A characterization of eventual periodicity

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Abstract

In this article, we show that the Kamae-Xue complexity function for an infinite sequence classifies eventual periodicity completely. We prove that an infinite binary word $x_1x_2\cdots$ is eventually periodic if and only if $\Sigma(x_1x_2\cdots x_n)/n^3$ has a positive limit, where $\Sigma(x_1x_2\cdots x_n)$ is the sum of the squares of all the numbers of occurrences of finite words in $x_1x_2\cdots x_n$, which was introduced by Kamae-Xue as a criterion of randomness in the sense that $x_1x_2\cdots x_n$ is more random if $\Sigma(x_1x_2\cdots x_n)$ is smaller. In fact, it is known that the lower limit of $\Sigma(x_1x_2\cdots x_n)/n^2$ is at least $3/2$ for any sequence $x_1x_2\cdots$, while the limit exists as $3/2$ almost surely for the $(1/2,1/2)$ product measure. For the other extreme, the upper limit of $\Sigma(x_1x_2\cdots x_n)/n^3$ is bounded by $1/3$. There are sequences which are not eventually periodic but the lower limit of $\Sigma(x_1x_2\cdots x_n)/n^3$ is positive, while the limit does not exist.

1 Introduction

In [1], a criterion of randomness for binary words is introduced. As stated in Definitions 1 and 3, let

$$\Sigma(x_1x_2\cdots x_n) = \sum_{\xi\in\cup_{k=1}^{\infty}\{0,1\}^k} |x_1x_2\cdots x_n|_\xi^2,$$

where

$$|x_1x_2\cdots x_n|_\xi := \#\{i : 1 \le i \le n-k+1,\ x_ix_{i+1}\cdots x_{i+k-1} = \xi\}$$

is the number of occurrences of a finite word ξ in $x_1x_2\cdots x_n$. Since the function $f(x) = x^2$ is convex, the value $\sum_{\xi\in\{0,1\}^k} |x_1x_2\cdots x_n|_\xi^2$ for any $k = 1, 2, \cdots$ is smaller if the values $|x_1x_2\cdots x_n|_\xi$ for $\xi \in \{0,1\}^k$ deviate less as a whole from the mean value $(n-k+1)/2^k$, that is, the sequence $x_1x_2\cdots x_n$ is more random. In fact, it is proved in [1] that

$$\liminf_{n\to\infty} \frac{\Sigma(x_1x_2\cdots x_n)}{n^2} \ge \frac{3}{2}$$

holds for any $x_1x_2\cdots \in \{0,1\}^\infty$, while

$$\lim_{n\to\infty} \frac{\Sigma(X_1X_2\cdots X_n)}{n^2} = \frac{3}{2}$$

holds with probability 1 if $X_1X_2\cdots X_n$ is the i.i.d. process with $P(X_i = 0) = P(X_i = 1) = 1/2$.

In this article, we study the opposite case that $\Sigma(x_1x_2\cdots x_n)$ increases in the order of n^3 and prove that $x_1x_2\cdots \in \{0,1\}^\infty$ is eventually periodic if and only if

$$\lim_{n\to\infty} \frac{\Sigma(x_1x_2\cdots x_n)}{n^3} \text{ exists and } > 0.$$

It is easy to see that if $x = x_1x_2\cdots \in \{0,1\}^\infty$ contains few 1s, or precisely speaking, if $x = 0^{k_1}10^{k_2}1\cdots$ with $\liminf_{n\to\infty} k_{n+1}/k_n > 1$, then we have

$$\liminf_{n\to\infty} \frac{\Sigma(x_1x_2\cdots x_n)}{n^3} > 0.$$

Since this $x_1x_2\cdots$ is not eventually periodic, it follows from our result that $\lim_{n\to\infty} \Sigma(x_1x_2\cdots x_n)/n^3$ does not exist.

There are many characterizations of eventual periodicity. The most famous one might be the result due to Morse and Hedlund concerning the complexity. That is, $x_1x_2\cdots$ is eventually periodic if and only if for some $k \ge 1$ the number of words of size k appearing in $x_1x_2\cdots$ is smaller than $k+1$ ([3]). Another characterization concerning the return time is obtained in [2]. Here, we add one more characterization which concerns both the complexity and the return time.

