Société Mathématique de France
Astérisque 49 (1977) p.109-114

MUTUAL SINGULARITY OF SPECTRA OF DYNAMICAL SYSTEMS GIVEN BY "SUMS OF DIGITS" TO DIFFERENT BASES

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0. Summary

In [3] , it was proved that if $(p,q) = 1$ and a and b are irrational numbers, then the following two arithmetic functions α and β have mutually singular spectral measures :

$$\alpha(n) = \exp(2\pi i\, a\, s_p(n)), \quad \beta(n) = \exp(2\pi i\, b\, s_q(n)) \qquad (n \in N) ,$$

where $s_p(n)$ $(s_q(n))$ is the sum of digits in the p-adic (q-adic) representation of n . Here we prove a slightly stronger result that the two shift dynamical systems corresponding to the strictly ergodic sequences α and β have mutually singular spectral measures. That is to say that for any $f \in L_2(\mu_\alpha)$ and $g \in L_2(\mu_\beta)$ such that $\int f d\mu_\alpha = \int g d\mu_\beta = 0$, where μ_α and μ_β are the measures on $\mathbb{T}^{\mathbb{N}}$ ($\mathbb{T}$ being the unit circle in the complex plane) for which α and β are generic with respect to the shift, respectively, the spectral measures $\Lambda_{\alpha,f}$ and $\Lambda_{\beta,g}$ are mutually singular, where $\Lambda_{\alpha,f}(\Lambda_{\beta,g})$ is the measure Λ on $\mathbb{R}/\mathbb{Z}$ determined by the relation
$(T^n f, f)_{\mu_\alpha} = \int e^{2\pi i\lambda n}\, d\Lambda(\lambda)$ $((T^n g, g)_{\mu_\beta} = \int e^{2\pi i\lambda n}\, d\Lambda(\lambda))$ for all $n \in \mathbb{N}$

(T denoting the shift as well as the isometry on L_2 induced by the shift).

1. Mutual singularity of spectra and disjointness

Given two dynamical systems $X = (X,\mu,S)$ and $Y = (Y,\nu,T)$. We consider, in the obvious way, $L_2(\mu)$ and $L_2(\nu)$ as subspaces of $L_2(\mu\times\nu)$. For $f \in L_2(\mu\times\nu)$, $H(f)$ denotes the closed subspace of $L_2(\mu\times\nu)$ spanned by f, $(S\times T) f$, $(S\times T)^2 f,\ldots$. The following theorem is essentielly due to A.N. Kolmogorov.

Theorem A.

X and Y have mutually singular spectral measures if and only if

(1) X and Y are disjoint in the sense of H. Furstenberg, and

(2) for any $f \in L_2(\mu)$ and $g \in L_2(\nu)$ such that
$\int f d\mu = \int g d\nu = 0$, $f \in H(f+g)$.

Proof :

We prove only that the mutual singularity of spectra implies the disjointness, since the other parts follows easily from [4]. Assume that X and Y are not disjoint. Then there exists a probability measure $\xi \neq \mu \times \nu$ on $X \times Y$ which is $S \times T$-invariant and satisfies that $\xi|_X = \mu$ and $\lambda|_Y = \nu$. Take $f \in L_2(\mu)$ and $g \in L_2(\nu)$ such that $\int f d\mu = \int g d\nu = 0$ and $(f,g)_\xi \neq 0$. Since

$$\frac{1}{N}\left|\left|\sum_1^N e^{-2\pi i\lambda n} S^n f\right|\right|_\mu^2 d\lambda \to \Lambda_{x,f}$$

$$\frac{1}{N}\left|\left|\sum_1^N e^{-2\pi i\lambda n} T^n g\right|\right|_\nu^2 d\lambda \to \Lambda_{y,g}$$
(weakly)

and the property of the <u>affinity</u> ρ [2], we have

$$\rho(\Lambda_{x,f}\ ,\ \Lambda_{y,g})$$

$$\geq \overline{\lim_{N}} \int \frac{1}{N}\, \Big|\Big| \sum_{1}^{N} e^{-2\pi i \lambda n}\ S^{n} f \Big|\Big|_{\mu}\ \Big|\Big| \sum e^{-2\pi i \lambda n}\ T^{n} g \Big|\Big|_{\nu}\ d\lambda$$

