IOP Publishing | London Mathematical Society Nonlinearity

Nonlinearity **34** (2021) 3411–3428 https://doi.org/10.1088/1361-6544/abe91a

Spectral properties of pattern sequences of general degrees

Yu Zheng[1], **Li Peng**[2,*] **and Teturo Kamae**[3]

[1] School of Information and Mathematics, Yangtze University, Jingzhou 434023, People's Republic of China

[2] School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, People's Republic of China

[3] Advanced Mathematical Institute, Osaka City University, 558-8585 Japan

E-mail: zhengyu@yangtzeu.edu.cn, lpeng@hust.edu.cn and kamae@apost.plala.or.jp

Received 23 July 2020, revised 21 February 2021
Accepted for publication 23 February 2021
Published 10 May 2021

Abstract

We study the spectral measures associated with pattern sequences of general degrees. They are either discrete, purely singular or absolutely continuous. We obtain a necessary and sufficient condition for the discreteness. We also obtain a sufficient condition to be absolutely continuous or to be the Lebesgue measure.

Keywords: spectral measure, pattern sequences, noncorrelated

Mathematics Subject Classification numbers: 11B85, 47B15.

1. Introduction

Let $k \geqslant 1$ be an integer. A subset P of $\{0,1\}^k$ is called a *pattern set* of degree k if $\emptyset \neq P \subset \{0,1\}^k \backslash \{0^k\}$. Let $\mathbb{N}$ be the set of nonnegative integers. We call $\alpha \in \{-1,1\}^{\mathbb{N}}$ the *pattern sequence* related to P if $\alpha(n) = (-1)^{\#(P,n)}$ for any $n \in \mathbb{N}$, where

$$\#(P,n) = \#\{i \in \mathbb{N};\ (n)_i(n)_{i+1}\ldots(n)_{i+k-1} \in P\}, \tag{1.1}$$

and $(n)_i \in \{0,1\} \quad (i \in \mathbb{N})$ are the digits in the dyadic expansion of n, that is, $n = \sum_{i=0}^{\infty}(n)_i 2^i$. Note that we read lower digits first. We let α_P denote this α. It is easy to see that α_P is uniformly recurrent, that is, any finite block appearing in α_p repeats within a bounded gap.

The *correlation function* $\gamma_P : \mathbb{Z} \to [-1,1]$ is defined as

$$\gamma_P(-r) = \gamma_P(r) = \lim_{N\to\infty}(1/N)\sum_{n=0}^{N-1}\alpha_P(n)\alpha_P(n+r) \quad (r = 0,1,2,\ldots).$$

*Author to whom any correspondence should be addressed.
Recommended by Dr Vaughn Climenhaga.

For the given k and $h = 0, 1, \ldots, 2^{k-1} - 1$, we also define

$$\gamma_P(-r|h) = \gamma_P(r|h) = \lim_{N\to\infty}(1/N) \sum_{\substack{n=0 \\ n\equiv h \pmod{2^{k-1}}}}^{N-1} \alpha_P(n)\alpha_P(n+r),$$

where for any $r \geqslant 0$ and h, the above two limits exist since

$$\alpha_P(n)\alpha_P(n+r) = \alpha_P(n')\alpha_P(n'+r)$$

holds if $n \equiv n' \equiv l \pmod{2^N}$ with $0 \leqslant l < 2^N - r$ for $N = 1, 2, \ldots$.

Note that $\gamma_P(r) = \sum_{0\leqslant h<2^{k-1}} \gamma_P(r|h)$ and $\gamma_P(0) = 1$. We say that α_P is *noncorrelated* if $\gamma_P(r) = 0$ for $r = 1, 2, \ldots$. For $r_0 \in \mathbb{N}$, we also say that α_P is r_0-*correlated* if $\gamma_P(r) = 0$ for $r = r_0 + 1, r_0 + 2, \ldots$. Clearly, 0-correlated is equivalent to noncorrelated.

Since the function γ_P is positive definite, by Herglotz theorem, there is a probability Borel measure Λ_P on $[0, 1)$, called the *spectral measure* of α_P, such that

$$\int e^{2\pi i r x}\, d\Lambda_P(x) = \gamma_P(r) \quad (r = 0, 1, 2, \ldots).$$

Note that this equality for $r < 0$ follows from that for $r \geqslant 0$ since $\gamma_P(r)$ is real.

If α_P is noncorrelated, then Λ_P is the Lebesgue measure. If α_P is r_0-correlated, then Λ_P has a density with respect to the Lebesgue measure which is a linear combination of 1 and

$$\cos x, \cos 2x, \ldots, \cos(r_0 x)$$

with coefficients in $(-2, 2)$ (lemma 12).

The spectral measure Λ_P is said to be *continuous* if it has no atom, that is, $\Lambda_P(\{x\}) = 0$ for any $x \in [0, 1)$. It is said to be *absolutely continuous* if it has a density with respect to the Lebesgue measure. A continuous measure Λ on $[0, 1)$ is said to be *singular* if it has no absolutely continuous part, that is, it is supported by a Borel set with Lebesgue measure 0.

A necessary and sufficient condition for α_P to be noncorrelated is known in the case $k = 2, 3$ (Zheng *et al* [13], see section 5). The condition was just technical and no conceptional reason was given. Here, we rewrite this condition to be a conceptional one independent of k and apply for the general k. It is proved that this new condition is sufficient, but not necessary for $k \geqslant 4$. A more natural question is to ask when the spectral measure Λ_P is continuous, singular or absolutely continuous. We discuss this also.

In the case of degree 1 and $P = \{1\}$, α_P is well known as Thue–Morse sequence and is studied by many people, Allouche and Shallit [2], Mauduit and Sárközy [7], Müllner and Spiegelhofer [9], Mauduit and Rivat [8], Spiegelhofer [12], Zaks *et al* [14], Peng and Kamae [11], etc. In this case, Λ_P is known to be singular. In the case of degree 2 and $P = \{11\}$, α_P is also well known as Rudin–Shapiro sequence, Λ_P being the Lebesgue measure (Allouche and Liardet [1], etc). We point out that the Thue–Morse and Rudin–Shapiro sequences are standard examples of 2-automatic sequences, and so are pattern sequences.

General pattern sequences with patterns in G^+, where G is a finite abelian group instead of the multiplicative group $\{-1, 1\}$, and the d-adic representation instead of the binary are studied by Morton and Mourant [5, 6]. The pattern sequences are also discussed by Coquet *et al* [4], Boyd *et al* [3], etc. Pattern sequences with patterns of nonconstant length are discussed by Jakub Konieczny [15] (see remark 5). For a general reference to the ergodic theory and dynamical systems, we cite [10].

In the following, we always assume that $k \geqslant 2$.

Definition 1. The *language graph* of degree k is the directed graph $G_k = (\{0,1\}^{k-1}, E)$, where the edge set E is defined as

$$E = \{(a\xi, \xi b) \in \{0,1\}^{k-1} \times \{0,1\}^{k-1}; a \in \{0,1\}, \quad b \in \{0,1\}, \quad \xi \in \{0,1\}^{k-2}\}.$$

We identify the edge $(a\xi, \xi b)$ with $a\xi b \in \{0,1\}^k$.

Theorem 1. *Let P be a pattern set of degree k. Then the following statements on the pattern sequence α_P are equivalent:*

(a) $\gamma_P(2^{k-1}) = 1$.
(b) *α_P is periodic with a period 2^{k-1}.*
(c) *Λ_P is supported by a finite set.*
(d) *Λ_P is not continuous.*
(e) *Any circuit in the language graph G_k contains an even number of edges belonging to P.*

If any of these conditions is satisfied, then $\alpha_P(n)$ is equal to -1 to the number of edges belonging to P in a path from the vertex $(n)_0(n)_1 \ldots (n)_{k-2}$ to the vertex 0^{k-1}, which does not depend on the choice of the path.

