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# On maximal pattern complexity of some automatic words

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*Abstract*. The pattern complexity of a word for a given pattern $S$, where $S$ is a finite subset of $\{0, 1, 2, \ldots\}$, is the number of distinct restrictions of the word to $S + n$ (with $n = 0, 1, 2, \ldots$). The maximal pattern complexity of the word, introduced in the paper of T. Kamae and L. Zamboni [Sequence entropy and the maximal pattern complexity of infinite words. *Ergod. Th. & Dynam. Sys.* **22**(4) (2002), 1191–1199], is the maximum value of the pattern complexity of $S$ with $\#S = k$ as a function of $k = 1, 2, \ldots$. A substitution of constant length on an alphabet is a mapping from the alphabet to finite words on it of constant length not less than two. An infinite word is called a fixed point of the substitution if it stays the same after the substitution is applied. In this paper, we prove that the maximal pattern complexity of a fixed point of a substitution of constant length on $\{0, 1\}$ (as a function of $k = 1, 2, \ldots$) is either bounded, a linear function of $k$, or $2^k$.

## 1. *Introduction*

Let $\Sigma$ be a finite alphabet, and let $x = x_0x_1 \cdots \in \Sigma^{\mathbb{N}}$ be an infinite word on $\Sigma$ with indices in $\mathbb{N} := \{0, 1, 2, \ldots\}$.

A *(k-)window* $T = \{t_0 < t_1 < \cdots < t_{k-1}\}$ is a subset of $\mathbb{N}$. A word $u$ is said to be a $T$*-factor* or $T$*-subword* of a word $v = v_0v_1 \cdots$ if $u = v_{i+t_0}v_{i+t_1} \cdots v_{i+t_k}$ for some $i$. A word $u$ is called simply a *factor* or *subword* of a word $v = v_0v_1 \cdots$ if $u = v_iv_{i+1} \cdots v_{i+n}$ for some $i$ and $n$. In both cases, we say that the subword (or $T$-subword) $u$ *occurs* in $v$ at the position $i$. A subword occurring in a word at the position 0 is called a *prefix* of that word. An infinite word is said to be *recurrent* if each of its subwords occurs in it at least twice. The *length* of the word $u = u_0u_1 \cdots u_{n-1}$ is $n$ and is denoted by $|u|$.

Let us denote the set of $T$-factors (respectively, factors) of a word $x$ by $P_x(T)$ (respectively, $F_x$) and the set of all finite words on $\Sigma$ by $\Sigma^*$. A classical complexity measure for an infinite (or finite) word $x$ on $\Sigma$ is the *subword complexity*, that is, the function $f_x(n) = |F_x \cap \Sigma^n|$. A survey on subword complexity can be found in [**5**].

We consider another complexity measure, namely the function $p^*_x(n) = \sup_{|T|=n} |P_x(T)|$ introduced in [**10**].

A *substitution* $\varphi : \Sigma^* \to \Sigma^*$ is a mapping that obeys the identity $\varphi(xy) = \varphi(x)\varphi(y)$ for all words $x,\ y \in \Sigma^*$. It is determined by the values $\varphi(a)$ for $a \in \Sigma$. A substitution is *binary* if $\Sigma = \{0, 1\}$. A substitution $\varphi : \Sigma^* \to \Sigma^*$ is *expanding* if for each $a \in \Sigma$ the inequality $|\varphi(a)| > 1$ holds.

Any expanding substitution $\varphi : \Sigma^* \to \Sigma^*$ generates the mapping $\Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}}$ that maps an infinite word $x = x_0x_1 \cdots$ to the word $\varphi(x_0)\varphi(x_1)\cdots$. We denote this mapping by the same letter $\varphi$ for convenience. When $\varphi(a)$ begins with $a$ for some $a \in \Sigma$, this mapping has a fixed point, i.e. an infinite word that satisfies $x = \varphi(x)$; it is exactly the word obtained as the limit $\lim_{n\to\infty} \varphi^n(a) = \varphi^\omega(a)$.

A substitution is said to be *of constant length* if the images of all letters are of the same length which is not less than two. Hence, such a substitution is expanding.

*Example 1.* The Sierpiński (Cantor's) word $s = 0101110101111\cdots$ is a fixed point of the substitution $\varphi$ of constant length three, where $\varphi_S$ on $\{0, 1\}$ is defined by $\varphi_S(0) = 010$ and $\varphi_S(1) = 111$.