2 Definitions and Lemmas

Definition 1. For $x_1x_2\cdots x_n \in \{0,1\}^n$, $\xi \in \{0,1\}^k$ with $1 \le k \le n$ and $i = 0,1,\cdots,n-k$, we denote

$$\xi \prec_i x_1x_2\cdots x_n \text{ if } \xi = x_{i+1}x_{i+2}\cdots x_{i+k}$$

and

$$\xi \prec x_1x_2\cdots x_n \text{ if } \xi \prec_i x_1x_2\cdots x_n \text{ for some } i = 0,1,\cdots,n-k.$$

We call ξ a *factor* or *suffix* of $x_1x_2\cdots x_n$, respectively, if $\xi \prec x_1x_2\cdots x_n$ or $\xi \prec_{n-k} x_1x_2\cdots x_n$. We also denote

$$|x_1x_2\cdots x_n|_\xi = \#\{i : 0 \le i \le n-k,\ \xi \prec_i x_1x_2\cdots x_n\}$$

and $|x_1x_2\cdots x_n| = n$.

Definition 2. For $\eta = a_1 \cdots a_k \in \{0,1\}^k$ and $\ell = 1, 2, \cdots$, we denote

$$\eta^\ell = \underbrace{a_1 \cdots a_k}_{1} \underbrace{a_1 \cdots a_k}_{2} \cdots \underbrace{a_1 \cdots a_k}_{\ell}.$$

In the same way, we define $\eta^\infty \in \{0,1\}^\infty$. We call η *primitive* if there is no ξ such that $\eta = \xi^\ell$ for some $\ell \geq 2$.

Definition 3 ([1])**.** Define $\Sigma^n : \{0,1\}^n \to \mathbb{R}$ by

$$\Sigma^n(x_1 x_2 \cdots x_n) = \sum_{\xi \in \{0,1\}^+} |x_1 x_2 \cdots x_n|_\xi^2,$$

where $\{0,1\}^+ = \bigcup_{k=1}^\infty \{0,1\}^k$. We write $\Sigma^n = \Sigma$ as a function from $\{0,1\}^+$ to $\mathbb{R}$.

Definition 4. For $x_1 x_2 \cdots x_n \in \{0,1\}^n$, define

$$\Lambda(x_1 x_2 \cdots x_n) = \max\{|\eta|^2 (\ell+1)^3 : \eta^\ell \prec x_1 x_2 \cdots x_n\}$$

Lemma 1. *For any $x_1 x_2 \cdots x_n \in \{0,1\}^n$, it holds that*

$$\Sigma(x_1 x_2 \cdots x_n) \geq \frac{\Lambda(x_1 x_2 \cdots x_n)}{48}.$$

Proof Let $M = \Lambda(x_1 x_2 \cdots x_n)$. Then, there exist positive integers k, ℓ and $\eta \in \{0,1\}^k$ with $\eta^\ell \prec x_1 x_2 \cdots x_n$ such that $k^2(\ell+1)^3 = M$. Then, we have

$$\sum_{\xi;\ \xi \prec \eta} |\eta^\ell|_\xi^2 \geq \ell \sum_{\xi;\ \xi \prec \eta} |\eta^\ell|_\xi \geq \frac{k^2 \ell^2}{2},$$

since $|\eta^\ell|_\xi \geq \ell$ if $\xi \prec \eta$ and $\sum_{\xi;\ \xi \prec \eta} |\eta^\ell|_\xi \geq k^2 \ell / 2$. In the same way, for any $i = 1, \cdots, \ell - 1$, we have

$$\sum_{\xi;\ \xi \not\prec \eta^i \text{ and } \xi \prec \eta^{i+1}} |\eta^\ell|_\xi^2 \geq \frac{k^2 (\ell - i)^2}{2}.$$

Therefore, we have

$$\begin{aligned} \Sigma(x_1 x_2 \cdots x_n) &\geq \sum_{i=0}^{\ell-1} \sum_{\xi;\ \xi \not\prec \eta^i \text{ and } \xi \prec \eta^{i+1}} |\eta^\ell|_\xi^2 \\ &\geq \sum_{i=0}^{\ell-1} \frac{k^2(\ell-i)^2}{2} \geq \frac{k^2 \ell^3}{6} \geq \frac{k^2(\ell+1)^3}{48} = \frac{M}{48}. \end{aligned}$$