The statement (a) implies (b) since (a) implies that $\alpha_P(n) = \alpha_P(n + 2^{k-1})$ for $n \in \mathbb{N}$ with density 1. Together with the fact that α_P is uniformly recurrent, we have (b). Clearly, (b) implies (a). Hence, (a) and (b) are equivalent. It is clear that (b) implies (c), and (c) implies (d). Hence, to prove the equivalence of all the statements, it is sufficient to prove that (d) implies (e), and (e) implies (b).

Theorem 2. *Let P be a pattern set of degree k. Let $r \equiv 0 \pmod{2^{k-1}}$. Then, we have $\gamma_P(2r) = \gamma_P(r)$.*

Theorem 3. *Let P be a pattern set of degree k. Assume that Λ_P is continuous. Then, Λ_P is either absolutely continuous or singular. Moreover, in the former case, the density function is bounded by 2^{k-1}.*

Definition 2. Let a pattern set P of degree k be given. A pair (i, j) of integers with $1 \leqslant i < j \leqslant k$ is said to be an *odd position* (resp. an *even position*) of P if

$$\begin{aligned}&\text{for any } \xi \in \{0,1\}^{k-j}, \quad \eta \in \{0,1\}^{j-i-1}, \quad \zeta^0 \in \{0,1\}^{i-1}, \quad \zeta^1 \in \{0,1\}^{i-1},\\ &\#\{a \in \{0,1\}; \quad \zeta^0 0\eta a\xi \in P\} + \#\{a \in \{0,1\}; \quad \zeta^1 1\eta a\xi \in P\}\\ &\text{is always odd (resp. always even, respectively).}\end{aligned}$$

Here, 'odd' implies either 1 or 3, and 'even' implies either 0, 2 or 4 (figure 1).

Definition 3. Let a pattern set P of degree k be given. A pair (i, j) of integers with $1 \leqslant i < j \leqslant k$ is said to be a *good position* of P if the following three conditions are satisfied:

(a) (i, j) is an odd position,
(b) Any other pair (i', j') with $1 \leqslant i' < j' \leqslant k$ such that $j' - i' = j - i$ is either an odd position or an even position.
(c) The total number of odd positions (i', j') with $1 \leqslant i' < j' \leqslant k$ such that $j' - i' = j - i$, is odd and all of them satisfy that $i' \leqslant i$.

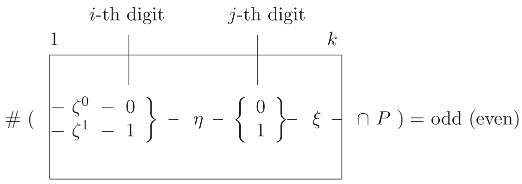

Figure 1. Odd (even) position.

In particular, $(1, k)$ is a good position if it is an odd position.

Theorem 4. *Let P be a pattern set of degree k. Then the following statements hold:*

(*a*) *If P has a good position* (i, j), *then* α_P *is* $(2^{i-1} - 1)$*-correlated. Hence, the spectral measure* Λ_P *has a density with respect to the Lebesgue measure which is a linear combination of*

$$1, \cos\, x, \cos\, 2x, \ldots, \cos(2^{i-1} - 1)x$$

whose coefficients are in $(-2, 2)$.

(*b*) *The pattern sequence* α_P *is noncorrelated if P has a good position* (i, j) *with* $i = 1$. *In the cases* $k = 2, 3$, *the converse also holds.*

Remark 5. Pattern sequences related to pattern sets consisting of patterns of nonconstant lengths are discussed in [15]. Here, the pattern sets $Q \subset \cup_{k\geqslant 2}\{0, 1\}^k$ consisting of finitely many patterns $\xi = \xi_1\xi_2 \ldots \xi_k \in \{0, 1\}^k$ with various k satisfying $\xi_1 = \xi_k = 1$ are considered. It is proved that the pattern sequence α_Q is noncorrelated if

$$Q \supset 1\{0, 1\}^{l-2}1,$$

where l is the maximum length of $\xi \in Q$. This also follows from our theorem 4. In this case, let

$$P = \{\xi \in \{0, 1\}^l; \sum_{i=1}^{l} 1_Q(\xi_1\xi_2 \cdots \xi_i) \text{ is odd}\}.$$

Then, we have $\alpha_P = \alpha_Q$. Moreover for any $\eta \in \{0, 1\}^{l-2}$, we have

$$\#(P \cap \{0\eta 0, 0\eta 1, 1\eta 0, 1\eta 1\} \text{ is odd}$$

since $0\eta 0 \notin P, 0\eta 1 \notin P$ and $\sum_{i=1}^{l} 1_Q(\xi_1\xi_2 \cdots \xi_i)$ for $\xi = 1\eta 0$ and $\xi = 1\eta 1$ differ just by 1 as $1\eta 0 \notin P$ and $1\eta 1 \in P$.

Therefore, $(1, l)$ is an odd position of P, and hence, a good position. Thus, by theorem 4, α_P, and hence, α_Q is noncorrelated.

Remark 6. It can happen that a pattern sequence α_P is noncorrelated even if P only has a good position (i, j) with $i \geqslant 2$. This gives an example that the converse in (b) of theorem 4 fails in general (see example 8). On the other hand, whether P needs to have a good position when α_P is noncorrelated or not is still open.

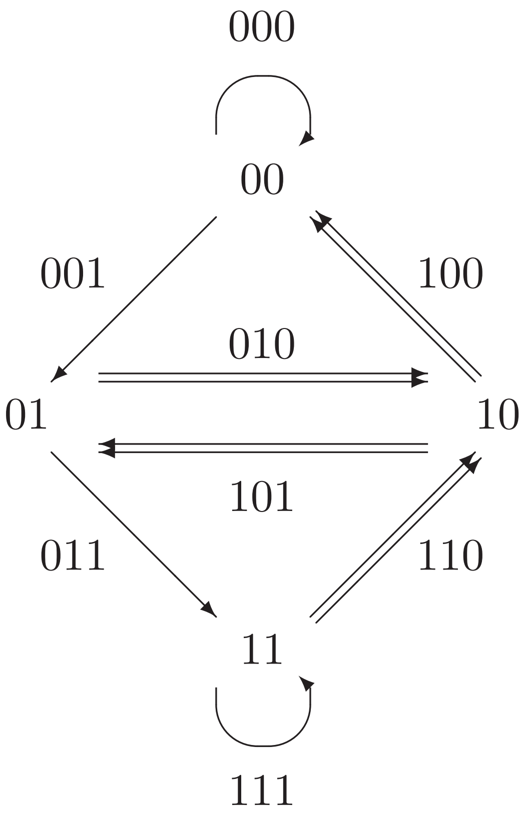

Figure 2. G_3 with P marked by double vectors.

2. Examples

In this section, we give some examples of pattern sequences and show how to apply theorems 1, 3 and 4 to get properties of periodicity, singularity and absolutely continuity.

Example 1. Let $P = \{100, 010, 110, 101\}$. Figure 2 is the language graph G_3 together with the marked edges belonging to P. It is easily seen that any circuit contains an even number of marked edges. Hence, by theorem 1, α_P is periodic with a period 4. In fact, $\alpha_P = (1(-1)11)^\infty$.

Example 2. Let $P = \{010, 111\}$. Then we have

$$\gamma_P(1|0) = \frac{1}{4}, \qquad \gamma_P(1|1) = \gamma_P(1|2) = -\frac{1}{4}$$

$$\gamma_P(1|3) = -\frac{1}{16} + \frac{1}{4}\gamma_P(1|3),$$

and hence, $\gamma_P(1|3) = -\frac{1}{12}$ and $\gamma_P(1) = -\frac{1}{3}$. We also have

$$\gamma_P(4|0) = \gamma_P(4|1) = \frac{1}{4}\gamma_P(1) = -\frac{1}{12}$$

$$\gamma_P(4|2) = \gamma_P(4|3)$$

$$= \frac{1}{4}(-\gamma_P(1|0) - \gamma_P(1|1) + \gamma_P(1|2) + \gamma_P(1|3)) = -\frac{1}{12},$$

and hence, $\gamma_P(4) = -\frac{1}{3}$. Therefore, by theorem 2, we have

$$\gamma_P(2^k) = -\frac{1}{3} \quad (k = 2, 3, \ldots)$$

Since $\limsup_{r\to\infty} |\gamma_P(r)| > 0$, Λ_P is not absolutely continuous, and hence is singular by theorem 3.

