *and* $C > 0$ *such that*

$$G_E(y) - G_E(x) < C(y-x)^{1-\epsilon}$$

for any $x, y \in E$ *with* $x < y$*, then* $\dim_H E \geq (1-\epsilon)s^*$.

Lemma 1. *It holds that*
(1) $\lim_{k\to\infty} \frac{\log n_k}{-\log \delta_{k-1}} = (t-1)s^*$ *and* $\lim_{k\to\infty} \frac{\log c_k}{\log \delta_{k-1}} = t-1$,
(2) $f(1) \leq t-1$ *and* $f(0) \geq (t-1)s^*$.

Proof (1) Since

$$\lim_{k\to\infty} \frac{\log n_k}{\log N_{k-1}} = \lim_{k\to\infty} \frac{\log n_k}{\sum_{i=0}^{k-1} \log n_{k-i}} = \frac{1}{t^{-1} + t^{-2} + \cdots} = t-1$$

and $\lim_{k\to\infty} \frac{\log N_{k-1}}{-\log \delta_{k-1}} = s^*$, we have $\lim_{k\to\infty} \frac{\log n_k}{-\log \delta_{k-1}} = (t-1)s^*$.

Since

$$\begin{aligned}
\lim_{k\to\infty} \frac{\log \delta_k}{\log \delta_{k-1}} &= \lim_{k\to\infty} \frac{-\log \delta_k / \log N_k}{-\log \delta_{k-1} / \log N_{k-1}} \frac{\log N_k}{\log N_{k-1}} \\
&= \frac{1/s^*}{1/s^*} \frac{1 + t^{-1} + t^{-2} + \cdots}{t^{-1} + t^{-2} + \cdots} = t,
\end{aligned}$$

we have

$$\lim_{k\to\infty} \frac{\log c_k}{\log \delta_{k-1}} = \lim_{k\to\infty} \frac{\log \delta_k - \log \delta_{k-1}}{\log \delta_{k-1}} = t-1.$$

(2) By the definition, $a_k(n_k, k) = 1$ $(k = 1, 2, \cdots)$. On the other hand, since $\delta_{k-1}^{t-b(n_k,k)} \leq \delta_{k-1}$, we have $b(n_k, k) \leq t-1$. Therefore, $f_k(1) \leq t-1$ $(k = 1, 2, \cdots)$, and hence, $f(1) \leq t-1$.

For any small $\epsilon > 0$, define $i_0 \in \{1, 2, \cdots, n_k\}$ by $i_0 - 1 = \lfloor n_k - n_k^{1-\epsilon} \rfloor$. Then, we have

$$\delta_{k-1}^{t-b(i_0,k)} = \sum_{i=1}^{i_0} d_k^i \delta_{k-1} \geq i_0 c_k \delta_{k-1}.$$

Therefore,

$$\begin{aligned}
f(0) &= \lim_{\epsilon\to 0} \lim_{k\to\infty} b(i_0, k) = \lim_{\epsilon\to 0} \lim_{k\to\infty} \left(t - \frac{\log \sum_{i=1}^{i_0} d_k^i \delta_{k-1}}{\log \delta_{k-1}} \right) \\
&\geq \lim_{\epsilon\to 0} \lim_{k\to\infty} \left(t - \frac{\log i_0 c_k \delta_{k-1}}{\log \delta_{k-1}} \right) = \lim_{\epsilon\to 0} \lim_{k\to\infty} \left(t - \frac{\log (n_k - n_k^{1-\epsilon}) c_k \delta_{k-1}}{\log \delta_{k-1}} \right) \\
&= \lim_{k\to\infty} \left(t - \frac{\log n_k c_k \delta_{k-1}}{\log \delta_{k-1}} \right) = t - \lim_{k\to\infty} \frac{\log n_k + \log c_k + \log \delta_{k-1}}{\log \delta_{k-1}} \\
&= t - ((t-1)s^* + t - 1 + 1) = (t-1)s^*.
\end{aligned}$$

□

The following lemma in a weaker sense is used in [8, 9].

Lemma 2. *Assume* $(*1)$*. Assume that there exist* k_0 *and* $\epsilon > 0$ *such that for any* $k \geq k_0$ *and* $i_1, i_2 = 1, 2, \cdots, n_k$ *with* $i_1 < i_2$*, it holds that*

$$\frac{\log(\delta_{k-1}(i_2 - i_1)/n_k)}{\log(\delta_{k-1}\sum_{i=i_1+1}^{i_2} d_k^i)} > 1 - \epsilon.$$

Then, we have $\dim_H E \geq s^*(1-\epsilon)$.

