Selection rules preserving normality

(to appear in Israel J. Math.)

Xiaowen Wang* and Teturo KAMAE†

Abstract

We discuss a way of choosing subsequences (selection rule) from binary one-sided infinite sequences such that whether the i-th place is chosen or not is decided by the information before it. It is realized by a countable automaton with input $\{0, 1\}$ in a way that the i-th place is chosen if and only if the finite subsequence from the beginning to the $(i-1)$-place is accepted by the automaton. We characterize selection rules which preserve normality, that is, the subsequence of a normal number chosen by it is always a normal number if the set of chosen places has a positive lower density. Our result is a common generalization of known results for special automata.

1 Introduction

R. von Mises [1] introduced the notion of probability as random sequences of letters called "collectives" in the beginning of the 20th century. A *collective* is by definition a sequence $\alpha(0)\alpha(1)\cdots$ over a finite alphabet satisfying that the frequencies of symbols (or blocks of symbols) in it are preserved for all the subsequences $\alpha(s_0)\alpha(s_1)\cdots$ with $0 \le s_0 < s_1 < \cdots$ such that $i \in \{s_0, s_1, \cdots\}$ or not is determined whether $\alpha(0)\alpha(1)\cdots\alpha(i-1) \in \mathcal{L}$ or not, where $\mathcal{L}$ is a subset of finite sequences over the alphabet belonging to a largest possible class (von Mises did not specify exactly which class). In general, a way of choosing subsequences is called a *selection rule*, so is the above $\mathcal{L}$. It is an interesting problem to ask how far the normal numbers are random from this point of view. That is, to determine the class of selection rules which preserves the property of being normal numbers. This problem was proposed by B. Weiss in [4].

Let us restrict the alphabet to $\{0, 1\}$ just for simplicity (it can be any finite set of symbols). Let $\{0, 1\}^* := \cup_{n=0}^{\infty}\{0, 1\}^n$. Let $M = (\Sigma, \psi, \sigma_0, F)$ be a *countable automaton* over $\{0, 1\}$. That is, Σ is a set with the cardinality

*Department of Mathematics Huazhong University of Science and Technology, Wuhan 430074, PR. China (xiaowen@hust.edu.cn)

†Advanced Mathematical Institute, Osaka City University, 558-8585 Japan (kamae@apost.plala.or.jp)

at most countable, $\sigma_0 \in \Sigma$, $F \subset \Sigma$ and $\psi : \Sigma \times \{0,1\} \to \Sigma$. We call σ_0 the *initial* state and F the *final set* of M. For any $\alpha \in \{0,1\}^{\mathbb{N}}$ and M, where $\mathbb{N} = \{0,1,2,\cdots\}$, the *input-output sequence* $(\alpha,\gamma) \in \{0,1\}^{\mathbb{N}} \times \Sigma^{\mathbb{N}}$ is defined so that

$$\gamma(i) = \psi(\sigma_0,\ \alpha[0,i)) = \psi(\cdots\psi(\psi(\sigma_0,\alpha(0)),\alpha(1))\cdots,\alpha(i-1))\ (\forall i \in \mathbb{N}),$$

where we denote $\alpha[i,j) = \alpha(i)\alpha(i+1)\cdots\alpha(j-1)$ for any $i \le j$. In this setting, we define a subsequence of α by picking the places i such that $\gamma(i) \in F$, that is, $\alpha[0,i)$ is *accepted* by M. Hence, if

$$S = \{s_0 < s_1 < \cdots\} := \{i \in \mathbb{N};\ \gamma(i) \in F\}$$

is an infinite set, then the selected subsequence $\alpha[S] \in \{0,1\}^{\mathbb{N}}$ is defined to be $\alpha[S](i) = \alpha(s_i)$ $(\forall i)$. This set S is denoted by $S(M,\alpha)$.

Given a selection rule $\mathcal{L} \subset \{0,1\}^*$. Define an equivalence relation $\sim_{\mathcal{L}}$ on $\{0,1\}^*$ so that $\xi \sim_{\mathcal{L}} \eta$ if and only if

$$\{\zeta \in \{0,1\}^*;\ \xi\zeta \in \mathcal{L}\} = \{\zeta \in \{0,1\}^*;\ \eta\zeta \in \mathcal{L}\}.$$

Let $\pi_{\mathcal{L}}$ be the projection from $\{0,1\}^*$ to the equivalence classes $\{0,1\}^*/\sim_{\mathcal{L}}$. Let $\Sigma = \pi_{\mathcal{L}}(\{0,1\}^*)$, $\sigma_0 = \pi_{\mathcal{L}}(\epsilon)$ (ϵ being the empty sequence) and $F = \pi_{\mathcal{L}}(\mathcal{L})$. Define $\psi : \Sigma \times \{0,1\} \to \Sigma$ by $\psi(\sigma,a) = \pi_{\mathcal{L}}(\xi a)$ for any $\sigma \in \Sigma$ and $a \in \{0,1\}$, where ξ is any element in $\{0,1\}^*$ such that $\pi_{\mathcal{L}}(\xi) = \sigma$ (ψ is well defined). Then, $M = (\Sigma,\psi,\sigma_0,F)$ is known as the *minimum* automaton which *recognizes* $\mathcal{L}$, that is, the selection rule $\mathcal{L}$ is realized as the set of accepted sequences by M. In this case, the selection rule $\mathcal{L}$ is called the *selection rule* M as well.

For $\beta \in K^{\mathbb{N}}$ with a (possibly infinite) set K and $L \subset K$, we denote the subset $\{i \in \mathbb{N};\ \beta(i) \in L\}$ of $\mathbb{N}$ by $S(\beta,L)$. This is a special case of a countable automaton. That is, we consider $M = (\mathbb{N},\psi,0,F)$ such that $\psi(\sigma,a) = \sigma+1$ $(\forall(\sigma,a) \in \mathbb{N}\times\{0,1\})$ and $F = \{i \in \mathbb{N};\ \beta(i) \in L\}$. Then, we have $S(M,\alpha) = S(\beta,L)$ for any $\alpha \in \{0,1\}^{\mathbb{N}}$. In this special case, the selection rule M is called the *selection rule* (β,L) as well.