Example 3. Let $P = \{100, 010, 110, 011\}$. Then, $(1, 3)$ is an odd position since in P, there are 1 element with the second digit 0 and 3 elements with the second digit 1. Hence, it is a good position, and hence α_P is noncorrelated.

Example 4. Let $P = \{010, 011\}$. Then, $(1, 2)$ is an odd position since in P, there are 1 element with the 3rd digit 0 and 1 element with the 3rd digit 1. Moreover, $(2, 3)$ is an even position since the numbers of elements in P with the first 2 digits 00 or 01, 00 or 11, 10 or 01, 10 or 11 are 2, 0, 2, 0. Hence, it is a good position, and hence α_P is noncorrelated.

Example 5. Let $P = \{100, 010, 110, 001\}$. Then, $(1, 2)$ is an odd position since in P, there are 3 elements with the 3rd digit 0 and 1 element with the 3rd digit 1. Moreover, $(2, 3)$ is an even position since the numbers of elements in P with the first 2 digits 00 or 01, 00 or 11, 10 or 01, 10 or 11 are 2, 2, 2, 2. Hence, it is a good position, and hence α_P is noncorrelated.

Example 6. Let $P = \{100, 110, 001, 111\}$. Then, $(2, 3)$ is an odd position since the numbers of elements in P with the first 2 digits 00 or 01, 00 or 11, 10 or 01, 10 or 11 are 1, 3, 1, 3. Moreover, $(1, 2)$ is an even position since there are 2 elements with the 3rd digit 0 and 2 elements with the 3rd digit 1. Hence, $(2, 3)$ is a good position and α_P is 1-correlated. In fact, we have

$$\mathrm{d}\Lambda_P(x) = (1 - \cos 2\pi x)\mathrm{d}x.$$

Example 7. Let $P = \{010, 110, 001, 101, 011, 111\}$. Then, $(2, 3)$ is an odd position since the numbers of elements in P with the first 2 digits 00 or 01, 00 or 11, 10 or 01, 10 or 11 are 3, 3, 3, 3. Moreover, $(1, 2)$ is an even position since there are 2 elements with the 3rd digit 0 and 4 elements with the 3rd digit 1. Hence, $(2, 3)$ is a good position and α_P is 1-correlated. In fact, we have

$$\mathrm{d}\Lambda_P(x) = (1 + \cos 2\pi x)\mathrm{d}x.$$

Example 8. Let $P = \{0100, 0110, 1101, 1111\}$. Then, $(2, 4)$ is an odd position since the numbers of elements in P with the triple of the 1st, 2nd and 3rd digits 000 or 010, 000 or 110, 100 or 010, 100 or 110, 001 or 011, 001 or 111, 101 or 011, 101 or 111 are 1, 1, 1, 1, 1, 1, 1, 1. Moreover, $(1, 3)$ is an even position since the numbers of elements in P with the pair of the 2nd and 4th digits 00, 10, 01, 11 are 0, 2, 0, 2. Hence, $(2, 4)$ is a good position and α_P is 1-correlated. We can also prove that $(2, 4)$ is the only good position of P. Nevertheless, we can prove directly that $\gamma_P(1) = 0$, and hence, α_P is noncorrelated.

Example 9. Let $P = \{0011, 1011, 0111, 1111\}$. Then, $(3, 4)$ is an odd position since the numbers of elements in P with the triple of the 1st, 2nd and 3rd digits $ab0$ or $cd1$ is 1 for any $ab \in \{0, 1\}^2$ and $cd \in \{0, 1\}^2$. Moreover, $(2, 3)$ is an even position since the numbers of elements in P with the triple of the 1st, 2nd and 4th digits $a0c$ or $b1c$ is 0 for any $a, b \in \{0, 1\}$ and $c = 0$, and 2 for any $a, b \in \{0, 1\}$ and $c = 1$. Furthermore, $(1, 2)$ is an even position since the numbers of elements in P with the pair of the 3rd and 4th digits 00, 10, 01, 11 are 0, 0, 0, 4. Hence, $(3, 4)$ is a good position, and hence α_P is 3-correlated. Since $\alpha_P(n)$ does not depend on the first 2 digits, we have

$$\gamma_P(1) = 3/4, \qquad \gamma_P(2) = 1/2 \quad \text{and} \quad \gamma_P(3) = 1/4,$$

and hence,

$$\mathrm{d}\Lambda_P(x) = \left(1 + \frac{3}{2}\cos 2\pi x + \cos 4\pi x + \frac{1}{2}\cos 6\pi x\right)\mathrm{d}x.$$

Example 10. Let $k \geqslant 3$. Let f, g be functions from $\{0, 1\}^{k-2}$ to $\{0, 1\}$ satisfying that $f(0^{k-2}) + g(0^{k-2}) \geqslant 1$. Let P be a pattern set of degree k such that $P = \{f(\xi)\xi g(\xi); \xi \in \{0, 1\}^{k-2}\}$. Then, $(1, k)$ is an odd position, and hence, a good position. Thus, α_P is noncorrelated.

Example 11. Let

$$\begin{aligned} P = \{&100\,000, 001\,000, 101\,000, 111\,000, 000\,100, 100\,100, 010\,100, 001\,100,\\ &000\,010, 001\,010, 101\,010, 011\,010, 000\,110, 100\,110, 110\,110, 101\,110,\\ &100\,001, 011\,001, 000\,101, 100\,101, 010\,101, 101\,101, 011\,101, 111\,101,\\ &000\,011, 111\,011, 000\,111, 100\,111, 110\,111, 001\,111, 011\,111, 111\,111\}. \end{aligned}$$

Then, $(3, 6)$ is a good position, since any of $(1, 4), (2, 5)$ and $(3, 6)$ is an odd position. Hence, α_P is 3-correlated by theorem 4. It can be proved directly that $\gamma_P(r) = 0$ for $r = 1, 2, 3$. Hence, α_P is noncorrelated.

3. Proof of theorem 1

Definition 4. Let $\Omega = \{0, 1\}^{\mathbb{N}}$ be the set of infinite binary sequences which is also considered as the group of 2-adic integers. An element $\omega \in \Omega$ is denoted as $(\omega)_0(\omega)_1(\omega)_2\ldots$. Let $\lambda = (\frac{1}{2}, \frac{1}{2})^{\mathbb{N}}$ be the product of the probability measure $(\frac{1}{2}, \frac{1}{2})$ giving probability $\frac{1}{2}, \frac{1}{2}$ to 0 and 1, which is the Haar measure on the group of 2-adic integers. Let $T : \Omega \to \Omega$ be the adding machine, that is $T\omega = \omega + 1$. Thus, (Ω, λ, T) is a measure preserving dynamical system. Let P be a pattern set of degree k. Define a *cocycle* $C_P : \Omega \times \mathbb{Z} \to \{-1, 1\}$ by

$$C_P(\omega, r) = (-1)^{\#(P,\omega+r)-\#(P,\omega)} \text{ (see (1.1))},$$

where for almost all $\omega \in \Omega$, there exists i_0 such that $(\omega + r)_i = (\omega)_i$ for any $i \geqslant i_0$ so that

$$\begin{aligned} \lim_{N\to\infty} \big(&\#\{0 \leqslant i < N; (\omega + r)_i \cdots (\omega + r)_{i+k-1} \in P\} \\ &- \#\{0 \leqslant i < N; (\omega)_i \cdots (\omega)_{i+k-1} \in P\}\big) \end{aligned}$$

is well defined and is finite which we denote $\#(P, \omega + r) - \#(P, \omega)$, though each of $\#(P, \omega + r)$ or $\#(P, \omega)$ is not necessarily finite. Hence, the *skew product* of the adding machine by C_P is defined as the measure preserving dynamical system

$$\Omega_P := \left(\Omega \times \{-1, 1\}, \lambda \times \left(\frac{1}{2}, \frac{1}{2}\right), T_P\right), \tag{3.1}$$

where $T_P : \Omega \times \{-1, 1\} \to \Omega \times \{-1, 1\}$ is defined as $T_P(\omega, \tau) = (T\omega, \tau C_P(\omega, 1))$ and $(\frac{1}{2}, \frac{1}{2})$ is the probability measure on $\{-1, 1\}$ giving probability $\frac{1}{2}$, $\frac{1}{2}$ to -1 and 1. We denote the projection from $\Omega \times \{-1, 1\}$ to $\{-1, 1\}$ by ψ.