$$\geq \overline{\lim_{N}} \int \frac{1}{N}\, \Big| \Big(\sum_{1}^{N} e^{-2\pi i \lambda n}\ S^{n} f\ ,\ \sum_{1}^{N} e^{-2\pi i \lambda n}\ T^{n} g \Big)_{\xi} \Big|\ d\lambda$$

$$\geq \overline{\lim_{N}} \frac{1}{N}\, \Big| \int \Big(\sum_{1}^{N} e^{-2\ i\ n}\ S^{n} f\ ,\ {}_{1}^{N}\ e^{-2\ i\ n}\ T^{n} g \Big)_{\xi}\ d\lambda \Big|$$

$$= |(f,g)_{\xi}| > 0$$

Thus $\Lambda_{x,f}$ and $\Lambda_{y,g}$ are not mutually singular.

2. <u>Disjointness of α and β</u>

To prove the disjointness of the two dynamical systems given by α and β in §0 , it is sufficient to prove that any γ and δ in the orbit closures of α and β , respectively, with respect to the shift are independent of each other. The proof by J. Besineau [1] for the independency of α and β works well for these γ and δ . Thus, we have the disjointness of α and β .

3. <u>Mutual singularity of dynamical systems given by α and β</u>

Let (X,μ,S) be a dynamical system. Let f and g be in $L_2(\mu)$. Then we have

<u>Lemma</u>

(1) $\Lambda_{cf} = |c|^2\ \Lambda_f$, where c is a constant .

(2) $\Lambda_{f+g} \leq 2\Lambda_f + 2\Lambda_g$.

(3) $||\Lambda_f - \Lambda_g|| < ||f-g||^2 + 2||f||\ ||f-g||$, where $||\Lambda_f - \Lambda_g||$ is the total variance of the measure $\Lambda_f - \Lambda_g$.

Proof :

(1) is clear. To prove (2), we have

$$\Lambda_{f+g} = \underset{N}{\text{w-lim}} \frac{1}{N} \int ||\sum_{1}^{N} e^{-2\pi i n\lambda} S^n(f+g)||^2 \, d\lambda \leq$$

$$\leq \underset{N}{\text{w-lim}} \frac{2}{N} \int (||\sum_{1}^{N} e^{-2\pi i n\lambda} S^n f||^2 + ||\sum_{1}^{N} e^{-2\pi i n\lambda} S^n g||^2) \, d\lambda =$$

$$= 2\Lambda_f + 2\Lambda_g$$

(3) follows from the fact that

$$||\Lambda_f - \Lambda_g|| \leq \underline{\lim_N} \int \frac{1}{N} \left| ||\sum_{1}^{N} e^{-2\pi i n\lambda} S^n f||^2 - ||\sum_{1}^{N} e^{-2\pi i n\lambda} S^n g||^2 \right| d\lambda \leq$$

$$\leq \underline{\lim_N} \int \frac{1}{N} (||\sum_{1}^{N} e^{-2\pi i n\lambda} S^n(f-g)||^2 +$$

$$+ 2||\sum_{1}^{N} e^{-2\pi i n\lambda} S^n(f-g)|| \, ||\sum_{1}^{N} e^{-2\pi i\lambda n} S^n f||) \, d\lambda \leq$$

$$< ||f-g||^2 + 2||f-g|| \, ||f||$$

Because of this lemma, to prove the mutual singularity of dynamical systems given by α and β, it is sufficient to show that $\Lambda_{\alpha,f}$ and $\Lambda_{\beta,g}$ are mutually singular for f and g of the form

$$f(\gamma) = \gamma^{M_o}(T\gamma)^{M_1} \ldots (T^k\gamma)^{M_k} - C$$

$$g(\gamma) = \gamma^{N_o}(T\gamma)^{N_1} \ldots (T^k\gamma)^{N_k} - D$$

($k=1,2,\ldots$, M_i , $N_i \in \mathbb{Z}$; C,D are constants such that $\int f d\mu_\alpha = \int g d\mu_\beta = 0$)