Proof For $x, y \in E$ such that $x < y$ and $y - x$ is sufficiently small, there exist $\sigma \in D_{k-1}$ and $i_1, i_2 = 1, 2, \cdots, n_k$ with $k \geq k_0$ and $i_1 < i_2$ such that $x \in \mathbb{J}_{\sigma i_1}$ and $y \in \mathbb{J}_{\sigma i_2}$. Since $d_k^i \geq 2c_k$ $(i = 2, 3, \cdots, n_k)$, we have

$$(1/2) \sum_{i=i_1+1}^{i_2} d_k^i \leq \sum_{i=i_1+1}^{i_2} d_k^i - c_k \leq \frac{y-x}{\delta_{k-1}} \leq \sum_{i=i_1+1}^{i_2} d_k^i + c_k \leq 2 \sum_{i=i_1+1}^{i_2} d_k^i.$$

On the other hand, with $d_k' = \dfrac{1-c_k}{n_k - 1}$ we have

$$\frac{G_E(y) - G_E(x)}{\delta_{k-1}} \leq (i_2 - i_1)d_k' + c_k \leq 2(i_2 - i_1)d_k' \leq 4(i_2 - i_1)/n_k.$$

Hence, we have

$$\begin{aligned} &G_E(y) - G_E(x) \leq 4\delta_{k-1}(i_2 - i_1)/n_k \\ &= 4\delta_{k-1} \sum_{i=i_1+1}^{i_2} d_k^i \, \frac{\delta_{k-1}(i_2 - i_1)/n_k}{\delta_{k-1}\sum_{i=i_1+1}^{i_2} d_k^i} \leq 8(y-x) \, \frac{\delta_{k-1}(i_2 - i_1)/n_k}{\delta_{k-1}\sum_{i=i_1+1}^{i_2} d_k^i}. \end{aligned}$$

Since

$$\begin{aligned} &\log \, \frac{\delta_{k-1}(i_2 - i_1)/n_k}{\delta_{k-1}\sum_{i=i_1+1}^{i_2} d_k^i} = \log(\delta_{k-1}(i_2 - i_1)/n_k) - \log(\delta_{k-1} \sum_{i=i_1+1}^{i_2} d_k^i) \\ &= \log(\delta_{k-1} \sum_{i=i_1+1}^{i_2} d_k^i) \left(\frac{\log(\delta_{k-1}(i_2 - i_1)/n_k)}{\log(\delta_{k-1}\sum_{i=i_1+1}^{i_2} d_k^i)} - 1 \right) \\ &< \log(\delta_{k-1} \sum_{i=i_1+1}^{i_2} d_k^i)((1-\epsilon) - 1) \leq \log((y-x)/2)(-\epsilon) \end{aligned}$$

we have

$$\frac{\delta_{k-1}(i_2 - i_1)/n_k}{\delta_{k-1}\sum_{i=i_1+1}^{i_2} d_k^i} < C'(y-x)^{-\epsilon}$$

with some constant C'. Hence, we have

$$G_E(y) - G_E(x) < C''(y-x)^{1-\epsilon},$$

which completes the proof by Fact 1. □

3 Proofs of main results

Proof of Theorem 3

Take $x \in E$. Take an arbitrary $y \in E$ with $x \neq y$ which is sufficiently close to x. Then, there exist $k = 1, 2, \cdots$ such that $\pi_j(x) = \pi_j(y)$ for $j = 1, \cdots, k-1$ and $\pi_k(x) \neq \pi_k(y)$. Denote this k by $k_0(y)$.

Take any subsequence of $\{y\} \subset E$ converging to x such that

$$L(y) := \lim_{y \to x} \log |G_E(y) - G_E(x)| / \log |y - x|$$

exists. We may take a further subsequence of $\{y\}$ such that

$$a_x := \lim_{y \to x} a(\pi_{k_0(y)}(x),\ k_0(y)) \text{ and } a_y := \lim_{y \to x} a(\pi_{k_0(y)}(y),\ k_0(y))$$

exist.

Let $i_x = \pi_{k_0(y)}(x)$ and $i_y = \pi_{k_0(y)}(y)$. We denote $i_1 = \min\{i_x, i_y\}$, $i_2 = \max\{i_x, i_y\}$ and $k = k_0(y)$. By the same argument as in the proof of Lemma 2, we can deduce that

$$(1/2)\delta_{k-1} \sum_{i=i_1+1}^{i_2} d_k^i \leq |y - x| \leq 2\delta_{k-1} \sum_{i=i_1+1}^{i_2} d_k^i$$

and

$$(1/4)\delta_{k-1}(i_2 - i_1)/n_k \leq |G_E(y) - G_E(x)| \leq 4\delta_{k-1}(i_2 - i_1)/n_k.$$

If $a_x < a_y$, then we have

$$\delta_{k-1} \sum_{i=i_1+1}^{i_2} d_k^i = \delta_{k-1}^{t-f(a_y)+o(1)} - \delta_{k-1}^{t-f(a_x)+o(1)} = \delta_{k-1}^{t-f(a_y)+o(1)}$$

as $y \to x$. In the same way, if $a_x > a_y$, then we have

$$\delta_{k-1} \sum_{i=i_1+1}^{i_2} d_k^i = \delta^{t-f(a_x)+o(1)} - \delta^{t-f(a_y)+o(1)} = \delta^{t-f(a_x)+o(1)}$$

as $y \to x$.