Owing to Cobham's theorem [**1**], fixed points of substitutions of constant length form a subclass of automatic words.

In this paper, we show that the maximal pattern complexity of a fixed point of a binary substitution of constant length can be classified as either bounded, linear or $2^n$ as a function of $n = 1, 2, \ldots$. We also give an easy criterion to distinguish the different cases.

## 2. *Preliminaries*

Let $u$ be a finite word, and let $u^\omega$ denote its infinite concatenation, i.e. the word $uuu\cdots$. We say that an infinite word is *periodic* if it is of the form $pu^\omega$ for some finite words $p$ and $u$; an infinite word is *non-periodic* if it is not periodic. Note that a non-periodic fixed point of a binary substitution is recurrent.

The following is a known fact from [**10**].

LEMMA 1. *The maximal pattern complexity of a periodic word is bounded.*

Let us denote by $\overline{u}$ the infinite word obtained by exchanging zeros and ones in a word $u$ on the binary alphabet. A substitution $\varphi : \{0, 1\}^* \to \{0, 1\}^*$ of constant length is said to be *symmetric* if $\varphi(0) = \overline{\varphi(1)}$.

The *arithmetical complexity* $a_x(n)$ of an infinite word $x$ is the number of distinct words of the form $x_kx_{k+d}\cdots x_{k+(n-1)d}$ for arbitrary initial positions $k \geq 0$ and differences $d \geq 1$ (see [**3**]). We say that a binary infinite word is *arithmetically universal* (respectively, *pattern universal*) if its arithmetical complexity (respectively, maximal pattern complexity) is $2^n$.

The next theorem was proved in [**6**].

THEOREM 1. *Non-periodic fixed points of symmetric binary substitutions are arithmetically universal.*

A direct consequence of this theorem is the following lemma.

LEMMA 2. *Non-periodic fixed points of symmetric binary substitutions are pattern universal.*

*Proof.* Let $x$ be such a fixed point. Consider a sequence of words $\{u^{(n)}\}_{n=1,2,\dots}$ such that $f_{u^{(n)}}(n) = 2^n$ for each $n$. Since $x$ is arithmetically universal, $u^{(n)} = x_{k_n} x_{k_n+d_n} \cdots x_{k_n+l_n d_n}$ for some $k_n$, $d_n$ and $l_n$. For each $n$ and the pattern $T^n = \{0, d_n, 2d_n, \dots, (n-1)d_n\}$, we have $p_x(T^n) = 2^n$. Therefore, $p_x^*(n) \equiv 2^n$. □

*Example 2.* The Thue–Morse word $x_{\mathrm{TM}} = 01101001100101\cdots$ is a fixed point of the binary symmetric substitution $\varphi_{\mathrm{TM}}$ defined by $\varphi_{\mathrm{TM}}(0) = 01$ and $\varphi_{\mathrm{TM}}(1) = 10$. By Lemma 2, it is pattern universal.

Let $x$ and $y = y_0 y_1 \cdots$ be infinite words on $\Sigma \cup \{?\}$. Let us denote by $x \triangleleft y$ the result of replacing the $i$th occurrence of ? in $x$ by $y_i$. *Classical Toeplitz words* [**8**] are words of the form $\xi^\omega \triangleleft \xi^\omega \triangleleft \xi^\omega \triangleleft \cdots$ where $\xi$ is a finite word on $\Sigma \cup \{?\}$ not starting with '?'.

Subword and arithmetical complexities of Toeplitz words were studied in [**2, 4**]. The following statement is a simplified version of the theorem proved in [**7**].

THEOREM 2. *If a finite word $\xi = \xi_0\xi_1 \cdots \xi_{r-1}$ on $\Sigma \cup \{?\}$ with $r \geq 2$ and $\xi_0 \neq ?$ contains exactly one ?, then there exists $c \geq 0$ such that*

$$\lim_{k\to\infty} \frac{p_x^*(n)}{n} = c$$

*for a Toeplitz word $x = \xi^\omega \triangleleft \xi^\omega \triangleleft \cdots$. Moreover, $c$ is such that*

$$c = \#\Sigma + \max_{\substack{L \subset \{0,1,\dots,r-1\} \\ \ell \geq 2}} \frac{E(\xi, L)}{\ell - 1} \tag{1}$$

*with $\ell = \#L$ and*

$$E(\xi, L) = \#(\pi_\Sigma P_{\xi^\omega}(L) \cup \{a^\ell : a \in \Sigma\}) - \ell\#\Sigma,$$