Since the arithmetic mean corresponds to the expectation by the probability measure λ, we have the following lemma.

Lemma 7. *For any pattern set P, we have*

$$\gamma_P(r) = \int C_P(\omega, r)\mathrm{d}\lambda(\omega) = \int \psi(\omega, \tau)\psi(T_P^r(\omega, \tau))\mathrm{d}\left(\lambda \times \left(\frac{1}{2}, \frac{1}{2}\right)\right)(\omega, \tau)$$

for any $r \in \mathbb{Z}$.

Proof. Replacing the arithmetic mean by the expectation with respect to the probability measure λ, we have

$$\begin{aligned}\gamma_P(r) &= \lim_{N\to\infty}(1/N)\sum_{n=0}^{N-1}(-1)^{\#(P,n+r)-\#(P,n)} \\ &= \mathbb{E}((-1)^{\#(P,\omega+r)-\#(P,\omega)}) = \int C_P(\omega, r)\mathrm{d}\lambda(\omega)\end{aligned}$$

if $r \geqslant 0$. If $r < 0$, then

$$\begin{aligned}\gamma_P(r) = \gamma_P(-r) &= \lim_{N\to\infty}(1/N)\sum_{n=0}^{N-1}(-1)^{\#(P,n-r)-\#(P,n)} \\ &= \mathbb{E}((-1)^{\#(P,\omega-r)-\#(P,\omega)}) = \mathbb{E}((-1)^{\#(P,\omega')-\#(P,\omega'+r)}) \\ &= \int C_P(\omega, r)\mathrm{d}\lambda(\omega),\end{aligned}$$

where we put $\omega' = \omega - r$ and use the invariance of the Haar measure λ under the addition. Thus, our lemma follows since

$$\psi(\omega, \tau)\psi(T_P^r(\omega, \tau)) = \tau C_P(\omega, r)\tau = C_P(\omega, r).$$

□

As stated before, to prove theorem 1, it is sufficient to prove that (d) implies (e), and (e) implies (b).

(d) implies (e): assume that there exists a circuit in G_k containing an odd number of edges belonging to P. Since G_k is strongly connected, there exists a circuit in G_k from 0^{k-1} to 0^{k-1} containing an odd number of edges belonging to P. Let $0^{k-1}\xi 0^{k-1}$ with $\xi = \xi_1\xi_2 \ldots \xi_n$ be one such circuit, that is, an odd number of edges among

$$0^{k-1}\xi_1, 0^{k-2}\xi_1\xi_2, \ldots, \xi_n 0^{k-1},$$

belong to P. Let $Q = 2^{k-1}\sum_{i=1}^{n}\xi_i 2^{i-1}$.

$$
\begin{array}{rl}
 & \qquad\quad M \qquad\qquad\qquad M+n+2k-3 \\
 & \qquad\quad | \qquad\qquad\qquad\qquad\quad | \\
\omega & = *** \, 0\cdots 0 \;\; 0 \;\cdots 0 \;\; 0\cdots 0 \;\; *** \\
Q2^M & = 0\cdots 0 \; \underbrace{0\cdots 0}_{k-1} \;\; \xi_1\cdots\xi_n \; \underbrace{0\cdots 0}_{k-1} \;\; 0\cdots
\end{array}
$$

Figure 3. $C_P(\omega, Q2^M) = -1$.

We prove that Λ_P is continuous. To the contrary, suppose that Λ_P is not continuous and has an atom at ν (say). Then, there exists an eigenfunction, say f with eigenvalue $\nu(|\nu| = 1)$, belonging to the L^2-space spanned by $\psi \circ T_P^r(r = 0, 1, 2, \ldots)$ in $L^2(\Omega \times \{-1, 1\}, \lambda \times (\frac{1}{2}, \frac{1}{2}))$. We may assume that $|f(\omega, \tau)| = \text{constant} > 0$ almost surely, since otherwise, we may take an ergodic measure with this property in the ergodic decomposition of $\lambda \times (\frac{1}{2}, \frac{1}{2})$ in place of $\lambda \times (\frac{1}{2}, \frac{1}{2})$ if it is not ergodic. In any case, we let $\mathbb{P}$ denote this measure. Note that the marginal distribution of $\mathbb{P}$ to Ω is λ since (Ω, λ, T) is ergodic. We may also assume that the above constant is 1. Since f is Borel measurable, we have $f(\omega + Q2^M, \tau) \to f(\omega, \tau)$ in probability as $M \to \infty$ (w.r.t. $\mathbb{P}$). It is easy to see that there exists a subsequence $\{N_1 < N_2 < \ldots\} \subset \{1, 2, \ldots\}$ such that $\lim_{n\to\infty} \nu^{Q2^{N_n}} = \beta$, where $\beta = \mathrm{e}^{\mathrm{i}\theta}$ with $-2\pi/3 \leqslant \theta \leqslant 2\pi/3$. Since $\psi(T_P^r(\omega, -\tau)) = -\psi(T_P^r(\omega, \tau))$ holds for any $r = 0, 1, 2, \ldots$, we have $f(\omega, -\tau) = -f(\omega, \tau)$.

Let

$$E_M = \{\omega \in \Omega; (\omega)_M = (\omega)_{M+1} = \ldots = (\omega)_{M+n+2k-3} = 0\}.$$

Then, $\mathbb{P}(E_M) = \lambda(E_M) = 2^{-n-2k+2}$. If $\omega \in E_M$, then we have $C_P(\omega, Q2^M) = -1$ (see figure 3) since $0^{k-1}\xi 0^{k-1}$ contains an odd number of elements belonging to P.

Since

$$\psi(T_P^{Q2^M}(\omega, -\tau)) = -\psi(T_P^{Q2^M}(\omega, \tau))$$

and f belongs to the L^2-space spanned by $\psi \circ T_P^r(r = 0, 1, 2, \ldots)$,

$$f(T_P^{Q2^M}(\omega, \tau)) = f(\omega + Q2^M, -\tau) = -f(\omega + Q2^M, \tau)$$

holds if $\omega \in E_M$, hence with probability at least $2^{-n-2k+2}$ for any M.

Since $f(\omega + Q2^M, \tau) \to f(\omega, \tau)$ in probability as $M \to \infty$, for any $\varepsilon > 0$, we have

$$
\begin{aligned}
&\liminf_{M\to\infty} \mathbb{P}(|f(T_P^{Q2^M}(\omega, \tau)) + f(\omega, \tau)| < \varepsilon) \\
\geqslant &\liminf_{M\to\infty} \mathbb{P}\Big(f(T_P^{Q2^M}(\omega, \tau)) = -f(\omega + Q2^M, \tau) \\
&\qquad\qquad\qquad \text{and } |f(\omega + Q2^M, \tau) - f(\omega, \tau)| < \varepsilon\Big) \\
\geqslant &\liminf_{M\to\infty} \mathbb{P}(f(T_P^{Q2^M}(\omega, \tau)) = -f(\omega + Q2^M, \tau)) \\
&\qquad\qquad\qquad - \limsup_{M\to\infty} \mathbb{P}(|f(\omega + Q2^M, \tau) - f(\omega, \tau)| \geqslant \varepsilon) \\
\geqslant &2^{-n-2k+2}.
\end{aligned}
$$

Let ε be such that $0 < \varepsilon < 1/2$. Since $|f(\omega,\tau)| = 1$ and $\beta = e^{i\theta}$ with $-2\pi/3 \leqslant \theta \leqslant 2\pi/3$, $|f(T_P^{Q2^M}(\omega,\tau)) + f(\omega,\tau)| < \varepsilon$ implies that $|f(T_P^{Q2^M}(\omega,\tau)) - \beta f(\omega,\tau)| > 1 - \varepsilon > \varepsilon$. Hence, we have

$$\liminf_{M\to\infty} \mathbb{P}(|f(T_P^{Q2^M}(\omega,\tau)) - \beta f(\omega,\tau)| > \varepsilon)$$
$$\geqslant \liminf_{M\to\infty} \Big(\mathbb{P}(|f(T_P^{Q2^M}(\omega,\tau)) + f(\omega,\tau)| < \varepsilon) \geqslant 2^{-n-2k+2}.$$

On the other hand, since f is an eigenfunction with the eigenvalue ν, with probability 1, we have

$$f(T_P^{Q2^M}(\omega,\tau)) = \nu^{Q2^M} f(\omega,\tau).$$

Therefore, $f(T_P^{Q2^{N_n}}(\omega,\tau))$ converges to $\beta f(\omega,\tau)$ as $n \to \infty$ with probability 1 (w.r.t. $\mathbb{P}$). This contradicts the above inequality.