On the other hand, if $a_x < a_y$, then we have

$$\delta_{k-1}(i_2 - i_1)/n_k = \delta_{k-1}(n_k^{1-a_x+o(1)} - n_k^{1-a_y+o(1)})/n_k = \delta_{k-1} n_k^{-a_x+o(1)}$$

as $y \to x$. If $a_x > a_y$, then we have

$$\delta_{k-1}(i_2 - i_1)/n_k = \delta_{k-1}(n_k^{1-a_y+o(1)} - n_k^{1-a_x+o(1)})/n_k = \delta_{k-1} n_k^{-a_y+o(1)}$$

as $y \to x$.

Therefore, if $a_x < a_y$, then we have

$$\begin{aligned}&\lim_{y\to x} \log|G_E(y) - G_E(x)|/\log|y-x| \\ = &\lim_{y\to x} \frac{\log \delta_{k-1} - a_x \log n_k}{(t - f(a_y))\log \delta_{k-1}} = \frac{1 + a_x(t-1)s^*}{t - f(a_y)} ,\end{aligned}$$

and if $a_x > a_y$, then we have

$$\begin{aligned}&\lim_{y\to x} \log|G_E(y) - G_E(x)|/\log|y-x| \\ = &\lim_{y\to x} \frac{\log \delta_{k-1} - a_y \log n_k}{(t - f(a_x))\log \delta_{k-1}} = \frac{1 + a_y(t-1)s^*}{t - f(a_x)} .\end{aligned}$$

Therefore, the infimum value of them is $1/(t-f(a_x))$ and it is attained when $0 = a_y < a_x$. Moreover, since $a_x \geq \underline{a}(x)$, the infimum value of

$$\lim_{y\to x} \log|G_E(y) - G_E(x)|/\log|y-x|$$

taken when $a_x \neq a_y$ is $1/(t - f(\underline{a}(x)))$.

To complete the proof, we compare this value with the possible values taken when $a_x = a_y$. Let $a_x = a_y$. Since d_k^i is nondecreasing in i, we have

$$2|y-x| \geq \delta_{k-1} \sum_{i=i_1+1}^{i_2} d_k^i \geq \delta_{k-1}^{t-b(i_2,k)} \frac{i_2 - i_1}{i_2} \geq \delta_{k-1}^{t-b(i_2,k)} \frac{i_2 - i_1}{n_k}$$

and

$$(1/4)|G_E(y) - G_E(x)| \leq \delta_{k-1} \frac{i_2 - i_1}{n_k} .$$

Since $b(i_2, k) \to f(a_x) = f(a_y)$ as $y \to x$, we have

$$\frac{\log|G_E(y) - G_E(x)|}{\log|y-x|} \geq \frac{\log \delta_{k-1} + o(1) + \theta}{(t - f(a_x) + o(1))\log \delta_{k-1} + \theta}$$

as $y \to x$, where $\theta = \log((i_2 - i_1)/n_k) \leq 0$. Therefore,

$$\lim_{y\to x} \frac{\log|G_E(y) - G_E(x)|}{\log|y-x|} \geq \min\left\{\frac{1}{t - f(a_x)},\ 1\right\} \geq \frac{1}{t - f(\underline{a}(x))}.$$

□

Proof of Theorem 4

Take arbitrary a_1, a_2 with $0 \leq a_1 < a_2 \leq 1$. For $j = 0, 1, 2, \cdots$ and $\sigma \in D_j$ $(j = 1, 2, \cdots)$, let

$$\begin{aligned}E^{\sigma}_{a_1,a_2} = \{&x \in E;\ \Pi_j(x) = \sigma,\ \underline{a}(x) < a_2 \\ &\text{and } a(\pi_k(x), k) \geq a_1 \text{ for any } k = j+1, j+2, \cdots\}.\end{aligned}$$

For any $x \in E^\sigma_{a_1,a_2}$ and $k_0 > j$, let $k(x, k_0)$ be the minimum $k \geq k_0$ such that $a(\pi_k(x), k) \in [a_1, a_2)$. Then, we have

$$E^\sigma_{a_1,a_2} = \bigcup_{k=k_0}^{\infty} \{x \in E^\sigma_{a_1,a_2};\ k(x, k_0) = k\}.$$

Moreover, $\{x \in E^\sigma_{a_1,a_2};\ k(x, k_0) = k\}$ is covered by $(n_{j+1}n_{j+2}\cdots n_{k-1})^{1-a_1}$ number of intervals of length $\delta_{k-1}^{t-f_k(a_2)} - \delta_{k-1}^{t-f_k(a_1)}$ (both with negligible errors).