□

Lemma 2. *For any* $x_1x_2\cdots \in \{0,1\}^\infty$,

$$\liminf_{n\to\infty} \frac{\Sigma(x_1x_2\cdots x_n)}{n^3} > 0 \text{ if and only if } \liminf_{n\to\infty} \frac{\Lambda(x_1x_2\cdots x_n)}{n^3} > 0,$$

and

$$\limsup_{n\to\infty} \frac{\Sigma(x_1x_2\cdots x_n)}{n^3} > 0 \text{ if and only if } \limsup_{n\to\infty} \frac{\Lambda(x_1x_2\cdots x_n)}{n^3} > 0.$$

Proof By Lemma 1, the "if" parts are clear. Let us prove the "only if" parts. Let $M_n = \Lambda(x_1x_2\cdots x_n)$. Assume that there exist i, m, k with $1 \le i+1 < i+k+1 < i+m < i+k+m \le n$ such that

$$x_{i+1}x_{i+2}\cdots x_{i+m} = x_{i+k+1}x_{i+k+2}\cdots x_{i+k+m}.$$

Let k be the minimum with this property. Let $\eta = x_{i+1}x_{i+2}\cdots x_{i+k}$ and $\ell = \lfloor m/k \rfloor$. Then, $k^2(\ell+1)^3 \le M_n$ holds since $\eta^\ell \prec x_{i+1}x_{i+2}\cdots x_{i+m}$. Hence, $k \ge (k(\ell+1))^3/M_n > m^3/M_n$. It follows that $|x_1x_2\cdots x_n|_\xi \le n/(m^3/M_n)$ for any $\xi \in \{0,1\}^m$. Therefore for any $1 \le m \le n$,

$$\begin{aligned}\sum_{\xi\in\{0,1\}^m} |x_1x_2\cdots x_n|_\xi^2 &\le \frac{n}{m^3/M_n} \sum_{\xi\in\{0,1\}^m} |x_1x_2\cdots x_n|_\xi \\ &\le \frac{n}{m^3/M_n}\cdot n = \frac{n^2M_n}{m^3}.\end{aligned} \tag{2.1}$$

By (2.1), we have

$$\begin{aligned}\frac{\Sigma(x_1x_2\cdots x_n)}{n^3} &= \frac{1}{n^3}\sum_{m=1}^{n}\sum_{\xi\in\{0,1\}^m} |x_1x_2\cdots x_n|_\xi^2 \\ &\le \frac{1}{n^3}\sum_{1\le m\le M_n^{1/3}} (n-m)^2 + \frac{1}{n^3}\sum_{M_n^{1/3}<m} \frac{n^2M_n}{m^3} \\ &\le \frac{1}{n^3}\cdot M_n^{1/3}\cdot n^2 + \frac{M_n}{n}\cdot\frac{1}{2(M_n^{1/3}-1)^2} \\ &= \left(\frac{M_n}{n^3}\right)^{1/3}\left(1+\frac{1}{2\left(1-M_n^{-1/3}\right)^2}\right).\end{aligned}$$

Therefore, the upper (lower) limit of M_n/n^3 is 0 implies that the upper (respectively, lower) limit of $\Sigma(x_1x_2\cdots x_n)/n^3$ is 0. □

Definition 5. For $\omega \in \{0,1\}^n$, $\xi \in \{0,1\}^k$ with $k \le n$ and $m = 1, 2, \cdots, n$, we denote

$$|\omega|_{\xi,m} = \#\{i;\ n-m-k+1 \le i \le n-k,\ \xi \prec_i \omega\}.$$

Lemma 3. *Let $\omega \in \{0,1\}^n$ and $\eta \in \{0,1\}^m$ with $n, m \geq 1$. Then, we have*

$$\Sigma(\omega\eta) - \Sigma(\omega) = \sum_{\xi \in \{0,1\}^+} 2|\omega\eta|_{\xi,m}|\omega|_\xi + |\omega\eta|_{\xi,m}^2.$$

Proof Clear from the fact that $|\omega\eta|_\xi = |\omega|_\xi + |\omega\eta|_{\xi,m}$. □