We apply the selection rules to the binary normal numbers to see whether the selected subsequence is again a normal number or not. Let $\alpha \in \{0,1\}^{\mathbb{N}}$ be a (binary) *normal number*, that is, for any $k = 1,2,\cdots$ and $\xi \in \{0,1\}^k$, it holds that

$$\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1}\#\{i;\ i<n,\ \alpha(i)\alpha(i+1)\cdots\alpha(i+k-1) = \xi\} = 2^{-k}.$$

If an infinite set $S \subset \mathbb{N}$ has a zero lower density, then for any $\lambda < 1$, there exists $S' \subset S$ such that S' has an upper density zero and a relative upper density in S larger than λ (see Appendix 1), where the latter statement implies that

$$\limsup_{n\to\infty} \frac{\#(S' \cap [0,n))}{\#(S \cap [0,n))} > \lambda.$$

Hence, if we modify α at places in S', this will not affect whether α itself is a normal number but may affect whether $\alpha[S]$ is a normal number. Because of this singularity, when we ask whether the normality is preserved by a selection rule M or not we will assume not only that $S(M,\alpha)$ is an infinite set, but also that it has a positive lower density. Thus, we make the following definition.

Definition 1. We say that the selection rule M *preserves normality* if $\alpha[S(M,\alpha)]$ is a normal number whenever α is a normal number and $S(M,\alpha)$ has a positive lower density.

The following theorem, which is proved by T. Kamae and B. Weiss [6] as a generalization of the case of finite automata (Y. N. Agafonov [3]). Note that the condition that the selection rule $\mathcal{L} \subset \{0,1\}^*$ satisfies that $\mathcal{L}/\sim_{\mathcal{L}}$ is finite is equivalent to that the selection rule is realized by a countable automaton $(\Sigma, \psi, \sigma_0, F)$ with F finite.

Theorem 1. ([6]) *Let $M = (\Sigma, \psi, \sigma_0, F)$ be a countable automaton such that F is a finite set. Then, M preserves normality.*

In the case that the selection rule M uses only the information i out of $\alpha(0)\alpha(1)\cdots\alpha(i-1)$, there exists a $\gamma \in \{0,1\}^{\mathbb{N}}$ such that

$$S(M,\alpha) = S(\gamma, \{1\}) \tag{1.1}$$

holds for any α. We say in [5] that γ is *completely deterministic*, if any pseudo-generic measure of γ with respect to the shift on $\{0,1\}^{\mathbb{N}}$, that is, μ_γ^U (see (2.1)) for any U, has Kolmogorov-Sinai's entropy 0. This implies that the set of the places $i \in \mathbb{N}$ such that $\gamma(i)$ is determined by a function of $\gamma(i-k)\cdots\gamma(i-1)$ tends to 1 in lower density as k tends to infinity.

Theorem 2. ([5]) *The automaton M, which can be identified with $\gamma \in \{0,1\}^{\mathbb{N}}$ so that (1.1) holds for any α, preserves normality if and only if γ is completely deterministic.*

Remark 1. By the discussions before Definition 1, the requirement that $S(\gamma,\{1\})$ has a positive lower density is actually a necessary condition for $\alpha[S(\gamma,\{1\})]$ to be a normal number whenever α is a normal number.

In Theorem 1 and 2, the selection rules which are proved to preserve normality are somehow restricted. We want to find a common and natural generalization of them. For this purpose, we prove the following Theorem 3 which generalizes Theorem 1 and the "if" part of Theorem 2. The key idea is the notion of "conditionally deterministic" (Definition 2) which is a generalization of "completely deterministic" in Theorem 2. In fact, "completely deterministic" is equivalent to "conditionally deterministic" with respect to any binary normal number α (Corollary 1). We reprove Theorem 1 and the "if" part of Theorem 2 using Theorem 3.

The "only if" part of Theorem 2 follows from the fact that if γ is not completely deterministic, then a shift invariant probability measure μ (say) having a positive Kolmogorov-Sinai's entropy is related to γ. By H. Furstenberg ([2]), there exists a correlated joining of μ with the coin-tossing measure, which implies that there exists a normal number α such that the distribution of $\alpha[S(\gamma,\{1\})]$ is not the coin-tossing measure, that is, $\alpha[S(\gamma,\{1\})]$ is not a normal number. To find a reasonable necessary condition for to preserve the normality among a general class of selection rules is still open.

Let K be a finite set and $\beta \in K^{\mathbb{N}}$. Consider the pair $(\alpha,\beta) \in (\{0,1\}\times K)^{\mathbb{N}}$. Let

$$M_n = M_n(\alpha,\beta) := (1/n)\sum_{i=0}^{n-1} \delta_{T^i(\alpha,\beta)} \quad (n = 1, 2, \cdots),$$

where δ_x denotes the unit measure at a point x in an arbitrary set and T denotes the shift on the space $(\{0,1\} \times K)^{\mathbb{N}}$. The same notation T will be used for the shift on the other spaces, say $(\{0,1\} \times K)^{\mathbb{Z}}$, $\{0,1\}^{\mathbb{N}}$, $\{0,1\}^{\mathbb{Z}}$, $K^{\mathbb{N}}$, etc. Since the base spaces are compact, the family M_n of probability measures is relatively compact in the weak sense so that for any infinite set $V \subset \mathbb{N}$, there exists an infinite set $U \subset V$ such that M_n converges as $n \to \infty$ satisfying $n \in U$. We denote this limit measure by $\mu^U_{(\alpha,\beta)}$. That is,

$$\mu^U_{(\alpha,\beta)} = \lim_{\substack{n\to\infty \\ n\in U}} (1/n)\sum_{i=0}^{n-1} \delta_{T^i(\alpha,\beta)}. \tag{1.2}$$

If α is a binary normal number, then the marginal distribution of $\mu^U_{(\alpha,\beta)}$ to the first coordinate $\{0,1\}^{\mathbb{N}}$ is the product measure $(1/2, 1/2)^{\mathbb{N}}$, where $(1/2, 1/2)$ denotes the measure on $\{0,1\}$ having mass $1/2$ at both 0 and 1.