Take any β and $\eta > 0$ such that $\beta > \dfrac{1-a_1}{t-f(a_2)}(s^* + 2\eta)$. Then, there exists $k_0 > j$ such that $N_{k-1} \leq \delta_{k-1}^{-s^*-\eta}$ and $\beta > \dfrac{1-a_1}{t-f_k(a_2)}(s^* + 2\eta)$ for any $k \geq k_0$. Then, we have

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} (\delta_{k-1}^{t-f_k(a_2)} - \delta_{k-1}^{t-f_k(a_1)})^\beta (n_{j+1}n_{j+2}\cdots n_{k-1})^{1-a_1} \\
&\leq \sum_{k=k_0}^{\infty} \delta_{k-1}^{(t-f_k(a_2))\beta} N_{k-1}^{1-a_1} \leq \sum_{k=k_0}^{\infty} \delta_{k-1}^{(t-f_k(a_2))\beta} \delta_{k-1}^{-(1-a_1)(s^*+\eta)} \\
&= \sum_{k=k_0}^{\infty} \delta_{k-1}^{(t-f_k(a_2))(\beta - \frac{1-a_1}{t-f_k(a_2)}(s^*+\eta))} \leq \sum_{k=k_0}^{\infty} \delta_{k-1}^{(1-a_1)\eta} \to 0
\end{aligned}$$

as $k_0 \to \infty$. Hence, $\dim_H E^\sigma_{a_1,a_2} \leq \dfrac{1-a_1}{t-f(a_2)} s^*$ for any $\sigma \in D$. Since

$$\{x \in E;\ \underline{a}(x) \in (a_1,\ a_2)\} \subset \cup_{\sigma \in D} E^\sigma_{a_1,a_2},$$

we have

$$\dim_H \{x \in E;\ \underline{a}(x) \in (a_1,\ a_2)\} \leq \frac{1-a_1}{t-f(a_2)}\ s^*. \tag{3.1}$$

Therefore,

$$\lim_{\epsilon \to 0} \dim_H \{x \in E;\ \underline{a}(x) \in (a_0 - \epsilon,\ a_0 + \epsilon)\} \leq \frac{1-a_0}{t-f(a_0)}\ s^*$$

holds for any $a_0 \in (0, 1)$.

Let us prove this inequality for $a_0 = 0$ and $a_0 = 1$.

Let $a_0 = 0$. Then for any $a_2 \in (0, 1)$, by the same argument to deduce (3.1), we have

$$\{x \in E;\ \underline{a}(x) \in [0,\ a_2)\} \subset \cup_{\sigma \in D} E^\sigma_{0,a_2},$$

so that

$$\dim_H \{x \in E;\ \underline{a}(x) \in [0,\ a_2)\} \leq \frac{1}{t-f(a_2)}\ s^*,$$

and hence,

$$\lim_{\epsilon\to 0}\dim_H\{x\in E;\ \underline{a}(x)\in[0,\epsilon)\}\le\frac{1}{t-f(0)}\ s^*.$$

Let $a_0=1$. Then for any $a_1\in(0,1)$, by the same argument to deduce (3.1), we have

$$\{x\in E;\ \underline{a}(x)\in(a_1,\ 1]\}\subset\cup_{\sigma\in D}\overline{E}^{\sigma}_{a_1}$$

and

$$\dim_H\{x\in E;\ \underline{a}(x)\in(a_1,\ 1]\}\le\frac{1-a_1}{t-f(1)}\ s^*,$$

where

$$\overline{E}^{\sigma}_{a_1}=\{x\in E;\ \Pi_j(x)=\sigma,\ a(\pi_k(x),k)\ge a_1 \text{ for any } k=j+1,j+2,\cdots\}.$$

Hence, we have

$$\lim_{\epsilon\to 0}\dim_H\{x\in E;\ \underline{a}(x)\in(1-\epsilon,\ 1]\}=0.$$

Together with these results, we have

$$\lim_{\epsilon\to 0}\dim_H\{x\in E;\ \underline{a}(x)\in(a_0-\epsilon,\ a_0+\epsilon)\}\le\frac{1-a_0}{t-f(a_0)}\ s^* \tag{3.2}$$

for any $a_0\in[0,1]$.

To complete the proof, it is sufficient to prove

$$\dim_H\{x\in E;\ \underline{a}(x)=a_0\}\ge\frac{1-a_0}{t-f(a_0)}\ s^* \tag{3.3}$$

for any $a_0\in[0,1]$.

For $a_0=1$, (3.3) is trivial.