*where for a set $S \subset (\Sigma \cup \{?\})^*$, $\pi_\Sigma S$ is the subset of $\Sigma^*$ containing all elements obtained from some element in $S$ by replacing each occurrence of the letter ? by an arbitrary letter in $\Sigma$.*

*Example 3.* Let $\xi = 01?$ and $x = \xi^\omega \triangleleft \xi^\omega \triangleleft \cdots$. Let $L = \{0, 1, 2\}$. Then

$$P_{\xi^\omega}(L) = \{01?, 1?0, ?01\} \quad \text{and} \quad \pi_{\{0,1\}} P_{\xi^\omega}(L) = \{010, 011, 100, 110, 001, 101\}.$$

Hence, $\pi_{\{0,1\}} P_{\xi^\omega}(L) \cup \{000, 111\} = \{0, 1\}^3$ and $E(\xi, L) = \#\{0, 1\}^3 - 3 \times 2 = 2$. Therefore, $\#\{0, 1\} + E(\xi, L)/(\ell - 1) = 3$. Moreover, if $\#L = 2$, then $E(\xi, L) \leq \#\{0, 1\}^2 - 2 \times 2 = 0$. Thus, $c = 3$ in (1) and $\lim_{n\to\infty} p_x^*(n)/n = 3$. This $x$ is also a fixed point of the substitution $0 \mapsto 010,\ 1 \mapsto 011$.

*Example 4.* Let $\xi = 010?$ and $x = \xi^\omega \triangleleft \xi^\omega \triangleleft \cdots$. Let $L = \{0, 1, 2, 3\}$. Then

$$P_{\xi^\omega}(L) = \{010?, 10?0, 0?01, ?010\}$$

and

$$\pi_{\{0,1\}} P_{\xi^\omega}(L) = \{0100, 0101, 1000, 1010, 0001, 0010\}.$$

Hence, $\#(\pi_{\{0,1\}} P_{\xi^\omega}(L) \cup \{0000, 1111\}) = 8$ and $E(\xi, L) = 8 - 4 \times 2 = 0$. Let $L = \{0, 1, 2\}$. Then

$$P_{\xi^\omega}(L) = \{010, 10?, 0?0, ?01\} \quad \text{and} \quad \pi_{\{0,1\}} P_{\xi^\omega}(L) = \{010, 100, 101, 000, 001\}.$$

Hence, $\#(\pi_{\{0,1\}} P_{\xi^\omega}(L) \cup \{000, 111\}) = 6$ and $E(\xi, \{0, 1, 2\}) = 6 - 3 \times 2 = 0$. By symmetry, $E(\xi, L) = 0$ for any $L$ with $\#L = 3$. Moreover, $E(\xi, L) \leq 0$ for any $L$

with $\#L = 2$ by the same reasoning as in Example 3. Thus, $c = 2$ in (1) and $\lim_{n\to\infty} p_x^*(n)/n = 2$. This $x$ is also a fixed point of the substitution $0 \mapsto 0100$, $1 \mapsto 0101$.

A result that we shall also need is the next theorem, which is proved in [**9**].

THEOREM 3. *Either an infinite binary word is pattern universal or its maximal pattern complexity is bounded by some polynomial.*

The *Hamming distance* $d_H(u, v)$ between two words $u$ and $v$ of the same length is the number of positions at which they differ from each other. Let us denote the number of occurrences of a symbol $a$ in a word $u$ by $|u|_a$.

## 3. *Fixed points*

In this section, we prove some links between the fixed points of binary substitutions of constant length and their languages. First of all, we prove that some of these fixed points are Toeplitz words as well.

LEMMA 3. *Let $\varphi$ be a binary substitution of constant length such that $\varphi(0) = pas$ and $\varphi(1) = p\bar{a}s$, where $s$ is a word, $p$ is a non-empty word and $a$ is a symbol. Suppose that the infinite word $x$ is a fixed point of $\varphi$. Then $x$ is a Toeplitz word $\xi^\omega \lhd \xi^\omega \lhd \cdots$ with $\xi = p?s$ or $\xi = \varphi(p)p?s\varphi(s)$.*

Deep results on this topic can be found in [**4**].

*Proof.* In the case where $a = p_0$, let $\xi = p?s$.