Let $P := \tilde{\mu}^U_{(\alpha,\beta)}$ be the T-invariant extension of $\mu^U_{(\alpha,\beta)}$ to $(\{0,1\} \times K)^{\mathbb{Z}}$ and $X_i,\ Y_i$ $(i \in \mathbb{Z})$ be the projections

$$X_i(\omega_1,\omega_2) = \omega_1(i)\ , \quad Y_i(\omega_1,\omega_2) = \omega_2(i), \tag{1.3}$$

for any $(\omega_1,\omega_2) \in \{0,1\}^{\mathbb{Z}} \times K^{\mathbb{Z}} = (\{0,1\} \times K)^{\mathbb{Z}}$. They are considered as random variables on $((\{0,1\} \times K)^{\mathbb{Z}},\ P)$.

Definition 2. We say that (α,β) is *conditionally deterministic* if for any $U \subset \mathbb{N}$ such that $\mu^U_{(\alpha,\beta)}$ exists, it holds that Y_0 is a measurable function of $(X_i, Y_i)_{i=-\infty}^{-1}$ under the probability measure $\tilde{\mu}^U_{(\alpha,\beta)}$.

Theorem 3. *For a normal number $\alpha \in \{0,1\}^{\mathbb{N}}$ and a countable automaton M, let $(\alpha, \gamma) \in \{0,1\}^{\mathbb{N}} \times \Sigma^{\mathbb{N}}$ be the input-output sequence. Assume that there exists a mapping π from Σ to a finite set K such that $(\alpha, \beta) \in \{0,1\}^{\mathbb{N}} \times K^{\mathbb{N}}$, where $\beta(i) = \pi(\gamma(i))$ $(\forall i \in \mathbb{N})$, is conditionally deterministic. Assume further that there exists $L \subset K$ such that $\pi^{-1}L = F$. Then, $\alpha[S(M, \alpha)]$ is a normal number if $S(M, \alpha)$ has a positive lower density.*

The following Corollary will be proved later.

Corollary 1. *For any $\gamma \in \{0,1\}^{\mathbb{N}}$ such that $\{i \in \mathbb{N};\ \gamma(i) = 1\}$ has a positive lower density, γ is completely deterministic if and only if (α, γ) is conditionally deterministic for any binary normal number α.*

Let $M = (\Sigma, \psi, \sigma_0, F)$ be a countable automaton. Forgetting F, we denote $M' = (\Sigma, \psi, \sigma_0)$. It is considered as a graph with the set of vertices Σ and the set of directed edges

$$\Gamma_\psi = \{(\sigma, \psi(\sigma, a));\ \sigma \in \Sigma,\ a \in \{0,1\}\}$$

together with the specified initial vertex σ_0. Note that we admit multiple edges between 2 vertices. Consider a sequence of random variables $R_0, R_1, R_2, \cdots$ such that

$$R_i = \psi(\sigma_0,\ X_0 X_1 \cdots X_{i-1}) \ \ (i = 0, 1, 2, \cdots), \tag{1.4}$$

where $X_0 X_1 X_2 \cdots$ is a sequence of independent random variables with $P(X_i = 0) = P(X_i = 1) = 1/2$ $(\forall i \in \mathbb{N})$. We call it the *random walk* on M'.

Definition 3. We say that a normal number α *simulates* the random walk on M' if for any $k = 1, 2, \cdots$ and a path $\zeta = \zeta_0 \zeta_1 \cdots \zeta_k$ of the graph with $\zeta_0 = \sigma_0$, we have

$$\begin{aligned} & P(R_0 R_1 \cdots R_k = \zeta) \\ & = \lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} 1_{\psi(\sigma_0, \alpha[0,i)) \psi(\sigma_0, \alpha[0,i+1)) \cdots \psi(\sigma_0, \alpha[0,i+k)) = \zeta}}{\sum_{i=0}^{n-1} 1_{\psi(\sigma_0, \alpha[0,i)) = \sigma_0}}. \end{aligned} \tag{1.5}$$

Definition 4. We say that $\alpha \in \{0,1\}^{\mathbb{N}}$ is *positively recurrent* on M' if

$$\liminf_{n \to \infty} \frac{1}{n} \#\{i \in \mathbb{N};\ 0 \le i < n,\ \psi(\sigma_0, \alpha[0,i)) = \sigma_0\} > 0.$$

Let $\xi \in \{0,1\}^k$ with $k \ge 1$. We say that M' is *weakly renewal* with the *renewal word* ξ if there exists a finite partition $\{\Sigma_1, \cdots, \Sigma_k\}$ of Σ such that for any $i = 1, \cdots, k$, there exists $\sigma_i \in \Sigma$ such that $\psi(\sigma, \xi) = \sigma_i$ for any $\sigma \in \Sigma_i$.