Let us prove (3.3) for $a_0\in(0,1)$. Take a sufficiently large k_0. For any $k\ge k_0$, let u_k,v_k satisfies that $0\le u_k\le a_0\le v_k\le 1$ and

$$(1/3)n_k^{1-a_0}<\#\{i;\ u_k\le a(i,k)\le v_k\}<(1/2)n_k^{1-a_0}. \tag{3.4}$$

Then, we have $\lim_{k\to\infty}u_k=\lim_{k\to\infty}v_k=a_0$ since otherwise, there exists $\delta>0$ such that there exists an arbitrary large k with the property that either

$$\#\{i;\ u_k\le a(i,k)\le v_k\}\ge n_k^{1-a_0}-n_k^{1-a_0-\delta}$$

or

$$\#\{i;\ u_k\le a(i,k)\le v_k\}\ge n_k^{1-a_0+\delta}-n_k^{1-a_0},$$

which contradicts (3.2) as $\lim_{k\to\infty}n_k=\infty$. Take any $\sigma\in D^{k_0}$ and define

$$H^{\sigma}_{k_0}=\{x\in E\cap\mathbb{J}_\sigma;\ u_k\le a(\pi_k(x),k)\le v_k\ (\forall k=k_0+1,k_0+2,\cdots)\}.$$

Then, we have $H^{\sigma}_{k_0} \subset \{x \in E;\ \underline{a}(x) = a_0\}$.

For $j = 1, 2, \cdots$, define m_j and n'_j by

$$\{m_j + i;\ i = 1, 2, \cdots, n'_j\} = \{\pi_{k_0+j}(x);\ x \in H^{\sigma}_{k_0}\}.$$

Let $[U, V]$ be a basic interval of E of level $k_0 + j$. Let U' be the right end points of the m_j-th basic intervals of level $k_0 + j + 1$ contained in $[U, V]$ if $m_j \geq 1$ and $U' = U$ if $m_j = 0$. Let V' be the right end points of the $(m_j + n'_j)$-th basic intervals of level $k_0 + j + 1$ contained in $[U, V]$. We replace each $[U, V]$ by $[U', V']$ and construct basic intervals of $H^{\sigma}_{k_0}$ of level j. We define $p_j = U' - U$, $q_j = V' - U$ and $\delta'_j = q_j - p_j$ for $j = 0, 1, 2, \cdots$. Let $c'_j = \delta'_j / \delta'_{j-1}$ for $j = 1, 2, \cdots$. Finally, define

$$d^{i\,\prime}_j = d^{m_j+i}_{k_0+j} \delta_{k_0+j-1} / \delta'_{j-1} \quad (i = 1, 2, \cdots, n'_j). \tag{3.5}$$

Then, it can be easily verified that

$$H^{\sigma}_{k_0} = \mathcal{C}(\mathbb{J}'_{\sigma}, \{n'_j\}, \{c'_j\}, \{d^{i\,\prime}_j\}),$$

where $\mathbb{J}'_{\sigma} = [U + p_0, U + q_0]$ if U is the left endpoint of $\mathbb{J}_{\sigma}$.

Denoting $k = k_0 + j$, we have $(1/3) n_k^{1-a_0} < n'_j < (1/2) n_k^{1-a_0}$, and hence,

$$N'_j = n'_1 \cdots n'_j = N_k^{1-a_0+o(1)} \quad (\text{as } j \to \infty).$$

Let $b_0 = f(a_0)$. Take a, a' which are sufficiently close to a_0 with $0 < a < a' < a_0$. Let $b = f(a)$ and $b' = f(a')$. Then, for any sufficiently large k, the average distance between the neighbouring basic intervals of E of level k with the index between u_k and v_k is not less than the same value with the index between a and a'. Hence we have

$$\begin{aligned}\delta'_j = q_j - p_j &\geq (\delta_k^{t-b'+o(1)} - \delta_k^{t-b+o(1)}) \frac{n_k^{1-a_0+o(1)}}{n_k^{1-a+o(1)} - n_k^{1-a'+o(1)}} \\ &= \delta_k^{t-b'+o(1)} n_k^{a-a_0+o(1)} \quad (\text{as } j \to \infty).\end{aligned}$$

In the same way, for any c, c' with $a_0 < c < c' < 1$ and $d' = f(c')$ we have

$$\delta'_j \leq \delta_k^{t-d'+o(1)} n_k^{c-a_0+o(1)} \quad (\text{as } j \to \infty).$$

Since $n_k = \delta_k^{-(1-(1/t))s^*+o(1)}$, for any $\epsilon > 0$, there exists b', d' with

$$b_0 - \epsilon < b' < b_0 < d' < b_0 + \epsilon$$

such that

$$\delta_k^{t-b'+o(1)} \leq \delta'_j \leq \delta_k^{t-d'+o(1)} \quad (\text{as } j \to \infty).$$

Since

$$\begin{aligned}\frac{1-a_0}{t-b'}\ s^* &= \frac{1-a_0}{t-b'}\lim_{k\to\infty}\frac{\log N_k}{-\log\delta_k} = \lim_{j\to\infty}\frac{\log N_j'}{-\log\delta_k^{t-b'}}\\ &\le \dim_H \mathcal{C}^*(\mathbb{J}_\sigma', \{n_j'\}, \{c_j'\}) = \lim_{j\to\infty}\frac{\log N_j'}{-\log\delta_j'}\\ &\le \lim_{j\to\infty}\frac{\log N_j'}{-\log\delta_k^{t-d'}} = \frac{1-a_0}{t-d'}\lim_{k\to\infty}\frac{\log N_k}{-\log\delta_k} = \frac{1-a_0}{t-d'}\ s^*,\end{aligned}$$

and $\epsilon > 0$ is arbitrary, we have

$$\dim_H \mathcal{C}^*(\mathbb{J}_\sigma', \{n_j'\}, \{c_j'\}) = \frac{1-a_0}{t-b_0}\ s^*.$$