Lemma 4. *Let $\omega \in \{0,1\}^n$ and $\eta \in \{0,1\}^k$ satisfy that $|\omega\eta^\ell|_{\eta^\ell} = 1$. Assume that η is primitive and $\omega_n \neq \eta_k$ (i.e., the last letters of ω and η are different). Then, for $\ell = 2, 3, \cdots$, we have*

$$0 \leq \Sigma(\omega\eta^{\ell+2}) - 2\Sigma(\omega\eta^{\ell+1}) + \Sigma(\omega\eta^\ell) - 2k^2\ell < 2k^4 + 3k. \tag{2.2}$$

Proof Put $\sigma = \omega\eta^\ell$. Denote

$$\begin{aligned}\Sigma(\sigma\eta^2) - 2\Sigma(\sigma\eta) + \Sigma(\sigma) = &\sum_{\substack{\xi \in \{0,1\}^+;\\ \xi \prec \eta^{\ell+2}, |\xi| \geq k}} |\sigma\eta^2|_\xi^2 - 2|\sigma\eta|_\xi^2 + |\sigma|_\xi^2 \\ &+ \sum_{\substack{\xi \in \{0,1\}^+;\\ \xi \prec \eta^{\ell+2}, |\xi| < k}} |\sigma\eta^2|_\xi^2 - 2|\sigma\eta|_\xi^2 + |\sigma|_\xi^2 \\ &+ \sum_{\substack{\xi \in \{0,1\}^+;\\ \xi \not\prec \eta^{\ell+2}}} |\sigma\eta^2|_\xi^2 - 2|\sigma\eta|_\xi^2 + |\sigma|_\xi^2 \\ =: &\, S_1 + S_2 + S_3.\end{aligned}$$

Let $\xi \prec \eta^{\ell+1}$ with $|\xi| \geq k$. Since η is primitive, if $\xi \prec_i \eta^{\ell+1}$, then $\xi \prec_j \eta^{\ell+1}$ holds if and only if $i \equiv j \pmod{k}$ and $j + |\xi| \leq |\eta^{\ell+1}|$. Therefore, $|\sigma\eta|_\xi - |\sigma|_\xi = |\sigma\eta|_{\xi,k} = 1$. Hence, $|\sigma\eta|_\xi^2 - |\sigma|_\xi^2 = 2|\sigma|_\xi + 1$. In the same way, $|\sigma\eta^2|_\xi^2 - |\sigma\eta|_\xi^2 = 2|\sigma\eta|_\xi + 1$. Thus,

$$|\sigma\eta^2|_\xi^2 - 2|\sigma\eta|_\xi^2 + |\sigma|_\xi^2 = 2(|\sigma\eta|_\xi - |\sigma|_\xi) = 2.$$

If $\xi \prec \eta^{\ell+2}$ but not $\xi \prec \eta^{\ell+1}$, then by the assumptions that $|\omega\eta^\ell|_{\eta^\ell} = 1$, η is primitive and $\omega_n \neq \eta_k$, $|\sigma\eta^2|_\xi = 1$ and $|\sigma\eta|_\xi = |\sigma|_\xi = 0$ hold. Hence,

$$|\sigma\eta^2|_\xi^2 - 2|\sigma\eta|_\xi^2 + |\sigma|_\xi^2 = 1.$$

Therefore,

$$S_1 = 2((\ell k + 1) + \ell k + \cdots + (\ell k - k + 2)) + k^2 = 2k^2\ell + 3k,$$

since the number of $\xi \prec \eta^{\ell+1}$ with $|\xi| \geq k$ is equal to the number of pairs of positions $(i, j) \in \{1, 2, \cdots, (\ell+1)k\}^2$ in $\eta^{\ell+1}$ with $1 \leq i \leq k$ and $j - i \geq k$. Also, the number of ξ with $\xi \not\prec \eta^{\ell+1}$ and $\xi \prec \eta^{\ell+2}$ is equal to the number of pairs of positions $(i, j) \in \{1, 2, \cdots, (\ell+2)k\}^2$ in $\eta^{\ell+2}$ with $1 \leq i \leq k$ and $(\ell+1)k + 1 \leq j \leq (\ell+2)k$.

