Local time of self-affine sets of Brownian motion type and the jigsaw puzzle problem

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Abstract

Let $\Omega \subset [0,1] \times [0,1]$ be the solution of the set equation:

$$\Omega = \cup_{i=1}^{k} (\varphi_{I_i,1} \times \varphi_{J_i,\tau_i})(\Omega),$$

where for an interval $I = [a,b] \subset [0,1]$ and $\tau \in \{-1,1\}$, $\varphi_{I,\tau} : [0,1] \to I$ is the linear map such that $\varphi_{I,1}(0) = a$, $\varphi_{I,1}(1) = b$, $\varphi_{I,-1}(0) = b$, $\varphi_{I,-1}(1) = a$, and $\{I_i;\ i = 1, \cdots, k\}$ is a partition of $[0,1]$ with $|J_i| = |I_i|^{1/2}$. Thus, Ω is a graph of a Borel function f_Ω almost surely and it is called a self-affine set of Brownian motion type. Let λ be the Lebesgue measure on $[0,1]$ and let $\mu_\Omega = \lambda \circ f_\Omega^{-1}$. The density $\rho_\Omega = \frac{d\mu_\Omega}{d\lambda}$, if it exists, is called the local time of Ω and it has been studied. It is known that $\dim_H \Omega = 3/2$ if ρ_Ω exists. In the present study, ρ_Ω is obtained by solving the so-called jigsaw puzzle on $\{J_i, \tau_i;\ i = 1, \cdots, k\}$, i.e., the problem of decomposing ρ_Ω into a sum of its self-similar images with the support J_i and the orientation τ_i for $i = 1, \cdots, k$.

1 Introduction

Assume that

$$(\#1) \qquad \begin{array}{l} k \text{ is an integer where } k \geq 2, \\ 0 = s_0 < s_1 < \cdots < s_k = 1, \text{ and} \\ I_i = [s_{i-1}, s_i] \ (i = 1, \cdots, k) \text{ are intervals.} \end{array}$$

In addition, assume further

$$(\#2) \qquad \begin{array}{l} J_i \ (i = 1, \cdots, k) \text{ are closed intervals in } [0,1] \\ \text{such that } |J_i| = |I_i|^{1/2} \ (i = 1, \cdots, k), \\ \text{and } \tau_i \in \{-1, 1\} \ (i = 1, \cdots, k), \end{array}$$

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where $|I|$ denotes the length $b-a$ of the interval $I=[a,b]$.

For an interval $I=[a,b]$ where $0 \le a < b \le 1$ and $\tau \in \{-1,1\}$, we define a linear map $\varphi_{I,\tau} : [0,1] \to I$ by

$$\varphi_{I,\tau}(x) = \begin{cases} a(1-x)+bx & (\tau = 1) \\ b(1-x)+ax & (\tau = -1). \end{cases}$$

We may denote $\varphi_{I,1}$ simply by φ_I. Therefore, $\varphi_I \times \varphi_{J,\tau}$ with intervals $I=[a,b]$, $J=[c,d]$ in $[0,1]$ is the linear bijection $[0,1]\times[0,1] \to I \times J$ such that $\varphi_I \times \varphi_{J,1}$ maintains the orientation, whereas $\varphi_I \times \varphi_{J,-1}$ reverses the vertical orientation. The former map is called the upward map and the latter map is called the downward map, which are denoted by upward and downward arrows, respectively.

Definition 1. Let $\Omega = \Omega(I_1,\cdots,I_k; J_1,\cdots,J_k; \tau_1,\cdots,\tau_k)$ be the compact set in $[0,1]\times[0,1]$ that satisfies the following set equation [2]:

$$\Omega = \bigcup_{i=1}^{k} (\varphi_{I_i} \times \varphi_{J_i,\tau_i})(\Omega). \tag{1.1}$$

We refer to Ω as a self-affine set of *Brownian motion type*.