In the case where $a \neq p_0$, let us consider the substitution $\varphi^2$. It can easily be seen that $\varphi^2$ satisfies the condition of the previous case and that $x$ is its fixed point. □

Let us consider two binary substitutions $\varphi$ and $\psi$ of constant length, defined by

$$\begin{aligned} \varphi(0) &= ps^{(0)}, \quad \varphi(1) = ps^{(1)}, \\ \psi(0) &= s^{(0)}p, \quad \psi(1) = s^{(1)}p, \end{aligned} \tag{2}$$

where $p$, $s^{(0)}$ and $s^{(1)}$ are some finite words. Such substitutions are said to be *conjugate* to each other.

CLAIM 1. *If the substitutions $\varphi$ and $\psi$ are conjugate to each other, then so are the substitutions $\varphi^2$ and $\psi^2$.*

*Proof.* Assume that (2) holds. Let us define

$$p' = \varphi(p)p = p\psi(p),$$
$$t^{(0)} = s^{(s_0^{(0)})}ps^{(s_1^{(0)})} \cdots ps^{(s^{(0)}_{|s^{(0)}|-1})},$$
$$t^{(1)} = s^{(s_0^{(1)})}ps^{(s_1^{(1)})} \cdots ps^{(s^{(1)}_{|s^{(1)}|-1})}.$$

Then we have the conjugacy between $\varphi^2$ and $\psi^2$ as follows:

$$\varphi^2(0) = p't^{(0)}, \quad \varphi^2(1) = p't^{(1)},$$
$$\psi^2(0) = t^{(0)}p', \quad \psi^2(1) = t^{(1)}p'.$$ □

The following statements show the relation between fixed points of conjugate substitutions.

CLAIM 2. *Let the substitutions $\varphi$ and $\psi$ be of the form (2), let $u$ and $v$ be words, and let $a$ be a letter. Then*

$$ua \in F_v \text{ implies } \psi(u) \in F_{\varphi(v)};$$
$$au \in F_v \text{ implies } \varphi(u) \in F_{\psi(v)}.$$

*Proof.* We have

$$\varphi(ua) = ps^{(u_0)}ps^{(u_1)} \cdots ps^{(u_n)}ps^{(a)} = p\psi(u)s^{(a)};$$
$$\psi(au) = s^{(a)}ps^{(u_0)}ps^{(u_1)} \cdots ps^{(u_n)}p = s^{(a)}\varphi(u)p.$$

So the statement is proved. □

CLAIM 3. *In Claim 2, if $v$ is a non-constant infinite word which is a fixed point of $\varphi$, then for any fixed point $u$ of $\psi$ we have $F_u \subset F_v$. Moreover, if in the above $u$ is non-constant and recurrent, then $F_u = F_v$.*

*Proof.* Let $u$ be a fixed point of $\psi$. Since $u_0 \in F_v$, there exists $a \in \{0, 1\}$ such that $u_0 a \in F_v$. Therefore, by Claim 2, $\psi(u_0) \in F_{\varphi(v)} = F_v$. In the same way, we can prove that $\psi^n(u_0) \in F_v$ for any $n = 1, 2, \ldots$. This implies $F_u \subset F_v$.

If $u$ is non-constant and recurrent, then $av_0 \in F_u$ for some $a \in \{0, 1\}$. Therefore $\varphi(v_0) \in F_u$, and by the same argument as above we obtain $F_v \subset F_u$. □

Let us define the *direct product* of two infinite words (or words of the same length) $x = x_0x_1\cdots$ and $y = y_0y_1\cdots$ on $\{0, 1\}$ to be the word $x \otimes y = \langle x_0, y_0\rangle\langle x_1, y_1\rangle \cdots$ on the alphabet $\{0, 1\} \times \{0, 1\}$. Let $\delta$ be the mapping $\{0, 1\} \times \{0, 1\} \to \{0, 1\}$ defined by $\delta(i, j) = 1$ if $i = j$ and $\delta(i, j) = 0$ if $i \neq j$. By applying $\delta$ coordinatewise, we can define the mapping $(\{0, 1\} \times \{0, 1\})^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$, which we also denote by $\delta$.