Thus, Λ_P is continuous and (d) implies (e).

(e) implies (b), and the last statement: since G_k is strongly connected and any circuit contains an even number of edges belonging to P, any pair of vertices determines the parity of the number of elements belonging to P contained in the paths connecting them. Take any vertex $\xi \in \{0,1\}^{k-1}$ of G_k. Then, the parity from ξ to 0^{k-1} is determined. Since $\alpha_P(n)$ is -1 to the number of elements belonging to P contained in the sequence $(n)_0(n)_1(n)_2\ldots$, which ends with 0^∞, it coincides with -1 to the number from $(n)_0(n)_1\ldots(n)_{k-2}$ to 0^{k-1}. Thus, it is determined by $(n)_0(n)_1\ldots(n)_{k-2}$, and hence, α_P is periodic with a period 2^{k-1}. □

4. Proofs of theorems 2 and 3

Definition 5. Let P be a pattern set of degree k. For any pair

$$\alpha = \alpha_1\alpha_2\ldots\alpha_{k-1} \in \{0,1\}^{k-1} \quad \text{and} \quad \beta = \beta_1\beta_2\ldots\beta_{k-1} \in \{0,1\}^{k-1},$$

define

$$\sigma(\alpha,\beta) = \#\{i \in \{1,2,\ldots,k-1\}; \epsilon_i \in P\},$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_{k-1} \in \{0,1\}^k$ are as follows

$$\epsilon_i = \alpha_i \ldots \alpha_{k-1}\beta_1\ldots\beta_i \quad (i = 1,2,\ldots,k-1).$$

For $n = 0,1,\ldots,2^{k-1}-1$, let $\varphi(n) = \alpha_1\alpha_2\ldots\alpha_{k-1} \in \{0,1\}^{k-1}$ be such that $\sum_{i=1}^{k-1}\alpha_i 2^{i-1} = n$, and define a matrix

$$\Sigma^P = ((-1)^{\sigma(\varphi(n),\varphi(m))})_{n,m=0,1,\ldots,2^{k-1}-1}.$$

Finally we define

$$\Gamma^P = {}^t\Sigma^P\Sigma^P,$$

where ${}^t\Sigma^P$ is the transpose of Σ^P.

Lemma 8. *Let P be a pattern set of degree k. Let r be a nonnegative integer and d be an integer with $d \geqslant k-1$. Then, we have*

$$\gamma_P(r2^d) = \frac{1}{2^{k-1}} \sum_{h=0}^{2^{k-1}-1} \gamma_P(r|h)\Gamma^P_{h,h+r}, \tag{4.1}$$

where $h+r$ is considered modulo 2^{k-1}, that is, $\Gamma^P_{h,h+r} := \Gamma^P_{h,h'}$ with $h' \equiv h+r \pmod{2^{k-1}}$ and $h' \in \{0,1,\ldots,2^{k-1}-1\}$ (the convention also applies to $\Sigma^P_{h,h+r}$ and $\varphi(h+r)$).

Proof. Let $n \equiv h \pmod{2^{k-1}}$ and $\lfloor i2^{k-1-d} \rfloor \equiv l \pmod{2^{k-1}}$ for any $n \in \mathbb{N}$ and $i \in \{0,1,\ldots,2^d-1\}$, where $h, l \in \{0,1,\ldots,2^{k-1}-1\}$. Then, we have

$$\#(P, i+n2^d) = \#(P, i|_d) + \sigma(\varphi(l), \varphi(h)) + \#(P,n)$$
$$\#(P, i+n2^d+r2^d) = \#(P, i|_d) + \sigma(\varphi(l), \varphi(h+r)) + \#(P,n+r),$$

where

$$\#(P, i|_d) = \#\{j \in \mathbb{N}; 0 \leqslant j \leqslant d-k,\ (i)_j(i)_{j+1}\ldots(i)_{j+k-1} \in P\}.$$

Hence, for any $n \in \mathbb{N}$ with $n \equiv h \pmod{2^{k-1}}$, we have

$$\alpha_P(i+n2^d)\alpha_P(i+n2^d+r2^d) = (\Sigma^P)_{l,h}(\Sigma^P)_{l,h+r}\alpha_P(n)\alpha_P(n+r).$$

Therefore,

$$\sum_{i=0}^{2^d-1} \alpha_P(i+n2^d)\alpha_P(i+n2^d+r2^d) = 2^{d-k+1}\Gamma^P_{h,h+r}\alpha_P(n)\alpha_P(n+r).$$

Thus we have

$$\begin{aligned}
\gamma_P(r2^d) &= \lim_{N\to\infty} \frac{1}{N2^d} \sum_{n=0}^{N-1} \sum_{i=0}^{2^d-1} \alpha_P(i+n2^d)\alpha_P(i+n2^d+r2^d) \\
&= \lim_{N\to\infty} \frac{1}{N2^d} \sum_{h=0}^{2^{k-1}-1} \sum_{\substack{n=0 \\ n\equiv h \ (\mathrm{mod}\ 2^{k-1})}}^{N-1} \sum_{i=0}^{2^d-1} \alpha_P(i+n2^d)\alpha_P(i+n2^d+r2^d) \\
&= \lim_{N\to\infty} \frac{1}{N2^d} \sum_{h=0}^{2^{k-1}-1} \sum_{\substack{n=0 \\ n\equiv h \ (\mathrm{mod}\ 2^{k-1})}}^{N-1} 2^{d-k+1}\Gamma^P_{h,h+r}\alpha_P(n)\alpha_P(n+r) \\
&= \frac{2^{d-k+1}}{2^d} \sum_{h=0}^{2^{k-1}-1} \Gamma^P_{h,h+r} \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=0 \\ n\equiv h \ (\mathrm{mod}\ 2^{k-1})}}^{N-1} \alpha_P(n)\alpha_P(n+r) \\
&= \frac{1}{2^{k-1}} \sum_{h=0}^{2^{k-1}-1} \gamma_P(r|h)\Gamma^P_{h,h+r}.
\end{aligned}$$

□

Proof of theorem 2. Let $r \equiv 0 \pmod{2^{k-1}}$. Then, we have $r = r'2^{k-1}$ and $2r = r'2^k$ with $r' \in \mathbb{N}$. Let us apply (4.1) for r (and $2r$) with r' in place of r and $d = k-1$ ($d = k$, respectively). Since the right-hand side of (4.1) does not depend on d, we have

$$\gamma_P(r) = \gamma_P(r'2^{k-1}) = \gamma_P(r'2^k) = \gamma_P(2r).$$

□

Let P be a pattern set of degree k such that Λ_P is continuous. Denote $\tilde{\gamma}_P(r) = \gamma_P(r2^{k-1})$ and let $\tilde{\Lambda}_P$ be the probability Borel measure on $[0, 1)$ such that

$$\int e^{2\pi i r x} d\tilde{\Lambda}_P(x) = \tilde{\gamma}_P(r) \quad (r = 0, 1, 2, \ldots).$$

Then by theorem 2, $\tilde{\Lambda}_P$ is S-invariant, where S is the transformation on $[0, 1)$ such that $Sx = \{2x\}$.