Theorem 4. *A normal number $\alpha \in \{0,1\}^{\mathbb{N}}$ simulates the random walk on M' if α is positively recurrent on M'. Moreover, if the random walk on M' is not positively recurrent, that is, the expectation of $\min\{n \geq 1;\ R_n = \sigma_0\}$ is infinity, then for any normal number α, it holds that*

$$\limsup_{n\to\infty} \frac{1}{n} \#\{i \in \mathbb{N};\ 0 \leq i < n,\ \psi(\sigma_0, \alpha[0,i)) = \sigma_0\} = 0.$$

Theorem 5. *A countable automaton $M = (\Sigma, \psi, \sigma_0, F)$ preserves normality if M' is weakly renewal.*

2 Proof of Theorem 3

For a normal number $\alpha \in \{0,1\}^{\mathbb{N}}$, define $\beta \in K^{\mathbb{N}}$ as in Theorem 3. Since $S(M,\alpha) = S(\beta, L)$, it is sufficient to prove that $\alpha[S(\beta,L)]$ is a normal number if $S(\beta, L)$ has a positive lower density, which we assume. For any infinite set $U' \subset \mathbb{N}$, take an infinite subset $U \subset U'$ such that

$$\mu^U_{(\alpha,\beta)} = \lim_{\substack{n\to\infty \\ n\in U}} (1/n) \sum_{i=0}^{n-1} \delta_{T^i(\alpha,\beta)} \tag{2.1}$$

exists. Let $\Omega = (\{0,1\} \times K)^{\mathbb{Z}} = \{0,1\}^{\mathbb{Z}} \times K^{\mathbb{Z}}$ and $P = \tilde{\mu}^U_{(\alpha,\beta)}$. Define random variables X_n, Y_n for $n \in \mathbb{Z}$ on the probability space (Ω, P) as (1.3). Since the marginal of P to the first component comes from the normal number α, $\cdots, X_{-1}, X_0, X_1, \cdots$ are independent random variables such that $P(X_n = 0) = P(X_n = 1) = 1/2$ $(\forall n \in \mathbb{Z})$. Moreover, since $S(\beta, L)$ has a positive lower density, we have $P(Y_n \in L) > 0$ $(\forall n \in \mathbb{Z})$.

Shannon's conditional entropy $H(Z|W)$ of random variables Z, W taking finitely many values z_i $(i = 1, 2, \cdots, I)$, w_j $(j = 1, 2, \cdots, J)$ (respectively) is defined as

$$H(Z|W) = -\sum_{i,j} P(Z = z_i,\ W = w_j) \log \frac{P(Z = z_i,\ W = w_j)}{P(W = w_j)}$$

and $H(Z|W_1, W_2, \cdots)$ is defined as $\lim_{n\to\infty} H(Z|W_1, W_2, \cdots, W_n)$. Refer [7] for further properties of entropy.

Since (α, β) is conditionally deterministic, Y_0 is determined almost surely by $(X_n, Y_n)_{n=-\infty}^{-1}$. Moreover, since $\cdots, X_{-2}, X_{-1}, X_0$ are independent, we have

$$\begin{aligned}
H(X_0) &= H(X_0|\cdots, X_{-2}, X_{-1}) \\
&\leq H((X_0, Y_0)|\cdots, (X_{-2}, Y_{-2}), (X_{-1}, Y_{-1})) \\
&= H(Y_0|\cdots, (X_{-2}, Y_{-2}), (X_{-1}, Y_{-1})) \\
&\quad + H(X_0|\cdots, (X_{-2}, Y_{-2}), (X_{-1}, Y_{-1}), Y_0) \\
&= H(X_0|\cdots, (X_{-2}, Y_{-2}), (X_{-1}, Y_{-1}), Y_0) \leq H(X_0).
\end{aligned}$$

Therefore,

$$H(X_0|\cdots,(X_{-2},Y_{-2}),(X_{-1},Y_{-1}),Y_0) = H(X_0)$$

holds, and hence

$$X_0 \text{ is independent of } (\cdots,(X_{-2},Y_{-2}),(X_{-1},Y_{-1}),Y_0). \tag{2.2}$$

Since $P(Y_{n+1} \notin L, Y_{n+2} \notin L, \cdots) = P(Y_n \notin L, Y_{n+1} \notin L, \cdots)$ as P is invariant under the shift, we have

$$P(Y_n \in L, Y_{n+1} \notin L, Y_{n+2} \notin L, \cdots) = 0$$

and hence,

$$P(Y_0 \in L \text{ and } Y_n \in L \text{ only for finitely many } n > 0) = 0. \tag{2.3}$$

For $k = 0, 1, 2, \cdots$, let

$$W_k = \{(\omega_1, \omega_2) \in \Omega;\ \omega_2(0) \in L \text{ and } \omega_2(n) \in L \text{ for at least } k \text{ number of } n > 0\}.$$

Then by (2.3),

$$P(\{(\omega_1,\omega_2);\ \omega_2(0) \in L\} \setminus W_k) = 0 \quad (k = 0, 1, 2, \cdots).$$

Moreover, since W_k is an open set whose boundary is exactly the above set inside $P(\cdot)$ having P-measure 0, it holds that

$$P(W_k) = \lim_{\substack{n\to\infty \\ n\in U}} (1/n) \sum_{i=0}^{n-1} \delta_{T^i(\alpha,\beta)\in W_k}.$$

For $(\omega_1, \omega_2) \in W_k$, let

$$\{0 = t_0 < t_1 < \cdots < t_{k-1} < \cdots\} = \{n \geq 0;\ \omega_2(n) \in L\}$$

which can be either a finite set or an infinite set. For $\xi = \xi_0\xi_1\cdots\xi_k \in \{0,1\}^{k+1}$ and $\xi' = \xi_0\xi_1\cdots\xi_{k-1} \in \{0,1\}^k$, define

$$W_\xi = \{(\omega_1,\omega_2) \in W_k;\ \omega_1(t_0)\omega_1(t_1)\cdots\omega_1(t_k) = \xi\}$$

and

$$W_{\xi'} = \{(\omega_1,\omega_2) \in W_k;\ \omega_1(t_0)\omega_1(t_1)\cdots\omega_1(t_{k-1}) = \xi'\}.$$

Since P is T-invariant and $1_{W_k} = 1_{Y_0 \in L}$ holds P-almost surely, we have $P(W_\xi) = (1/2)P(W_{\xi'})$ by the shifted statement of (2.2). Hence,

$$P(W_\xi) = (1/2)^k P(Y_0 \in L) \tag{2.4}$$

by the induction in k.