Let $\xi \prec \eta^{\ell+2}$ with $|\xi| < k$. Then, $1 \le |\sigma|_{\xi,k} = |\sigma\eta|_{\xi,k} = |\sigma\eta^2|_{\xi,k} \le k$ and $|\sigma\eta^2|_\xi - |\sigma\eta|_\xi = |\sigma\eta|_\xi - |\sigma|_\xi = |\sigma|_{\xi,k}$. Hence,

$$\begin{aligned}
|\sigma\eta^2|_\xi^2 - 2|\sigma\eta|_\xi^2 + |\sigma|_\xi^2 &= (|\sigma\eta^2|_\xi^2 - |\sigma\eta|_\xi^2) - (|\sigma\eta|_\xi^2 - |\sigma|_\xi^2) \\
&= (2|\sigma\eta|_\xi|\sigma\eta^2|_{\xi,k} + |\sigma\eta|_{\xi,k}^2) - (2|\sigma|_\xi|\sigma\eta|_{\xi,k} + |\sigma|_{\xi,k}^2) \\
&= 2(|\sigma\eta|_\xi - |\sigma|_\xi)|\sigma|_{\xi,k} = 2|\sigma|_{\xi,k}^2.
\end{aligned}$$

Therefore, $0 \le S_2 < 2k^4$.

If $\xi \prec \sigma$ with $\xi \not\prec \eta^{\ell+2}$, then it holds that $|\sigma|_\xi = |\sigma\eta|_\xi$ since $|\sigma\eta|_{\xi,k} = 0$ by the assumptions that $|\omega\eta^\ell|_{\eta^\ell} = 1$, η is primitive and $\omega_n \ne \eta_k$. If $\xi \not\prec \sigma$, $\xi \prec \sigma\eta$ and $\xi \not\prec \eta^{\ell+2}$, then we have $|\sigma|_\xi = 0$ and $|\sigma\eta|_\xi = 1$. Hence,

$$\sum_{\substack{\xi\in\{0,1\}^+;\\ \xi\not\prec\eta^{\ell+2}}} (|\sigma\eta|_\xi^2 - |\sigma|_\xi^2) = \#\{\xi \in \{0,1\}^+;\ \xi \not\prec \eta^{\ell+2},\ \xi \prec \sigma\eta,\ \xi \not\prec \sigma\} = kn.$$

In the same way, we have

$$\sum_{\substack{\xi\in\{0,1\}^+;\\ \xi\not\prec\eta^{\ell+2}}} (|\sigma\eta^2|_\xi^2 - |\sigma\eta|_\xi^2) = kn.$$

Therefore, we have $S_3 = 0$.

Thus, we have

$$0 \le \Sigma(\sigma\eta^2) - 2\Sigma(\sigma\eta) + \Sigma(\sigma) - 2k^2\ell < 2k^4 + 3k.$$

□

Lemma 5. *Assume that*

$$\limsup_{n\to\infty} \frac{\Sigma(x_1x_2\cdots x_n)}{n^3} > 0.$$

Then, there exists a primitive $\eta \in \{0,1\}^+$ *and* $0 \le \ell_1 \le \ell_2 \le \cdots$ *such that*

$$\eta^{\ell_n} \prec x_1x_2\cdots x_n \quad \text{and} \quad \limsup_{n\to\infty} \frac{\ell_n}{n} > 0.$$

Proof By Lemma 2, we have

$$\limsup_{n\to\infty} \frac{\Lambda(x_1x_2\cdots x_n)}{n^3} > 0.$$

Hence, there exist $\eta_n \in \{0,1\}^+$ and h_n for any sufficiently large n with ${\eta_n}^{h_n} \prec x_1x_2\cdots x_n$ such that

$$\limsup_{n\to\infty} \frac{|\eta_n|^2 h_n^3}{n^3} \ge \limsup_{n\to\infty} \frac{|\eta_n|^2(h_n+1)^3}{8n^3} > 0.$$