The following lemmas 9 and 10 are proved by Peng and Kamae [11] in the case of Thue–Morse system. The following proofs for our general case are similar to this.

Lemma 9. *For any $r, l \in \mathbb{Z}$, we have*

$$\lim_{n\to\infty} \tilde{\gamma}_P(r + l2^n) = \tilde{\gamma}_P(r)\tilde{\gamma}_P(l).$$

Proof. By lemma 7,

$$\tilde{\gamma}_P(r) = \gamma_P(r2^{k-1}) = \int C_P(\omega, r2^{k-1}) d\lambda(\omega)$$

Since

$$C_P\left(\omega, ((r + l2^n)2^{k-1}\right) = C_P(\omega, r2^{k-1})C_P(\omega + r2^{k-1}, l2^{n+k-1}),$$

we have

$$\tilde{\gamma}_P(r + l2^n) = \int C_P(\omega, r2^{k-1})C_P(\omega + r2^{k-1}, l2^{n+k-1}) d\lambda(\omega).$$

Let Ω_n be the set of $\omega \in \Omega$ such $(\omega + r2^{k-1})_i = (\omega)_i$ for any $i \geqslant n + k - 1$. Since $\lambda(\Omega_n) \geqslant 1 - r2^{k-1}/2^{n+k-1}$, $\lambda(\Omega_n) \to 1$ as $n \to \infty$. The random variables $C_P(\omega, r2^{k-1})$ and $C_P(\omega + r2^{k-1}, l2^{n+k-1})$ restricted to Ω_n are independent since the former depends only on $\{(\omega)_i; i = 0, 1, \ldots, 2^{n+k-1} - 1\}$ while the latter depends only on $\{(\omega)_i; i = 2^{n+k-1}, 2^{n+k-1} + 1, \ldots\}$. Therefore, using theorem 2, we have

$$\begin{aligned}
&\lim_{n\to\infty} \int C_P(\omega, r2^{k-1})C_P(\omega + r2^{k-1}, l2^{n+k-1}) d\lambda(\omega) \\
&= \lim_{n\to\infty} \int C_P(\omega, r2^{k-1}) d\lambda(\omega) \int C_P(\omega + r2^{k-1}, l2^{n+k-1}) d\lambda(\omega) \\
&= \lim_{n\to\infty} \int C_P(\omega, r2^{k-1}) d\lambda(\omega) \int C_P(\omega, l2^{n+k-1}) d\lambda(\omega) \\
&= \lim_{n\to\infty} \gamma_P(r2^{k-1})\gamma_P(l2^{n+k-1}) = \lim_{n\to\infty} \tilde{\gamma}_P(r)\tilde{\gamma}_P(l2^n) = \tilde{\gamma}_P(r)\tilde{\gamma}_P(l)
\end{aligned}$$

□

Lemma 10. *The measure preserving system* $([0,1), \tilde{\Lambda}_P, S)$ *is ergodic. Hence,* $\tilde{\Lambda}_P$ *is the Lebesgue measure or is singular.*

Proof. It is sufficient to prove that for any $k \in \mathbb{N}$,

$$I_N := \int \left| (1/N) \sum_{n=0}^{N-1} \mathrm{e}^{2\pi \mathrm{i} k S^n x} - \tilde{\gamma}_P(k) \right|^2 \mathrm{d}\tilde{\Lambda}_P(x) \to 0$$

as $N \to \infty$. We have

$$\begin{aligned} I_N &= (1/N^2) \sum_{n,m=0}^{N-1} \int \mathrm{e}^{2\pi ik(2^n - 2^m)x} \mathrm{d}\tilde{\Lambda}_P(x) \\ &\quad - (1/N)\tilde{\gamma}_P(k) \left(\sum_{n=0}^{N-1} \int \mathrm{e}^{2\pi \mathrm{i} k 2^n x} \mathrm{d}\tilde{\Lambda}_P(x) + \sum_{m=0}^{N-1} \int \mathrm{e}^{-2\pi \mathrm{i} k 2^m x} \mathrm{d}\tilde{\Lambda}_P(x) \right) + \tilde{\gamma}_P(k)^2 \\ &= (1/N^2) \sum_{n,m=0}^{N-1} \tilde{\gamma}_P(k2^n - k2^m) - 2\tilde{\gamma}_P(k)^2 + \tilde{\gamma}_P(k)^2. \end{aligned}$$

Since by theorem 2 and lemma 8,

$$\lim_{\substack{n,m \to \infty \\ |n-m| \to \infty}} \tilde{\gamma}_P(k2^n - k2^m) = \tilde{\gamma}_P(k)^2,$$

we have

$$\lim_{N \to \infty} (1/N^2) \sum_{n,m=0}^{N-1} \tilde{\gamma}_P(k2^n - k2^m) = \tilde{\gamma}_P(k)^2,$$

and hence, $\lim_{N \to \infty} I_N = 0$.

Since any two distinct ergodic measures for $([0,1), S)$ are mutually singular, and the Lebesgue measure is an ergodic measure for $([0,1), S)$, it follows that either $\tilde{\Lambda}_P$ is the Lebesgue measure or is singular. □

Proof of theorem 3. Let $g : [0,1) \to [0,1)$ be given by $g(x) = \{2^{k-1}x\}$. Then, g is a 2^{k-1}-to-1 measure preserving transformation from $([0,1), \Lambda_P)$ to $([0,1), \tilde{\Lambda}_P)$. By our assumption that Λ_P is continuous, $\tilde{\Lambda}_P$ is not discrete. Hence, either it is the Lebesgue measure or is singular. If $\tilde{\Lambda}_P$ is the Lebesgue measure, then there exists a nonnegative measurable function f on $[0,1)$ such that

$$\mathrm{d}\Lambda_P(x) = f(x)\mathrm{d}x \quad \text{and} \quad (1/2^{k-1}) \sum_{i=0}^{2^{k-1}-1} f(\{x + i2^{k-1}\}) = 1$$

for any $x \in [0,1)$. Therefore, Λ_P is absolutely continuous and the density function is bounded with the bound 2^{k-1}. On the other hand, if $\tilde{\Lambda}_P$ is singular, then it is supported by a Borel set, say B with $\lambda(B) = 0$. Since Λ_P is supported by $g^{-1}B$ and $\lambda(g^{-1}B) = \lambda(B) = 0$, Λ_P is singular. □

5. The cases $k = 2, 3$

It is known [13] that in the case of $k = 2$, α_P is noncorrelated if and only if P is one of

$$\{01\}, \{10\}, \{11\}, \{01, 10, 11\}.$$

Moreover, in the case of $k = 3$, the following necessary and sufficient condition is known.

Theorem 11. [13] *The pattern sequence α_P related to a pattern set P of degree 3 is noncorrelated if and only if one of the following three conditions is satisfied:*

(*a*) $a_1 = -a_4a_5$ *and* $a_2a_3 = -a_6a_7$,
(*b*) $a_1 = a_4a_5 = -1$, $a_2a_3 = a_6a_7 = 1$ *and* $a_2 = a_5a_6$,
(*c*) $a_1 = a_4a_5 = 1$, $a_2a_3 = a_6a_7 = -1$ *and* $a_2 = -a_5a_6$,

Where we denote $a_n = \alpha_P(n) \quad (\forall n \in \mathbb{N})$.

In this section, we prove the last statement of (b) of theorem 4 using the above characterizations.

If $k = 2$, then $(1, 2)$ is an odd position of P if and only if $\#P$ is odd. Hence, P has a good position $(1, 2)$ if and only if $\#P$ is odd, that is, P is one of

$$\{01\}, \{10\}, \{11\}, \{01, 10, 11\}.$$

Thus, theorem 4 follows in the case $k = 2$.

Let $k = 3$.