Let $S(M,\alpha) = S(\beta, L) = \{s_0 < s_1 < \cdots\} \subset \mathbb{N}$ and

$$V = \{i \in \mathbb{N};\ (s_i, s_{i+1}] \cap U \neq \emptyset\}. \tag{2.5}$$

Let us prove that for any $k = 0, 1, 2, \ldots$ and $\xi \in \{0,1\}^k$,

$$\lim_{\substack{n\to\infty \\ n\in V}} (1/n) \sum_{i=0}^{n-1} 1_{\alpha(s_i)\alpha(s_{i+1})\cdots\alpha(s_{i+k-1})=\xi} = 2^{-k}. \tag{2.6}$$

Take any extension $(\tilde{\alpha}, \tilde{\beta}) \in \{0,1\}^{\mathbb{Z}} \times K^{\mathbb{Z}}$ of (α, β) to a bi-infinite sequence, that is

$$\tilde{\alpha}(0)\tilde{\alpha}(1)\tilde{\alpha}(2)\cdots = \alpha,$$
$$\tilde{\beta}(0)\tilde{\beta}(1)\tilde{\beta}(2)\cdots = \beta.$$

Then, it is easy to see that

$$P = \lim_{\substack{n\to\infty \\ n\in U}} (1/n) \sum_{i=0}^{n-1} \delta_{T^i(\tilde{\alpha},\tilde{\beta})}. \tag{2.7}$$

On the other hand, since

$$\alpha(s_i)\alpha(s_{i+1})\cdots\alpha(s_{i+k-1}) = \xi \text{ if and only if } T^{s_i}(\tilde{\alpha},\tilde{\beta}) \in W_\xi$$

for any $\xi \in \{0,1\}^k$. By (2.4), (2.7) and the fact that W_ξ is an open set whose boundary has P-measure 0, we have

$$\lim_{\substack{n\to\infty \\ n\in V}} (1/n) \sum_{i=0}^{n-1} 1_{\alpha(s_i)\alpha(s_{i+1})\cdots\alpha(s_{i+k-1})=\xi} = \lim_{\substack{N\to\infty \\ N\in U}} \frac{\sum_{i=0}^{N-1} 1_{T^i(\tilde{\alpha},\tilde{\beta})\in W_\xi}}{\sum_{i=0}^{N-1} 1_{\tilde{\beta}(i)\in L}}$$
$$= \lim_{\substack{N\to\infty \\ N\in U}} \frac{(1/N)\sum_{i=0}^{N-1} 1_{T^i(\tilde{\alpha},\tilde{\beta})\in W_\xi}}{(1/N)\sum_{i=0}^{N-1} 1_{\tilde{\beta}(i)\in L}} = \frac{P(W_\xi)}{P(Y_0 \in L)} = (1/2)^k,$$

where $n = n(N) \in V$ is determined as a nondecreasing function of $N \in U$. For any infinite set $V' \subset \mathbb{N}$, let U' corresponds to V' in the sense of (2.5). Then, there exists an infinite set $U \subset U'$ such that $\mu^U_{\alpha,\beta}$ exists. Let V corresponds to U in the sense of (2.5). Then, (2.6) holds with this $V \subset V'$. This implies that

$$\lim_{n\to\infty} (1/n) \sum_{i=0}^{n-1} 1_{\alpha(s_i)\alpha(s_{i+1})\cdots\alpha(s_{i+k-1})=\xi} = 2^{-k}$$

for any $k = 0, 1, 2, \cdots$ and $\xi \in \{0,1\}^k$, and hence, $\alpha[S(M,\alpha)]$ is a normal number. □

Proof of Theorem 1: Let $M = (\Sigma, \psi, \sigma_0, F)$ be a countable automaton such that F is a finite set. Let α be a binary normal number. Let $K = F \cup \{\infty\}$, $L = F$ and $\pi : \Sigma \to K$ be

$$\pi(\sigma) = \begin{cases} \sigma & (\sigma \in F) \\ \infty & (\sigma \notin F) \end{cases} .$$

Define $(\alpha, \beta) \in \{0,1\}^{\mathbb{N}} \times K^{\mathbb{N}}$ as in Theorem 3 and use the notation in the proof of Theorem 3. Then, (α, β) becomes conditionally deterministic since if $Y_n \in L$ for some $n < 0$, then $Y_0 = \pi(\psi(Y_n, X_n X_{n+1} \cdots X_{-1}))$ and if $Y_n = \infty$ for all $n = \cdots, -2, -1$, then by the same reason as (2.3), we have $Y_0 = \infty$ almost surely. Therefore, Y_0 is determined by $(X_n, Y_n)_{n=-\infty}^{-1}$ and hence, (α, β) is conditionally deterministic. Thus, Theorem 1 follows from Theorem 3.

Proof of "if" part of Theorem 2: If $\gamma \in \{0,1\}^{\infty}$ is completely deterministic, then (α, γ) is conditionally deterministic for any normal number α. Thus, "if" part of Theorem 2 follows from Theorem 3. For the proof of "only if" part, refer to [5].

Proof of Corollary 1: Since "only if" is clear, we prove "if'" part. Assume that (α, γ) is conditionally deterministic for any normal number α. Then by Theorem 3, $\alpha[S(\gamma, \{1\})]$ is a normal number for any normal number α. Hence, by "only if" part of Theorem 2, γ is completely deterministic. □