Since $|\eta_n|^2 h_n^3/n^3 \leq 1/|\eta_n|$, $\liminf_{n\to\infty} |\eta_n| < \infty$. Therefore, there exist $\eta \in \{0,1\}^+$ and $0 \leq \ell_1 \leq \ell_2 \leq \cdots$ such that $\eta^{\ell_n} \prec x_1 x_2 \cdots x_n$ and

$$\limsup_{n\to\infty} \frac{\ell_n}{n} = \frac{1}{|\eta|^{2/3}} \left(\limsup_{n\to\infty} \frac{|\eta|^2 \ell_n^3}{n^3} \right)^{1/3} > 0.$$

If η is not primitive and $\eta = \xi^p$ with a primitive ξ, we may replace η by ξ and ℓ_n by $p\ell_n$. □

3 Main results

Theorem 1. *If $x = x_1 x_2 \cdots$ is eventually periodic with minimal period k, then it holds that*

$$\lim_{n\to\infty} \frac{\Sigma(x_1 x_2 \cdots x_n)}{n^3} = \frac{1}{3k}.$$

Proof Let $\eta \in \{0,1\}^k$ be primitive with $k \geq 1$. Let $x = \zeta\eta^\infty$ with $\zeta \in \{0,1\}^+ \cup \{\emptyset\}$, where $\emptyset$ is the empty word. Let $|\zeta| = h$. Then, for any $\xi \in \{0,1\}^+$ with $|\xi| = \ell$, we have

$$0 \leq |\zeta\eta^n|_\xi - |\eta^n|_\xi \leq h.$$

Hence,

$$0 \leq |\zeta\eta^n|_\xi^2 - |\eta^n|_\xi^2 \leq h(|\zeta\eta^n|_\xi + |\eta^n|_\xi).$$

Therefore, we have

$$\begin{aligned}
0 &\leq \sum_{\xi\in\{0,1\}^\ell} |\zeta\eta^n|_\xi^2 - \sum_{\xi\in\{0,1\}^\ell} |\eta^n|_\xi^2 \\
&\leq \begin{cases} h((h + kn - \ell + 1) + (kn - \ell + 1)) & \text{for } 1 \leq \ell \leq kn, \\ h(h + kn - \ell + 1) & \text{for } kn + 1 \leq \ell \leq h + kn, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Hence,

$$0 \leq \Sigma(\zeta\eta^n) - \Sigma(\eta^n) \leq h(h + 2kn)(h + kn).$$

Thus,

$$\lim_{n\to\infty} \frac{\Sigma(\zeta\eta^n)}{n^3} = \lim_{n\to\infty} \frac{\Sigma(\eta^n)}{n^3}$$

holds in the sense that if the limit exists in one side, then the limit exists in the other side, and they coincide. Now, we prove that

$$\lim_{n\to\infty} \frac{\Sigma(\eta^n)}{n^3} = \frac{k^2}{3},$$

which will complete the proof.

Assume that $|\xi| \geq k$ and $\xi \prec_i \eta^n$. Since η is primitive, $\xi \prec_j \eta^n$ holds if and only if $i \equiv j \pmod{k}$ and $0 \leq j \leq |\eta^n| - |\xi|$. Hence, for $\xi \prec \eta^n$ such that $|\xi| \geq k$, we have

$$-1 \leq |\eta^n|_\xi - (n - |\xi|/k) \leq 1.$$

Therefore, it holds that

$$\begin{aligned}
&\left| \sum_{\xi \in \{0,1\}^+;\ \xi \prec \eta^n,\ |\xi| \geq k} |\eta^n|_\xi^2 \ - \sum_{\xi \in \{0,1\}^+;\ \xi \prec \eta^n,\ |\xi| \geq k} \left(n - \frac{|\xi|}{k}\right)^2 \right| \\
&= \left| \sum_{\xi \in \{0,1\}^+;\ \xi \prec \eta^n,\ |\xi| \geq k} \left(|\eta^n|_\xi \ - \left(n - \frac{|\xi|}{k}\right)\right) \left(|\eta^n|_\xi \ + \left(n - \frac{|\xi|}{k}\right)\right) \right| \\
&\leq \sum_{\xi \in \{0,1\}^+;\ \xi \prec \eta^n,\ |\xi| \geq k} \left(|\eta^n|_\xi \ + \left(n - \frac{|\xi|}{k}\right)\right) \\
&\leq \sum_{\xi \in \{0,1\}^+;\ \xi \prec \eta^n,\ |\xi| \geq k} (2|\eta^n|_\xi \ + 1) \leq 2(kn)^2 + k(kn) \leq 3(kn)^2.
\end{aligned}$$