Take any $j = 1, 2, \cdots$ and $i_1, i_2 = 1, 2, \cdots, n_j'$ such that $i_1 < i_2$. Take $0 < a < a' < a_0$ close enough to a_0. Let $b = f(a)$ and $b' = f(a')$. Then, since d_k^i is nondecreasing in i, we have

$$\delta_{k-1} d_j^{m_j+i} \ge \frac{\delta_{k-1}^{t-b'+o(1)} - \delta_{k-1}^{t-b+o(1)}}{n_k^{1-a+o(1)} - n_k^{1-a'+o(1)}} = \delta_{k-1}^{t-b'+o(1)} n_k^{a-1+o(1)}$$

for any $i = 1, 2, \cdots, n_j'$ as $k \to \infty$, and hence

$$\delta_{j-1}' \sum_{i=i_1+1}^{i_2} {d_j^i}' = \delta_{k-1} \sum_{i=m_j+i_1+1}^{m_j+i_2} d_k^i \ge \delta_{k-1}^{t-b'+o(1)} n_k^{a-1+o(1)} (i_2 - i_1).$$

On the other hand, we have

$$\delta_{j-1}'(i_2 - i_1)/n_j' = \delta_{k-1}^{t-d'+o(1)} n_k^{a_0-1}(i_2 - i_1).$$

Hence,

$$\begin{aligned}&\frac{\log(\delta_{j-1}'(i_2-i_1)/n_j')}{\log(\delta_{j-1}'\sum_{i_1+1}^{i_2} {d_j^i}')}\\ &\ge \frac{(t-d'+o(1))\log\delta_{k-1} + \theta}{(t-b'+o(1))\log\delta_{k-1} + (a-a_0)\log n_k + \theta}\\ &\ge \min\left\{\frac{t-d'+o(1)}{t-b'+(a-a_0)s^*(t-1)+o(1)},\ 1\right\}\end{aligned}$$

as $k \to \infty$, where $\theta = \log(n^{a_0-1}(i_2 - i_1)) \le 0$. Therefore, for any $\epsilon > 0$, by taking a, a', c, c' sufficiently close to a_0, there exists j_0 such that

$$\frac{\log(\delta_{j-1}'(i_2-i_1)/n_j')}{\log(\delta_{j-1}'\sum_{i_1+1}^{i_2} {d_j^i}')} > 1 - \epsilon$$

for any $j \geq j_0$. Hence by Lemma 2, we have

$$\dim_H H_{k_0}^{\sigma} \geq \frac{1-a_0}{t-f(a_0)} s^*(1-\epsilon).$$

Thus, we have (3.3) for any $a_0 \in (0,1)$ since $\epsilon > 0$ is arbitrary.

Finally, we prove that

$$\dim_H \{x \in E;\ \underline{a}(x) = 0\} \geq \frac{1}{t-f(0)}\ s^*. \tag{3.6}$$

We define $H_{k_0}^{\sigma}$ exactly in the same way as above with $a_0 = 0$ and $u_k = 0$, and hence $m_j = 0$. We define $\mathbb{J}'_{\sigma}, \{n'_j\}, \{c'_j\}, \{{d_j^i}'\}$ in the same way as above. Then, we have

$$H_{k_0}^{\sigma} = \mathcal{C}(\mathbb{J}'_{\sigma}, \{n'_j\}, \{c'_j\}, \{{d_j^i}'\}).$$

We can also prove

$$\dim_H \mathcal{C}^*(\mathbb{J}'_{\sigma}, \{n'_j\}, \{c'_j\}) = \frac{1}{t-f(0)}\ s^*, \tag{3.7}$$

since $N'_j = N_k^{1+o(1)}$ and $\delta'_j = \delta_k^{t-f(0)+o(1)}$ with $k = k_0 + j$ as $j \to \infty$.

Take any $1 \leq i_1 < i_2 \leq n'_j$. By (3.5), we have

$${d_j^1}'\delta'_{j-1} = d_k^1 \delta_{k-1} = \delta_{k-1}^{t-f(0)+o(1)} \text{ and } \delta'_{j-1} = \delta_{k-1}^{t-f(0)+o(1)}.$$