3 Proof of Theorem 4

Let α be a normal number which is positively recurrent in M'. Let $M = (\Sigma, \psi, \sigma_0, \{\sigma_0\})$ and use the same notations $\Omega, \alpha, \beta, \gamma, \pi, U, P$ and X_n, Y_n $(n \in \mathbb{Z})$ as in Theorem 3 and Section 2 with respect to $V \subset \mathbb{N}$, and with the special restriction that $F = L = \{\sigma_0\}$. Then, X_n $(n \in \mathbb{Z})$ are independent random variables with $P(X_n = 0) = P(X_n = 1) = 1/2$ and Y_n $(n \in \mathbb{Z})$ takes 2 values σ_0 or ∞. Then Y_0 is a measurable function of $(X_i, Y_i)_{i=-\infty}^{-1}$ under $P = \tilde{\mu}^U_{(\alpha,\beta)}$ by the proof of Theorem 1. Denote $X^m = (X_n)_{n=-\infty}^{m}$ (same with Y). Since Y_0 is a measurable function of (X^{-1}, Y^{-1}) with respect to P, (X^{-1}, Y^{-1}) and (X^{-1}, Y^0) define the same partition on the probability space (Ω, P). Hence, using (2.2), it holds for any $k = 0, 1, 2, \cdots$ that

$$\begin{aligned} & H(X_0 X_1 \cdots X_k | X^{-1}, Y^{-1}) = H(X_0 X_1 \cdots X_k | X^{-1}, Y^0) \\ & = H(X_0 | X^{-1}, Y^0) + H(X_1 X_2 \cdots X_k | X^0, Y^0) \\ & = H(X_0) + H(X_1 X_2 \cdots X_k | X^0, Y^0). \end{aligned}$$

Repeating this, we have

$$\begin{aligned}
&H(X_0X_1\cdots X_k|X^{-1},Y^0)=H(X_0X_1\cdots X_k|X^{-1},Y^{-1})\\
&=H(X_0)+H(X_1)+\cdots+H(X_k)=H(X_0X_1\cdots X_k).
\end{aligned}$$

Hence, $X_0X_1\cdots X_k$ is independent of (X^{-1},Y^0).

Let $R_0R_1R_2\cdots$ on M' is defined in (1.4) with these X_n $(n\in\mathbb{Z})$. Then, using the above result, we have

$$P(R_0R_1\cdots R_k=\zeta)=P(R_nR_{n+1}\cdots R_{n+k}=\zeta|Y_n=\sigma_0)$$

for any $n=0,1,2,\cdots$ and any path $\zeta=\zeta_0\zeta_1\cdots\zeta_k$ in M' with $\zeta_0=\sigma_0$.

Hence, we have

$$\begin{aligned}
&P(R_0R_1\cdots R_k=\zeta)\\
&=P(\sigma_0\psi(\sigma_0,X_0)\cdots\psi(\sigma_0,X_0X_1\cdots X_{i-1})=\zeta|Y_0=\sigma_0)\\
&=\lim_{\substack{n\to\infty\\ n\in U}}\frac{(1/n)\sum_{i=0}^{n-1}1_{\psi(\sigma_0,\alpha[0,i))\psi(\sigma_0,\alpha[0,i+1))\cdots\psi(\sigma_0,\alpha[0,i+k))=\zeta}}{(1/n)\sum_{i=0}^{n-1}1_{\psi(\sigma_0,\alpha[0,i))=\sigma_0}}\\
&=\lim_{\substack{n\to\infty\\ n\in U}}\frac{\sum_{i=0}^{n-1}1_{\psi(\sigma_0,\alpha[0,i))\psi(\sigma_0,\alpha[0,i+1))\cdots\psi(\sigma_0,\alpha[0,i+k))=\zeta}}{\sum_{i=0}^{n-1}1_{\psi(\sigma_0,\alpha[0,i))=\sigma_0}}.
\end{aligned}$$

Here, we use the fact that α is positively recurrent to show

$$\lim_{\substack{n\to\infty\\ n\in U}}(1/n)\sum_{i=0}^{n-1}1_{\psi(\sigma_0,\alpha[0,i))=\sigma_0}>0.$$

Since this holds for some $U\subset V$ given an arbitrary V, we have (1.5). Thus, α simulates the random walk on M'.

To complete the proof, assume that a normal number α satisfies that

$$\limsup_{n\to\infty}\frac{1}{n}\#\{i\in\mathbb{N};\ 0\le i<n,\ \psi(\sigma_0,\alpha[0,i))=\sigma_0\}>0.$$

Then, there exists U such that $P(Y_0=\sigma_0)>0$ with $P=\tilde{\mu}^U_{(\alpha,\beta)}$. To prove that the random walk $R_0R_1R_2\cdots$ is positively recurrent, we may assume that the dynamical system (Ω,P,T) is ergodic, since otherwise, we take an ergodic component (Ω,P',T) of it with $P'(Y_0=\sigma_0)>0$. The distribution of X_n $(n\in\mathbb{Z})$ doesn't change under P' as it is already ergodic under P, and define the corresponding random walk $R_0R_1R_2\cdots$ under P'. Denoting $P'=P$ and using the ergodicity, we have

$$\begin{aligned}
&\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1}P(R_i=\sigma_0)=\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1}\frac{P(Y_0=\sigma_0,\ Y_i=\sigma_0)}{P(Y_0=\sigma_0)}\\
&=\frac{P(Y_0=\sigma_0)P(Y_0=\sigma_0)}{P(Y_0=\sigma_0)}=P(Y_0=\sigma_0)>0,
\end{aligned}$$

and hence, the random walk on M' is positively recurrent.

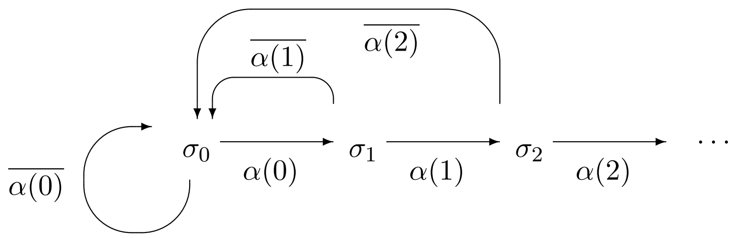

Figure 1: M' in Remark 2

Remark 2. The second half of Theorem 4 was suggested by H. Furstenberg. The converse is not true. In fact, let $\alpha \in \{0,1\}^{\mathbb{N}}$ be an arbitrary normal number. Let M' be as in Figure 1, then the random walk is positively recurrent, but α is not. Here, we denote $\overline{0} = 1$ and $\overline{1} = 0$.