Assume the condition (a) in theorem 10. Since $000 \notin P$ and $a_0 = 1$, we have

$$a_0a_1 = (-1)^{\#(P\cap\{100\})}.$$

Since 4 and 5 have common upper 2 digits 01, we have

$$a_4a_5 = (-1)^{\#(P\cap\{001,101\})}.$$

Since $a_1 = -a_4a_5$, we have $a_0a_1a_4a_5 = -1$, hence, $\#(P \cap \{000, 100, 001, 101\})$ is odd. In the same way, since $a_2a_3 = -a_6a_7$, $\#(P \cap \{010, 110, 011, 111\})$ is odd. These facts imply that $(1, 3)$ is an odd position of P, and hence, $(1, 3)$ is a good position.

Assume the condition (b) in theorem 10. Since $a_1 = a_4a_5 = -1$, both of $\#(P \cap \{000, 100\})$ and $\#(P \cap \{001, 101\})$ are odd. Since $a_2a_3 = a_6a_7 = 1$, both of $\#(P \cap \{010, 110\})$ and $\#(P \cap \{011, 111\})$ are even. Hence, both of $\#(P \cap \{000, 100, 010, 110\})$ and $\#(P \cap \{001, 101, 011, 111\})$ are odd. Therefore, $(1, 2)$ is an odd position of P.

We prove that $(2, 3)$ is an even position. Since $a_1 = -1$, $100 \in P$. Since $a_4a_5 = -1$, exactly one of 001 or 101 is in P. Since $a_2a_3 = a_6a_7 = 1$, we have 4 cases:

$$010 \in P, 110 \in P, 011 \in P, 111 \in P \tag{5.1}$$

$$010 \in P, 110 \in P, 011 \notin P, 111 \notin P \tag{5.2}$$

$$010 \notin P, 110 \notin P, 011 \in P, 111 \in P \tag{5.3}$$

$$010 \notin P, 110 \notin P, 011 \notin P, 111 \notin P \tag{5.4}$$

Together with $001 \in P$ or $101 \in P$, we have 8 cases. Among these cases, the condition that $a_2 = a_5a_6$ is satisfied only in the following 4 cases:

(a) $001 \notin P 101 \in P$ and (5.1)

(b) $001 \in P101 \notin P$ and (5.2)
(c) $001 \in P101 \notin P$ and (5.3)
(d) $001 \notin P101 \in P$ and (5.4)

The pattern set P for these 4 cases are

$$\{100, 010, 110, 101, 011, 111\}, \{100, 010, 110, 001\}, \{100, 001, 011, 111\}, \{100, 101\}$$

Any of these cases, $(1, 2)$ is a good position.

Assume the condition (c) in theorem 10. Since $a_1 = a_4a_5 = 1$, $100 \notin P$ and $\#(P \cap \{001, 101\})$ is even. Since $a_2a_3 = a_6a_7 = -1$, both of $\#(P \cap \{010, 110\})$ and $\#(P \cap \{011, 111\})$ are odd. Hence, both of $\#(P \cap \{000, 100, 010, 110\})$ and $\#(P \cap \{001, 101, 011, 111\})$ are odd. Therefore, $(1, 2)$ is an odd position of P.

We prove that $(2, 3)$ is an even position. Since $a_1 = 1$, $100 \notin P$. Since $a_4a_5 = 1$, either $\{001, 101\} \subset P$ or $\{001, 101\} \cap P = \emptyset$. Since $a_2a_3 = a_6a_7 = -1$, we have 4 cases:

$$010 \in P, 110 \notin P, 011 \in P, 111 \notin P \tag{5.5}$$

$$010 \in P, 110 \notin P, 011 \notin P, 111 \in P \tag{5.6}$$

$$010 \notin P, 110 \in P, 011 \in P, 111 \notin P \tag{5.7}$$

$$010 \notin P, 110 \in P, 011 \notin P, 111 \in P \tag{5.8}$$

Together with the condition for $\{001, 101\}$, we have 8 cases. Among these cases, the condition that $a_2 = -a_5a_6$ is satisfied only in the following 4 cases:

(a) $\{001, 101\} \cap P = \emptyset$ and (5.5)
(b) $\{001, 101\} \subset P$ and (5.6)
(c) $\{001, 101\} \subset P$ and (5.7)
(d) $\{001, 101\} \cap P = \emptyset$ and (5.8)

The pattern set P for these 4 cases are

$$\{010, 011\}, \{010, 001, 101, 111\}, \{110, 001, 101, 011\}, \{110, 111\}$$

Any of these cases, $(1, 2)$ is a good position.

Thus, we have proved that if α_P is noncorrelated, then P has a good position (i, j) with $i = 1$.

6. Proof of theorem 4

Lemma 12. *If $\alpha \in \{-1, 1\}^{\mathbb{N}}$ is r_0-correlated, then, Λ_α has a density with respect to the Lebesgue measure on $[0, 1)$ which is a linear combination of 1 and $\cos x, \cos 2x, \ldots, \cos(r_0x)$ whose coefficients are in $(-2, 2)$.*

Proof. Note that Λ_α is determined by the correlation function γ_α. Let $c_r = \gamma_\alpha(r)$ $(r \in \mathbb{N})$. Clearly, $c_0 = 1$ and $|c_r| \leqslant 1 \quad (r \in \mathbb{N})$. Since α is r_0-correlated, Λ_α is absolutely continuous. Hence, $|c_r| < 1$ holds for any $r > 0$. Moreover, $c_r = 0$ holds for any $r > r_0$. Let

$$f(x) = 1 + \sum_{r=1}^{r_0} 2c_r \cos 2\pi x \quad (x \in [0, 1)).$$

Then, $\int e^{2\pi i r x} f(x)dx = c_r$ holds for any $r \in \mathbb{N}$. Thus, $d\Lambda_\alpha(x) = f(x)dx$. □

Proof of theorem 4. Since $2^{i-1} - 1 = 0$ if $i = 1$, the 'if' part of (b) follows from (a) of theorem 4. The converse in the case $k = 2, 3$ is already proved in section 5. Hence, to prove theorem 4, it is sufficient to prove (a) of theorem 4.

Assume that a pattern set P of degree $k \geqslant 2$ has a good position (i, j). Let $d = j - i \geqslant 1$. Take any $r \in \mathbb{N}$ with $r \geqslant 2^{i-1}$. For any $n \in \mathbb{N}$, define $h_0 = h_0(n)$ by

$$h_0 = \max\{h \in \mathbb{N}; \quad (n)_h \neq (n+r)_h\}.$$

Then, $h_0 \geqslant i - 1$ since $r \geqslant 2^{i-1}$. Define $\overline{n}$ as

$$\overline{n} = \begin{cases} n + 2^{h_0+d} & \text{if } (n)_{h_0+d} = 0 \\ n - 2^{h_0+d} & \text{if } (n)_{h_0+d} = 1 \end{cases}.$$

Then, it is clear that $\overline{(\overline{n})} = n$. Moreover, it is easy to see that

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \in \mathbb{N}; \quad n < N \quad \text{and} \quad \overline{n} < N\} = 1. \tag{6.1}$$

We are going to prove that

$$\alpha_P(n)\alpha_P(n+r) + \alpha_P(\overline{n})\alpha_P(\overline{n}+r) = 0 \tag{6.2}$$

for any $n \in \mathbb{N}$. Then, $\gamma_P(r) = 0$ follows since by (6.1),

$$\begin{aligned} \gamma_P(r) &= \lim_{N\to\infty} \frac{1}{N} \sum_{0\leqslant n<N} \alpha_P(n)\alpha_P(n+r) \\ &= \lim_{N\to\infty} \frac{1}{2N} \sum_{0\leqslant n<N} (\alpha_P(n)\alpha_P(n+r) + \alpha_P(\overline{n})\alpha_P(\overline{n}+r)) = 0. \end{aligned}$$

Note that (6.2) is equivalent to

$$\alpha_P(n)\alpha_P(n+r)\alpha_P(\overline{n})\alpha_P(\overline{n}+r) = -1. \tag{6.3}$$