On the other hand, if $\xi \prec \eta^n$ and $|\xi| < k$, then we have $1 \leq |\eta^n|_\xi \leq kn$ and there are at most k^2 such ξ's. Therefore,

$$0 \leq \sum_{\xi \in \{0,1\}^+;\ \xi \prec \eta^n,\ |\xi| < k} |\eta^n|_\xi^2 = \Sigma(\eta^n) - \sum_{\xi \in \{0,1\}^+;\ \xi \prec \eta^n,\ |\xi| \geq k} |\eta^n|_\xi^2 \leq k^2(kn)^2.$$

Thus,

$$\lim_{n\to\infty} \frac{\Sigma(\eta^n)}{n^3} = \lim_{n\to\infty} \frac{1}{n^3} \sum_{\xi \in \{0,1\}^+;\ \xi \prec \eta^n,\ |\xi| \geq k} \left(n - \frac{|\xi|}{k}\right)^2.$$

Here, ξ as above corresponds to the pair (i, j), where i is the smallest i such that $\xi \prec_{i-1} \eta^n$ and $|\xi| = j$. This correspondence gives a bijection between the set of ξ as above and the set

$$\{(i, j) \in \{1, 2, \cdots, k\} \times \{k, k+1, \cdots, kn\};\ i + j - 1 \leq kn\}.$$

Hence, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{\Sigma(\eta^n)}{n^3} &= \lim_{n\to\infty}\frac{1}{n^3}\sum_{\xi\in\{0,1\}^+;\ \xi\prec\eta^n,\ |\xi|\geq k}\left(n-\frac{|\xi|}{k}\right)^2 \\
&= \lim_{n\to\infty}\frac{1}{n^3}\sum_{(i,j)\in\{1,2,\cdots,k\}\times\{k,k+1,\cdots,kn\};\ i+j-1\leq kn}\left(n-\frac{j}{k}\right)^2 \\
&= \lim_{n\to\infty}\frac{k}{n^3}\sum_{j\in\{k,k+1,\cdots,kn\}}\left(n-\frac{j}{k}\right)^2 \\
&= k^2\lim_{n\to\infty}\sum_{j\in\{1,2,\cdots,kn\}}\left(1-\frac{j}{kn}\right)^2\frac{1}{kn} \\
&= k^2\int_0^1(1-x)^2dx = \frac{k^2}{3}.
\end{aligned}$$

□

Theorem 2. *The infinite word $x_1x_2\cdots$ is eventually periodic if and only if $\lim_{n\to\infty}\Sigma(x_1x_2\cdots x_n)/n^3$ exists and is positive.*

Proof The "only if" part is proved in Theorem 1. Let us prove the "if" part. Suppose that $\lim_{n\to\infty}\Sigma(x_1x_2\cdots x_n)/n^3$ exists and takes a positive value, but $x_1x_2\cdots$ is not eventually periodic. By Lemma 5, there exist $k\geq 1$, $\eta=a_1a_2\cdots a_k\in\{0,1\}^k$ and $0\leq \ell_1\leq \ell_2\leq\cdots$ such that

$$\eta^{\ell_n}\prec x_1x_2\cdots x_n \quad\text{and}\quad A:=\limsup_{n\to\infty}\frac{\ell_n}{n}>0.$$

Here, we may also assume that η is primitive.

Take a subsequence $\{N\}$ of $\{1,2,\cdots\}$ and replace η by $a_i\cdots a_ka_1\cdots a_{i-1}$ for some i with $1\leq i\leq k$ if necessary, we may assume that η^{ℓ_N} is a suffix of $x_1x_2\cdots x_N$ and $x_{N-k\ell_N}\neq a_k$. Since $x_1x_2\cdots$ is not eventually periodic, we may also assume that $N-k\ell_N\to\infty$ as $N\to\infty$. Note that $kA\leq 1$.

Take $\delta>0$ with $1-kA<\delta<1$. Take ϵ with $0<\epsilon<1/2$ such that $(1-kA(1-\epsilon))(1+\epsilon)/(1-\epsilon)<\delta<1$. Take a sufficiently large N such that $\delta\ell_N\geq 2$ and $\ell_N/N>A(1-\epsilon)$ together with other requirements specified later.