4 Proof of Theorem 5

Let $M = (\Sigma, \psi, \sigma_0, F)$ be a countable automaton such that M' is weakly renewal satisfying the same conditions in Definition 4 with a renewal word ξ, a partition $\{\Sigma_1, \cdots, \Sigma_k\}$ of Σ and $\sigma_i \in \Sigma$ $(i = 1, \cdots, k)$. Let α be an arbitrary normal number and $(\alpha, \gamma) \in \{0,1\}^{\mathbb{N}} \times \Sigma^{\mathbb{N}}$ be the input-output sequence. Let $K = K_0 \cup K_1$ be such that

$$K_0 = \{(i,0);\ i = 1, \cdots, k,\ \Sigma_i \setminus F \neq \emptyset\}$$
$$K_1 = \{(i,1);\ i = 1, \cdots, k,\ \Sigma_i \cap F \neq \emptyset\}$$

and $\pi : \Sigma \to K$ be such that

$$\pi(\sigma) = \begin{cases} (i,0) & \text{if } \sigma \in \Sigma_i \setminus F \\ (i,1) & \text{if } \sigma \in \Sigma_i \cap F \end{cases}$$

Define $\beta \in K^{\mathbb{N}}$ by $\beta(i) = \pi(\gamma(i))$ $(\forall i \in \mathbb{N})$ and $L = K_1$.

We prove that (α, β) is conditionally deterministic. Then by Theorem 3, M preserves normality. Take any countable set $U \subset \mathbb{N}$ such that $\mu^U_{(\alpha,\beta)}$ in (1.2) exists. Let $P := \tilde{\mu}^U_{(\alpha,\beta)}$ be the T-invariant extension of $\mu^U_{(\alpha,\beta)}$ to $(\{0,1\} \times K)^{\mathbb{Z}}$ and $X_i,\ Y_i$ $(i \in \mathbb{Z})$ be the projections

$$X_i(\omega_1, \omega_2) = \omega_1(i)\ ,\quad Y_i(\omega_1, \omega_2) = \omega_2(i),$$

for any $(\omega_1, \omega_2) \in \{0,1\}^{\mathbb{Z}} \times K^{\mathbb{Z}} = (\{0,1\} \times K)^{\mathbb{Z}}$.

Let $\xi = \xi_0 \xi_1 \cdots \xi_{l-1}$ be the renewal word. Then with probability 1, there exists $n < -l$ such that $X_n X_{n+1} \cdots X_{n+l-1} = \xi$. Let $Y_n \in \{(i,0),(i,1)\}$ for some $i = 1, \cdots, k$. Then, Y_0 is determined as

$$Y_0 = \pi(\psi(\sigma_i, X_{n+l} X_{n+l+1} \cdots X_{-1})).$$

Thus, (α, β) is conditionally deterministic.

5 Examples

Example 1. ([6]) Let $M = (\mathbb{Z}, \psi, 0, \mathbb{Z}_+)$, where $\psi(\sigma, a) = \sigma + (-1)^a$. Since the random walk on M' is not positively recurrent, for any normal number $\alpha \in \{0,1\}^{\mathbb{N}}$, it holds by Theorem 4 that

$$\limsup_{n\to\infty} \frac{1}{n} \#\{i \in \mathbb{N};\ 0 \le i < n,\ \psi(\sigma_0, \alpha[0,i)) = \sigma_0\} = 0.$$

Therefore, the countable word

$$\gamma := 1_{\mathbb{Z}_+}(0) 1_{\mathbb{Z}_+}(\psi(0, \alpha(0)) 1_{\mathbb{Z}_+}(\psi(0, \alpha(0)\alpha(1)) \cdots$$

is completely deterministic ([5]) since it changes the values 0 and 1 so seldom. In fact, $\mu_\gamma^U(\{000\cdots, 111\cdots\}) = 1$ for any U. Thus, (α, γ) is conditionally deterministic and $\alpha[S(M, \alpha)]$ is a normal number if $S(M, \alpha)$ has a positive lower density by Theorem 3.

Example 2. Let $\tau \in \{0,1\}^{\mathbb{Z}}$ be a bi-countable word over $\{0,1\}$ such that $\{n \in \mathbb{Z};\ \tau(n) = 1\}$ is relatively dense, that is, there exists L such that for any $n \in \mathbb{Z}$, $[n, n+L)$ intersects with $\{n \in \mathbb{Z};\ \tau(n) = 1\}$. Let α be a normal number and $\beta \in \{0,1\}^{\mathbb{N}}$ be such that

$$\beta(n) = \tau\left(\sum_{i=0}^{n-1} (-1)^{\alpha(i)}\right) \quad (n \in \mathbb{N}).$$

Then, (α, β) is conditionally deterministic since for any U, Y_0 is determined by $(X_i, Y_i)_{i=-\infty}^{-1}$ under $\tilde{\mu}^U_{(\alpha,\beta)}$ since there exists $n < 0$ such that $(-1)^{X_n} + (-1)^{X_{n+1}} + \cdots + (-1)^{X_{-1}} = 0$ almost surely as the 1-dimensional random walk is recurrent. Then, $Y_0 = Y_n$. Moreover, $S(\beta, \{1\})$ has a positive lower density since $\{n \in \mathbb{Z};\ \tau(n) = 1\}$ is relative dense. Thus, $\alpha[S(\beta, \{1\})]$ is a normal number.