Since

$${d_j^1}'\delta'_{j-1} \leq \delta'_{j-1} \sum_{i=i_1+1}^{i_2} {d_j^i}' \leq \delta'_{j-1}$$

and

$${d_j^1}'\delta'_{j-1} \leq \delta'_{j-1}(i_2 - i_1)/n'_j \leq \delta'_{j-1},$$

we have

$$\frac{\log(\delta'_{j-1}(i_2-i_1)/n'_j)}{\log(\delta'_{j-1}\sum_{i=i_1+1}^{i_2} {d_j^i}')} = \frac{(t-f(0)+o(1))\log\delta_{k-1}}{(t-f(0)+o(1))\log\delta_{k-1}} = 1 + o(1)$$

as $j \to \infty$. Hence for any $\epsilon > 0$, there exists j_0 such that

$$\frac{\log(\delta'_{j-1}(i_2-i_1)/n'_j)}{\log(\delta'_{j-1}\sum_{i_1+1}^{i_2} {d_j^i}')} > 1 - \epsilon$$

for any $j \geq j_0$. Therefore by (3.7) and Lemma 2, we have

$$\dim_H H_{k_0}^{\sigma} \geq \frac{1}{t-f(a_0)} s^*(1-\epsilon).$$

Since $\epsilon > 0$ is arbitrary, we have (3.6). □

Proof of Corollary 1

By Theorem 4,

$$\dim_H E \geq \sup_{a\in[0,1]} \frac{1-a}{t-f(a)}\ s^*$$

holds clearly. We prove the opposite inequality. Take any $\eta > 0$. By Theorem 3, for any $a \in [0,1]$, there exists $\epsilon_a > 0$ such that

$$\dim_H\{x \in E;\ a-\epsilon_a < \underline{a}(x) < a+\epsilon_a\} < \frac{1-a}{t-f(a)}\ s^* + \eta.$$

Since E is a compact set, there exists a finite covering of $[0,1]$ consisted of intervals of the form $(a-\epsilon_a,\ a+\epsilon_a)$. It follows that

$$\dim_H E < \sup_{a\in[0,1]} \frac{1-a}{t-f(a)}\ s^* + \eta.$$

Since $\eta > 0$ is arbitrary, we have

$$\dim_H E \leq \sup_{a\in[0,1]} \frac{1-a}{t-f(a)}\ s^*,$$

which completes the proof. □

The following examples were essentially discussed in [9].

Example 1. Let real numbers $\lambda,\ t,\ s$ with $\lambda > 1,\ t > 1,\ 0 < s < 1$ be given. Let k_0 be a sufficiently large integer. Define

$$n_k = \lfloor \lambda^{t^{k_0+k}} \rfloor,\ c_k = \lambda^{-(1/s)t^{k_0+k}} \quad (k = 1, 2, \cdots).$$

Let p be a real number such that $0 < p < (1-s)(t-1)$. Define d_k^i $(i = 1, 2, \cdots, n^k;\ k = 1, 2, \cdots)$ by

$$d_k^1 + d_k^2 + \cdots + d_k^i = (n_k - i + 1)^{-\frac{p}{s(t-1)}}. \tag{3.8}$$

Then, we have a homogeneous Moran set $\mathcal{C}(\mathbb{J}_\emptyset, \{n_k\}, \{c_k\}, \{d_k^i\})$ satisfying the conditions $(*1)$ and $(*2)$ with this t, $s^* = s$ and the spacing function $f(a) = pa + t - 1 - p$.

To prove this, let $a_i = a(i,k),\ b_i = b(i,k)$ $(i = 1, 2, \cdots, n_k)$ for an arbitrary $k = 1, 2, \cdots$. Since $i-1 = n_k - n_k^{1-a_i}$, we have $1 - a_i = \dfrac{\log(n_k - i + 1)}{\log n_k}$. Moreover, by (3.8),

$$\delta_{k-1}(n_k - i + 1)^{-\frac{p}{s(t-1)}} = \delta_{k-1}^{t-b_i}.$$

Hence,

$$t - 1 - b_i = -\frac{p}{s(t-1)} \frac{\log(n_k - i + 1)}{\log \delta_{k-1}} = \frac{p}{s(t-1)} \frac{(1-a_i)\log n_k}{-\log \delta_{k-1}}$$

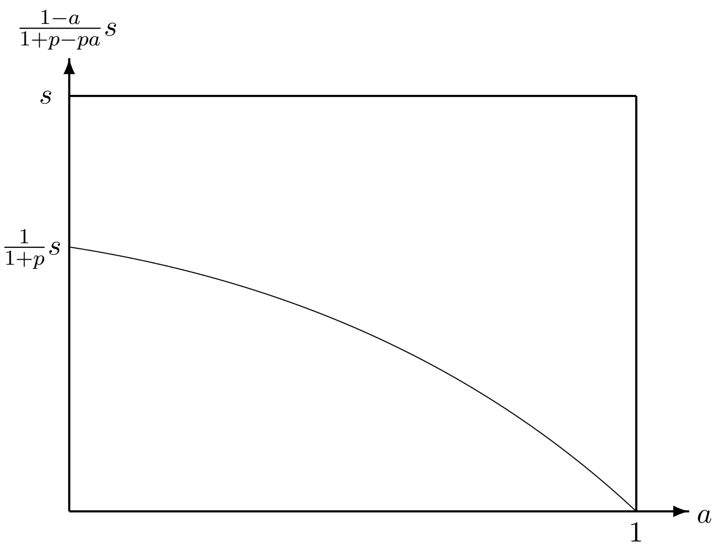

Figure 2: local dimension of E at a

Then, the following f_k becomes a pre-spacing function:

$$f_k(a) = t - 1 - \frac{p}{s(t-1)} \frac{(1-a)\log n_k}{-\log \delta_{k-1}} \quad (a \in [0,1]).$$