We assume that $N-k\ell_N$ is sufficiently large. Denote $n=N-k\ell_N$ and $\omega=x_1x_2\cdots x_n$. Then, $x_n\neq a_k$. Since n is sufficiently large, we may assume that $\ell_n/n<A(1+\epsilon)$. Hence,

$$\begin{aligned}
\ell_n &< A(1+\epsilon)n = A(1+\epsilon)(N-k\ell_N) \\
&< A(1+\epsilon)(N-kAN(1-\epsilon)) = (1+\epsilon)(1-kA(1-\epsilon))AN \\
&< (1+\epsilon)(1-kA(1-\epsilon))\frac{\ell_N}{1-\epsilon} < \delta\ell_N.
\end{aligned}$$

Take integers ℓ and ℓ' as functions of N such that

(1) $\delta^{-1}\ell_n < \ell < \ell + 2\ell' < \ell_N$ and

(2) ℓ/n and ℓ'/n are bounded away both from 0 and ∞.

Since $x_n \neq a_k$ and η is primitive, $|x\eta^\ell|_{\eta^\ell} \geq 2$ is possible only if $|x\eta|_{\eta^\ell} \geq 1$, and hence, only if $|x|_{\eta^{\ell-1}} \geq 1$. This is impossible since $\ell_n < \delta\ell < \ell - 1$ as $\delta < 1$ and N is sufficiently large. Thus, the assumptions in Lemma 4 are satisfied.

Adding (2.2) with replacing ℓ by $\ell, \ell+1, \cdots, \ell+\ell'-1$, we have

$$\begin{aligned} &\Sigma(\omega\eta^{\ell+\ell'+1}) - \Sigma(\omega\eta^{\ell+\ell'}) - \Sigma(\omega\eta^{\ell+1}) + \Sigma(\omega\eta^{\ell}) \\ &= 2k^2(\ell + (\ell+1) + \cdots + (\ell+\ell'-1)) + \ell' R \\ &= k^2\ell'(2\ell + \ell' - 1) + \ell' R \end{aligned} \tag{3.1}$$

for some R with $0 \leq R < 2k^4 + 3k$.

We further add (3.1) for the pairs $(\ell, \ell'), (\ell+1, \ell'), \cdots, (\ell+\ell'-1, \ell')$ in place of (ℓ, ℓ'), we get

$$\begin{aligned} \Sigma(\omega\eta^{\ell+2\ell'}) - 2\Sigma(\omega\eta^{\ell+\ell'}) + \Sigma(\omega\eta^{\ell}) &= \sum_{i=\ell}^{\ell+\ell'-1} \left(k^2\ell'(2i + \ell' - 1) + \ell' R_i\right) \\ &= 2k^2\ell'^2(\ell + \ell' - 1) + \ell'^2 \bar{R} \end{aligned} \tag{3.2}$$

with some $\bar{R}$, $0 \leq \bar{R} < 2k^4 + 3k$.

Taking a subsequence $\{n'\}$ of $\{n\}$ if necessary and denoting $\{n'\}$ by $\{n\}$, we may assume that $\lim_{n\to\infty} k\ell/n = \alpha > 0$ and $\lim_{n\to\infty} k\ell'/n = \beta > 0$. By the assumption

$$L := \lim_{h\to\infty} \frac{\Sigma(x_1 x_2 \cdots x_h)}{h^3} > 0$$

holds for $h = n + k(\ell + 2\ell')$, $h = n + k(\ell + \ell')$ and $h = n + k\ell$. Dividing (3.2) by n^3 and letting $n \to \infty$, we have

$$L(1 + \alpha + 2\beta)^3 - 2L(1 + \alpha + \beta)^3 + L(1 + \alpha)^3 = \frac{2\alpha\beta^2}{k} + \frac{2\beta^3}{k}.$$

Since ℓ, ℓ' can be arbitrary satisfying (1), (2) above, this should holds for any $\alpha, \beta > 0$ with $\alpha + 2\beta < A(1 - \epsilon)$, which is impossible since the left side is $6L((1 + \alpha)\beta^2 + \beta^3)$ and has a term of β^2 which the right side does not have. □

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