LEMMA 4. *Let the infinite words $x$ and $y$ be fixed points of a binary substitution $\varphi$ of constant length $m \geq 2$. Then the word $\delta(x \otimes y)$ is a fixed point of the substitution $\psi$ of constant length $m$ defined by*

$$\begin{aligned} \psi(0) &= \delta(\varphi(0) \otimes \varphi(1)), \\ \psi(1) &= 1^m. \end{aligned} \tag{3}$$

*Proof.* If $x_0 = y_0$, then we have $x = y$ and $\delta(x \otimes y) = 1^\omega$, which is a fixed point of $\psi$. Now suppose $x_0 \neq y_0$. We may assume that $x_0 = 0$ and $y_0 = 1$. Since $x$ and $y$ are fixed points of $\varphi$, the relations $\varphi(0)_0 = 0$ and $\varphi(1)_0 = 1$ hold. Hence we have $\psi(0)_0 = 0$.

Since (3) implies that $\psi\delta = \delta(\varphi \otimes \varphi)$, we have

$$\delta(x \otimes y) = \lim_{n\to\infty} \delta(\varphi \otimes \varphi)^n (0 \otimes 1) = \lim_{n\to\infty} \psi^n \delta(0 \otimes 1) = \lim_{n\to\infty} \psi^n(0).$$

Hence $\delta(x \otimes y)$ is a fixed point of $\psi$. □

## 4. *The main result*

Let us prove pattern universality for a specific case.

LEMMA 5. *A non-constant fixed point of a binary substitution $\psi$ of constant length $m$ defined by $\psi(1) = 1^m$ and $\psi(0) = 0u0s$ for some words $u$ and $s$ is pattern universal.*

*Proof.* Let $x = \psi^\omega(0)$ be such a point. We shall construct by induction a series of windows $T^n$ such that $F_x(T^n) \supset 0\Sigma^{n-1}$.

Suppose that we already have such an $n$-window $T^n$. Then there exists a finite word $p = \psi^c(0)$ with the property that $F_p(T^n) = F_x(T^n)$. Let us prove that $F_x(T^{n+1}) \supset 0\Sigma^n$ for $T^{n+1} = T^n \cup \{t_n\}$, where $t_n = |p\psi^c(u)|$.

Consider the word $p' = \psi(p) = p\psi^c(u)p\psi^c(s) \in F_x$. For each $k < |p|$ we have $p'_k = p'_{k+t_n}$. Therefore, $0\Sigma^{n-1}0 \subset F_x(T^{n+1})$.

That the infinite word $x$ is non-constant means that $x$ contains the symbol 1 and, consequently, an arbitrarily large word $11 \cdots 1$. So, a sufficiently large word of the form $x_k x_{k+d} \cdots x_{k+ld}$ contains 1 for each $d$. Let us choose $d = |u| + 1$ and $k$ such that $x_k = 0$ and $x_{k+d} = 1$.

Consider the word $p' = \psi^c(x_k x_{k+1} \cdots x_{k+d}) = p\psi^c(v)1^{|p|} \in F_x$, where $|v| = |u|$. For each $k < |p|$ we have $p'_{k+t_n} = 1$. Therefore, $0\Sigma^{n-1}1 \subset F_x(T^{n+1})$.

We have proved $p_x^*(n) \geq 2^{n-1}$. By Theorem 3 this means that $x$ is pattern universal. □

Consider the specific case of a substitution having two 'very' different fixed points.

LEMMA 6. *A non-periodic fixed point of a binary substitution $\varphi$ of constant length defined by $\varphi(0) = 0uas$ and $\varphi(1) = 1u\bar{a}s'$ for some words $u, s, s'$ and a symbol $a$ is pattern universal.*

*Proof.* Let us consider the fixed points $x = \varphi^\omega(0)$ and $y = \varphi^\omega(1)$ of $\varphi$. If $\varphi(0)$ does not contain 1 or $\varphi(1)$ does not contain 0, then we are in the setting of Lemma 5 and thus the statement is true.

Suppose now that $\varphi(0)$ contains 1 and $\varphi(1)$ contains 0. Note that this implies $F_x = F_y$ and, consequently, that $P_x(T) = P_y(T)$ for each window $T$.

In the case of the symmetric substitution $\varphi$, both $x$ and $y$ are pattern universal owing to Lemma 2.

Assume that $\varphi$ is not symmetric. Let $\delta$ be the mapping defined in the paragraph above Lemma 4, and let the substitution $\psi$ be defined by (3) with the $\varphi$ of the current lemma. Consider the word $z = \delta(x \otimes y)$ that is the fixed point of $\psi$ by Lemma 4. From our assumption and the definitions, it follows that $z$ is a non-constant word. One can make sure that $\psi$ also satisfies the conditions of Lemma 5, so that $p_z^*(n) \equiv 2^n$.