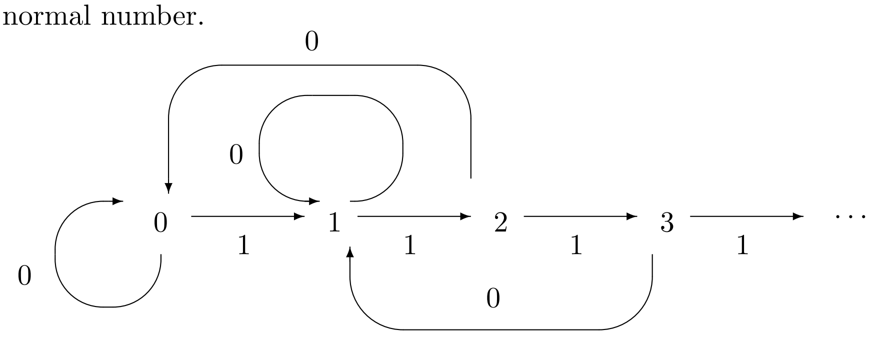

Figure 2: M' in Example 3

Example 3. Let $M = (\mathbb{N}, \psi, 0, F)$ be a countable automaton such that $\psi(\sigma, 1) = \sigma + 1$ ($\forall \sigma \in \mathbb{N}$), $\psi(\sigma, 0) = 0$ if σ is even and $\psi(\sigma, 0) = 1$ if σ is odd. Then, M' is weakly renewal with a renewal word 0. Therefore, M preserves normality.

6 Appendix

If an infinite set $S \subset \mathbb{N}$ has a zero lower density, then for any $\lambda < 1$, there exists $S' \subset S$ such that S' has a zero upper density and a relative upper density in S larger than λ.

Proof If S has a zero upper density, then $S' = S$ has the required property. Hence, assume that S has a positive upper density, say $\delta > 0$. Let η be an integer such that $\eta > \lambda/(1-\lambda)$. We define S' as $\cup_{i=0}^{\infty} S_i$. Let $S_0 = \emptyset$. Assume that a finite set S_i is already defined up to $i = 0, 1, \cdots, k-1$ $(k = 1, 2, \cdots)$. We will define S_k. Let l_k be the maximum element in $\cup_{i=0}^{k-1} S_i$ or 0 if $k = 1$. Take n such that
(1) $l_k/n < 1/(k+1)$,
(2) $0 < (1/n)\#(S \cap [0, n)) < \delta/\{(1+\eta)(k+1)\}$, and
(3) $\#(S \cap [n, (1+\eta)n)) \geq \eta\#(S \cap [0, n))$.
Such n exists since there exists n with (1)(2) as S has a zero lower density. If (3) is not satisfied, then take $(1+\eta)n$ instead of n. It satisfies (1). It also satisfies (2) since, by the assumption of (3) not being satisfied, we have $\#(S \cap [n, (1+\eta)n)) < \eta\#(S \cap [0, n))$, and hence

$$
\begin{aligned}
&(1/\{(1+\eta)n\})\#(S \cap [0, (1+\eta)n)) \\
&= (1/\{(1+\eta)n\})(\#(S \cap [0, n)) + \#(S \cap [n, (1+\eta)n)) \\
&< (1/\{(1+\eta)n\})(n\delta/\{(1+\eta)(k+1)\} + \eta\#(S \cap [0, n))) \\
&< (1/\{(1+\eta)n\})(n\delta/\{(1+\eta)(k+1)\} + n\delta\eta/\{(1+\eta)(k+1)\}) \\
&= \delta/\{(1+\eta)(k+1)\}.
\end{aligned}
$$

If it does not satisfies (3), then take $(1+\eta)^2 n$ instead of n. Then, (1)(2) are satisfied for this $(1+\eta)^2 n$ by the same reason as above. In this way, we can continue and find some $(1+\eta)^i n$ which satisfies (1)(2)(3) since S has an upper density $\delta > 0$.

Let this $(1+\eta)^i n$ be n_k. Define m_k with $n_k < m_k \leq (1+\eta)n_k$ such that $\#(S \cap [n_k, m_k)) = \eta\#(S \cap [0, n_k))$. Let $S_k = S \cap [n_k, m_k)$

Let $S' = \cup_{i=1}^{\infty} S_i$. For any $n > m_1$, let $n_{k-1} < n \leq n_k$. Then,

$$
\begin{aligned}
&(1/n)\#(S' \cap [0, n)) \leq (1/n)\#(S \cap [n_{k-1}, m_{k-1})) + (1/n)m_{k-2} \\
&\leq (1/n_{k-1})\eta\#(S \cap [0, n_{k-1})) + l_{k-1}/n_{k-1} < (\frac{\eta\delta}{1+\eta} + 1)\ \frac{1}{k}
\end{aligned}
$$

Hence, $(1/n)\#(S' \cap [0, n)) \to 0$ as $n \to \infty$ and S' has a zero upper density.

Since

$$
\#(S' \cap [n_k, m_k)) = \#(S \cap [n_k, m_k)) = \eta\#(S \cap [0, n_k)),
$$

we have

$$
\frac{\#(S' \cap [0, m_k))}{\#(S \cap [0, m_k))} \geq \frac{\#(S' \cap [n_k, m_k))}{\#(S \cap [0, n_k)) + \#(S \cap [n_k, m_k))} = \frac{\eta}{1+\eta} > \lambda
$$

for any $k = 1, 2, \cdots$. Thus, S' has a relative upper density in S larger than λ. $\square$

Acknowledgement: The authors thank Huazhong University of Science and Technology for inviting one of the authors and giving an opportunity of the collaboration. They thank Prof. Bo Tan for his useful helps to complete the paper. They also thank the anonymous referee for reading the paper carefully and giving them useful suggestions.

References

[1] R. von Mises, Grundlagen der Wahrscheinkeitsrechnung, Math. Z. 5 (1919) 52-99.

[2] H. Furstenberg, Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation, Math. Systems Theory 1-1 (1967), 1-49

[3] Y. N. Agafonov, Normal sequences and finite automata, Dokl. Akad. Nauk SSSR 179 (1968).

[4] B. Weiss, Normal sequences as collectives, Proc. Symp. on Topological Dynamics and Ergodic Theory, Univ. of Kentucky, 1971.

[5] Teturo Kamae, Subsequences of normal sequences, Israel J. Math 16 (1973), 121-149.

[6] Teturo Kamae & Benjamin Weiss, Normal numbers and selection rules, Israel J. Math 21 (1973), 101-110.

[7] Paul C. Shields, The Ergodic Theory of Discrete Sample Paths, Graduate Studies in Mathematics vol. 13, Amer. Math. Soc., 1996.