Since $\dfrac{\log n_k}{-s(t-1)\log \delta_{k-1}}$ converges to 1 as $k \to \infty$ uniformly in a, the spacing function f is determined as the limit of f_k so that

$$f(a) = t - 1 - p(1-a) = pa + t - 1 - p.$$

In this case, we have

$$\dim_H E = \sup_{a \in [0,1]} \frac{1-a}{t - f(a)}\ s = \sup_{a \in [0,1]} \frac{1-a}{1+p-pa}\ s = \frac{1}{1+p}\ s,$$

where "sup" is attained at $a = 0$. Figure 2 is the graph of $\dfrac{1-a}{1+p-pa}\ s$.

Example 2. We consider the same setting as Example 1, except for $\{d_k^i\}$. Let $w,\ p$ be real numbers such that $w > 1,\ 0 < p < (1-s)(t-1)$. Define d_k^i $(i = 1, 2, \cdots, n^k;\ k = 1, 2, \cdots)$ by

$$d_k^1 + d_k^2 + \cdots + d_k^i = \delta_{k-1}^{p(\log(n_k - i + 1)/\log n_k)^w}. \tag{3.9}$$

Then, we have a homogeneous Moran set $\mathcal{C}(\mathbb{J}_\emptyset, \{n_k\}, \{c_k\}, \{d_k^i\})$ satisfying the conditions $(*1)$ and $(*2)$ with this t, $s^* = s$ and the spacing function $f(a) = p(1-a)^w + t - 1$.

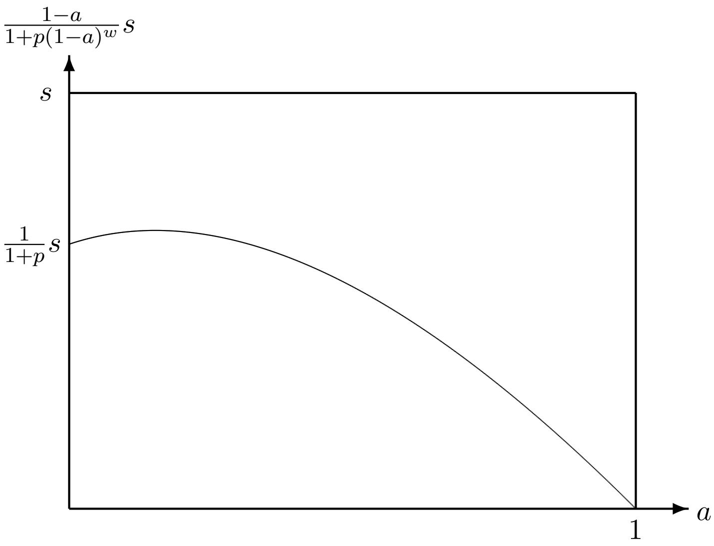

Figure 3: local dimension of E at a in the case $p(w-1) > 1$

To prove this, let $a_i = a(i,k),\ b_i = b(i,k)\ (i = 1, 2, \cdots, n_k)$ for an arbitrary $k = 1, 2, \cdots$. Since $1 - a_i = \dfrac{\log(n_k - i + 1)}{\log n_k}$, by (3.9), we have

$$\delta_{k-1}^{1+p(\log(n_k-i+1)/\log n_k)^w} = \delta_{k-1}^{t-b_i}.$$

Hence,

$$t - 1 - b_i = p\left(\frac{\log(n_k - i + 1)}{\log \delta_{k-1}}\right)^w = p(1 - a_i)^w$$

Then, the following f_k becomes a pre-spacing function:

$$f_k(a) = t - 1 - p(1-a)^w \quad (a \in [0, 1]).$$

Hence, $f(a) = t - 1 - p(1-a)^w$ is the spacing function.

In this case, we have

$$\begin{aligned}\dim_H E &= \sup_{a\in[0,1]} \frac{1-a}{t - f(a)} s = \sup_{a\in[0,1]} \frac{1-a}{1 + p(1-a)^w} s \\ &= \begin{cases} s/(1+p) & \text{if } p(w-1) \le 1 \\ \dfrac{w-1}{w}\left(\dfrac{1}{p(w-1)}\right)^{1/w} s & \text{if } p(w-1) > 1 \end{cases},\end{aligned}$$

where "sup" is attained at $a = 0$ in the former case and at $a = 1 - (p(w - 1))^{-1/w}$ in the latter case. Figure 3 is the graph of $\dfrac{1-a}{1+p(1-a)^w}s$ in the case of $p(w-1) > 1$.

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