For each window $T$ we have the inclusion

$$P_z(T) \subseteq \{\delta(u \otimes v) : u, v \in P_x(T)\}.$$

Hence, $p_z(T) = 2^{|T|}$ implies $p_x(T) \geq 2^{|T|/2}$. Thus $p_x^*(n) \equiv 2^n$ by virtue of Theorem 3. □

We have considered the cases of a substitution having two distinct fixed points. Now we are able to prove the main theorem.

THEOREM 4. *Suppose that an infinite non-periodic word $x$ is a fixed point of a binary substitution $\varphi$ of constant length. The following characterization holds for the maximal pattern complexity of $x$:*

$$\text{(Case 1)} \quad \text{if } d_H(\varphi(0), \varphi(1)) > 1, \text{ then } p_x(n) \equiv 2^n;$$

$$\text{(Case 2)} \quad \text{if } d_H(\varphi(0), \varphi(1)) = 1, \text{ then } p_x(n) \text{ is linear.}$$

*Proof.* We first prove Case 1.

*Case 1a.* Let $\varphi(0)_0 = 0$ and $\varphi(1)_0 = 1$. Then the substitution and the word satisfy the conditions of Lemma 6, and so the statement is true.

*Case 1b.* Let the words $\varphi(0)$ and $\varphi(1)$ have a common prefix. Consider a conjugate substitution $\varphi'$ of $\varphi$ such that $\varphi'(0)_0 \neq \varphi'(1)_0$.

By the choice of $\varphi'$ and the fact that $d_H(\varphi'(0), \varphi'(1)) = d_H(\varphi(0), \varphi(1)) > 1$, the substitution $\varphi'^2$ is of the form $\varphi'^2(0) = 0uas$ and $\varphi'^2(1) = 1u\overline{a}s'$; so it satisfies the conditions of Lemma 6 and has two fixed points.

Suppose that both of these fixed points are constant words. This would imply that $\varphi'(0) = aa \cdots a$ and $\varphi'(1) = bb \cdots b$ for some $a, b \in \Sigma_2$, and, owing to the conjugacy, the same must hold for $\varphi$. In this case, we would have a contradiction with the non-periodicity of $x$. So, let the word $x' = (\varphi'^2)^\omega(a)$ be non-constant for some $a \in \Sigma_2$.

Suppose that $x'$ is not recurrent. This would imply that $\varphi'^2(a) = a\overline{a} \cdots \overline{a}$ and $\varphi'^2(\overline{a}) = \overline{aa} \cdots \overline{a}$. Then we have the same statement for $\varphi'$ and, since $d_H(\varphi(0), \varphi(1)) = d_H(\varphi'(0), \varphi'(1))$, it contradicts the condition of Case 1. So $x'$ is recurrent and we have $F_x = F_{x'}$ by Claim 3 applied to the conjugate (by Claim 1) substitutions $\varphi'^2$ and $\varphi^2$. Hence, $x'$ is non-periodic.

Applying Lemma 6 to the word $x'$ and the substitution $\varphi'^2$, we obtain $p_{x'} \equiv 2^n$ and, consequently, $p_x \equiv 2^n$, which finishes the proof of Case 1.

Now consider Case 2. In this case, we have $\varphi(0) = pas$ and $\varphi(1) = p\overline{a}s$ for some words $p$ and $s$. If necessary, by considering $\varphi^2$ in place of $\varphi$, we may assume that $a = 0$. Moreover, since $x$ is non-periodic, $ps$ contains both of the letters 0 and 1.

*Case 2a.* Let $p$ be a non-empty word. By Lemma 3, the word $x$ is the Toeplitz word with one hole; hence, by Theorem 2, $p_x(n)$ is linear [**7**].

*Case 2b.* Let $\varphi(0) = 0s$ and $\varphi(1) = 1s$. Consider the conjugate substitution $\varphi'$ of $\varphi$ such that $\varphi'(0) = s0$ and $\varphi'(1) = s1$. Obviously, it has a fixed point $x'$ that is a Toeplitz word with one hole with a linear complexity $p_{x'}(n)$ (see [**7**]). Since $s$ contains both of the letters 0 and 1, $x$ is recurrent. Therefore, we have $F_x = F_{x'}$ by Claim 3. Thus, $p_x(n) = p_{x'}(n)$ is linear. □

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