Let ϕ and ψ are sequences such that

$$\phi(n) = \exp 2\pi i(M_o \, s_p(n) + M_1 \, s_p(n+1) + \ldots + M_r \, s_p(n+k)) - C$$

$$\psi(n) = \exp 2\pi i(N_o \, s_q(n) + N_1 \, s_q(n+1) + \ldots + N_r \, s_q(n+k)) - D$$

Then $\Lambda_{\alpha,f}$ and $\Lambda_{\beta,g}$ are the spectral measures Λ_ϕ and Λ_ψ of the sequences ϕ and ψ , respectively, in the sense of [2]. Let

$$\phi_L(n) = e^{2\pi i E a\, s_p([\frac{n}{p^L}])} A(n-p^L[\frac{n}{p^L}]) - C$$

$$\psi_L(n) = e^{2\pi i F b\, s_q([\frac{n}{q^L}])} B(n-q^L[\frac{n}{q^L}]) - D$$

where $E = \sum_{i=0}^{k} M_i$, $F = \sum_{i=0}^{k} N_i$ and

$$A(\ell) = \exp 2\pi i(M_o\, s_p(\ell) + \ldots + M_k\, s_p(\ell))$$

$$B(\ell) = \exp 2\pi i(N_o\, s_q(\ell) + \ldots + N_k\, s_q(\ell)) \quad .$$

Then, it is easy to see that ϕ_L and ψ_L converge to ϕ and ψ , respectively, as $L \to \infty$ in the sense of Besicovich norm. Therefore Λ_{ϕ_L} (Λ_{ψ_L}) converges to Λ_ϕ (Λ_ψ) in the sense of total variance (cf. Lemma). Therefore our conclusion follows from the statement that Λ_{ϕ_L} and Λ_{ψ_L} are mutually singular. The last statement can be proved in the following way.

<u>Case 1</u> : $E = F = 0$. Then ϕ_L and ψ_L are cyclic sequences whose cycles are coprime. Thus Λ_{ϕ_L} and Λ_{ψ_L} are mutually singular

<u>Case 2</u> : $E \neq 0$, $F = 0$. Since

$$(*) \quad d\Lambda_{\phi_L + C}(\lambda) = \left| \frac{1}{p^L} \sum_{\ell=0}^{p^L-1} A(\ell)\, e^{-2\pi i \lambda \ell} \right|^2 d\Lambda_\eta(p^L\lambda)$$

where $\eta(n) = e^{2\pi i\, Ea\, s_p(n)}$ is known [2] to have a continuous spectral measure, $\Lambda_{\phi_L + C}$ is continuous. This implies that $C = 0$ and Λ_{ϕ_L} is continuous. Since Λ_{ψ_L} is discrete, Λ_{ϕ_L} and Λ_{ψ_L} are mutually singular.

<u>Case 3</u> : $E = 0$, $F \neq 0$. Parallely as in case 2 .

Case 4 : $E \neq 0$, $F \neq 0$. Then as was shown in case 2, $C = D = 0$. Let η be as in case 2 and $\zeta(n) = e^{2\pi i\, Fb\, s_q(n)}$. It is known [3] that Λ_η and Λ_ζ are mutually singular. Since (*) and

$$d\Lambda_\eta(p^L\lambda) = \left| \frac{1}{p^L} \sum_{\ell=0}^{p^L-1} e^{2\pi i(Ea\, s_p(\ell) - \ell\lambda)} \right|^{-2} d\Lambda_\eta(\lambda) ,$$

Λ_{ϕ_L} is absolutely continuous with respect to Λ_η.

Parallely, Λ_{ψ_L} is absolutely continuous with respect to Λ_ζ. Thus Λ_{ϕ_L} and Λ_{ψ_L} are mutually singular. Thus we proved

Theorem B.

The two dynamical systems given by α and β in §0 have mutually singular spectral measures.

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