Moreover, (6.3) is equivalent to that $\sum_{l=0}^{\infty} K_l$ is odd, where

$$\begin{aligned} K_l = {} & 1_P((n)_l(n)_{l+1}\ldots(n)_{l+k-1}) + 1_P((n+r)_l(n+r)_{l+1}\ldots(n+r)_{l+k-1}) \\ & + 1_P((\overline{n})_l(\overline{n})_{l+1}\ldots(\overline{n})_{l+k-1}) + 1_P((\overline{n}+r)_l(\overline{n}+r)_{l+1}\ldots(\overline{n}+r)_{l+k-1}). \end{aligned}$$

If $l > h_0$, then we have

$$\begin{aligned} (n)_l(n)_{l+1}\ldots(n)_{l+k-1} &= (n+r)_l(n+r)_{l+1}\ldots(n+r)_{l+k-1}, \\ (\overline{n})_l(\overline{n})_{l+1}\ldots(\overline{n})_{l+k-1} &= (\overline{n}+r)_l(\overline{n}+r)_{l+1}\ldots(\overline{n}+r)_{l+k-1}. \end{aligned}$$

Hence, K_l is even. Also, if $l < h_0 - k + d + 1$, then since $l + k - 1 < h_0 + d$, we have

$$\begin{aligned} (n)_l(n)_{l+1}\ldots(n)_{l+k-1} &= (\overline{n})_l(\overline{n})_{l+1}\ldots(\overline{n})_{l+k-1}, \\ (n+r)_l(n+r)_{l+1}\ldots(n+r)_{l+k-1} &= (\overline{n}+r)_l(\overline{n}+r)_{l+1}\ldots(\overline{n}+r)_{l+k-1}. \end{aligned}$$

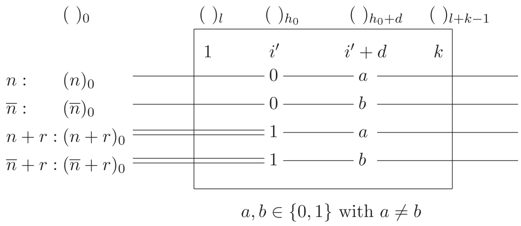

Figure 4. The position $(i', i' + d)$ is included in the sum $\sum \#\mathbb{S}_l$, if $l \geqslant 0$, that is, $l = h_0 - i' + 1 \geqslant 0$ in the above.

Hence, K_l is even. Moreover if $(h_0 - k + d + 1) \vee 0 \leqslant l \leqslant h_0$, then the four terms in the above two equality are different each other, where we denote $a \vee b := \max\{a, b\}$. Therefore, we have

$$\sum_{l=0}^{\infty} K_l \equiv \sum_{(h_0-k+d+1)\vee 0 \leqslant l \leqslant h_0} \#\mathbb{S}_l \ (\mathrm{mod}\, 2), \tag{6.4}$$

where

$$\begin{aligned}\mathbb{S}_l = \{&(n)_l(n)_{l+1} \ldots (n)_{l+k-1},\\ &(n+r)_l(n+r)_{l+1} \ldots (n+r)_{l+k-1},\\ &(\overline{n})_l(\overline{n})_{l+1} \ldots (\overline{n})_{l+k-1},\\ &(\overline{n}+r)_l(\overline{n}+r)_{l+1} \ldots (\overline{n}+r)_{l+k-1}\} \cap P.\end{aligned}$$

This implies that with the following notations:

$$\begin{aligned}\zeta^0 &:= (n)_l \ldots (n)_{h_0-1} = (\overline{n})_l \ldots (\overline{n})_{h_0-1}\\ \zeta^1 &:= (n+r)_l \ldots (n+r)_{h_0-1} = (\overline{n}+r)_l \ldots (\overline{n}+r)_{h_0-1}\\ \eta &:= (n)_{h_0+1} \ldots (n)_{h_0+d-1} = (n+r)_{h_0+1} \ldots (n+r)_{h_0+d-1}\\ &= (\overline{n})_{h_0+1} \ldots (\overline{n})_{h_0+d-1} = (\overline{n}+r)_{h_0+1} \ldots (\overline{n}+r)_{h_0+d-1}\\ \xi &:= (n)_{h_0+d+1} \ldots (n)_{h_0+k-1} = (n+r)_{h_0+d+1} \ldots (n+r)_{h_0+k-1}\\ &= (\overline{n})_{h_0+d+1} \ldots (\overline{n})_{h_0+k-1} = (\overline{n}+r)_{h_0+d+1} \ldots (\overline{n}+r)_{h_0+k-1}\\ &(n)_{h_0} = (\overline{n})_{h_0} = 0,\ (n+r)_{h_0} = (\overline{n}+r)_{h_0} = 1\\ &(n)_{h_0+d} = (n+r)_{h_0+d} \neq (\overline{n})_{h_0+d} = (\overline{n}+r)_{h_0+d},\end{aligned}$$

we have

$$\mathbb{S}_l = \{\zeta^0 0\eta 0\xi, \zeta^0 0\eta 1\xi, \zeta^1 1\eta 0\xi, \zeta^1 1\eta 1\xi\} \cap P.$$

As is seen in the figure 4, the position $(i', i' + d)$ with $1 \leqslant i' < i' + d \leqslant k$ is included in the sum of the right-hand side of (6.4) if $i' \leqslant h_0 + 1$. Since $i \leqslant h_0 + 1$ and $(i, i + d)$ is the largest odd position, any other odd position $(i', i' + d)$ is included in the sum (6.4). There are an odd number of odd positions and the others are even positions in the sum $\sum \#\mathbb{S}_l$, $\sum_{l=0}^{\infty} K_l$ is odd.

Thus, we have $\gamma_P(r) = 0$ for any $r \geqslant 2^{i-1}$, and α_P is $(2^{i-1} - 1)$-correlated. □

Acknowledgments

The authors thank the referees and the associate editor for their useful suggestions. The second named author is supported partially by NSFC No. 11831007.

References

[1] Allouche J-P and Liardet P 1991 Generalized Rudin–Shapiro sequences *Acta Arith.* **60** 1–27
[2] Allouche J-P and Shallit J 1999 The ubiquitous Prouhet–Thue–Morse sequence *Sequences and Their Applications* (Berlin: Springer) pp 1–16
[3] Boyd D W, Cook J H and Morton P 1989 On sequences of ± 1's defined by binary patterns *Diss. Math.* **283** 1–64
[4] Coquet J, Kamae T and Mendès France M 1977 Sur la mesure spectrale de certaines suites arithmétiques *Bull. Soc. Math. France* **79** 369–84
[5] Morton P 1990 Connections between binary patterns and paperfolding *Journal de Théorie des Nombres de Bordeaux* **2** 1–12
[6] Morton P and Mourant W J 1989 Paper folding, digit patterns and groups of arithmetic fractals *Proc. Lond. Math. Soc.* **3** 253–93
[7] Mauduit C and Sárközy A 1998 On finite pseudorandom binary sequences. II, the Champernowne, Rudin–Shapiro, and Thue–Morse sequences, a further construction *J. Number Theory* **73** 256–76
[8] Mauduit C and Rivat J 2010 Sur un problème de Gelfond: la somme des chiffres des nombres premiers *Ann. Math.* **171** 1591–646
[9] Müllner C and Spiegelhofer L 2017 Normality of the Thue–Morse sequence along Piatetski–Shapiro sequences, II *Isr. J. Math.* **220** 691–738
[10] Petersen K 1983 *Ergodic Theory* (Cambridge: Cambridge University Press)
[11] Peng L and Kamae T 2016 Spectral measure of the Thue–Morse sequence and the dynamical system and random walk related to it *Ergod. Theor. Dyn. Syst.* **36** 1247–59
[12] Spiegelhofer L 2018 The level of distribution of the Thue–Morse sequence (arXiv:1803.01689)
[13] Zheng Y, Peng L and Kamae T 2018 Characterization of noncorrelated pattern sequences and correlation dimensions *Discrete Continuous Dyn.* A **38** 5085–103
[14] Zaks M A, Pikovsky A S and Kurths J 1997 On the correlation dimension of the spectral measure for the Thue–Morse sequence *J. Stat. Phys.* **88** 1387–92
[15] Konieczny J 2021 Algorithmic classification of noncorrelated binary pattern sequences *J. Number Theory* **223** 229–54