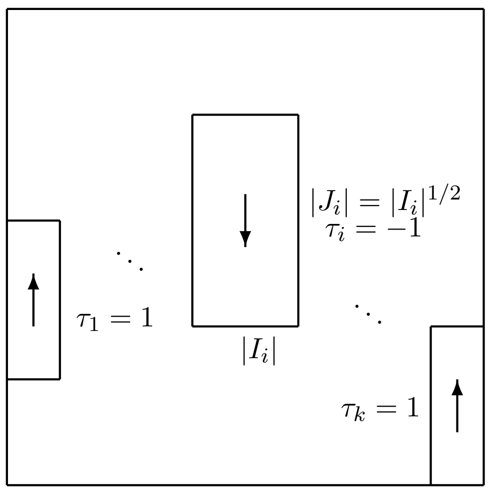

Figure 1: Set equation for Ω, where $\uparrow$ and $\downarrow$ correspond to the upward or downward linear maps, respectively, from the unit square to $I_i \times J_i$ $(i = 1,2,\cdots,k)$.

Self-affine sets of Brownian motion type were studied in [4], [5], and [7] from the perspective of stochastic or functional analysis. McMullen's notion of general Sierpiński carpets [3] is similar to our notion, but there are some differences, i.e., our notion has more freedom to choose the position and the orientation of J_i, whereas $\log|J_i|/\log|I_i|$ is not necessarily $1/2$ in McMullen's notion and there can be more than one rectangle on the same vertical line. In addition, continuous self-affine functions with $|I_i| = 1/k$ $(i = 1, \cdots, k)$ were studied previously [6], where it was proved that functions with different coprime k and k' that coincide are linear functions.

The self-affine sets were studied in a general framework by Feng and Hu [9]. They considered the iterated function system $\{S_1, \cdots, S_k\}$, where $S_i : \mathbf{R}^d \to \mathbf{R}^d$ $(i = 1, \cdots, k)$ are contracting injections, and they considered the attractor K with the probability measure $\mathbb{P}\circ\pi^{-1}$, where $\mathbb{P}$ is a shift-invariant measure on $\{1, \cdots k\}^\infty$ and $\pi : \{1, \cdots, k\}^\infty \to K$ such that $\pi(\sigma_1\sigma_2\cdots) = x$ with

$$\{x\} = \cap_{n=1}^{\infty} S_{\sigma_1} \circ \cdots \circ S_{\sigma_n} K.$$

In this context, our case is

$$\begin{aligned} &\mathbb{P} = (s_1 - s_0, \ \cdots, \ s_k - s_{k-1})^\infty, \text{ and} \\ &S_i = \varphi_{I_i} \times \varphi_{J_i,\tau_i} \quad (i = 1, \cdots, k). \end{aligned}$$

We provide actual examples to demonstrate their general theory, where we obtain the local times by solving jigsaw puzzle problems (Section 3). For the solution ρ of the jigsaw puzzle $\{J_i, \tau_i; \ i = 1, \cdots, k\}$, we refer to a piecewise continuous function $\rho : [0, 1] \to [0, \infty)$ with $\int \rho dx = 1$ such that the closure of $\{(x, y); \ 0 \le y \le \rho(x)\}$ is decomposed into self-similar images with the support J_i and the orientation τ_i modulo of the vertical isomorphism (Definition 4). This jigsaw puzzle is completely combinatorial and its solution is sensitive to the positions and the orientations of J_i and τ_i.

2 Basic notions and preliminary results

Let $\Omega = \Omega(I_1, \cdots, I_k; J_1, \cdots, J_k; \tau_1, \cdots, \tau_k)$ be as stated in Definition 1 with (#1) and (#2). For $l = 1, 2, \cdots$ and $i_1 \cdots i_l \in \{1, 2, \cdots, k\}^l$, let

$$\begin{aligned} \psi_{i_1\cdots i_l} &= (\varphi_{I_{i_1}} \times \varphi_{J_{i_1},\tau_{i_1}}) \circ \cdots \circ (\varphi_{I_{i_l}} \times \varphi_{J_{i_l},\tau_{i_l}}) \\ &= (\varphi_{I_{i_1}} \circ \cdots \circ \varphi_{I_{i_l}}) \times (\varphi_{J_{i_1},\tau_{i_1}} \circ \cdots \circ \varphi_{J_{i_l},\tau_{i_l}}) \end{aligned}$$

be the linear map from $[0, 1]\times[0, 1]$ to a small rectangle, such as $I_{i_1\cdots i_l}\times J_{i_1\cdots i_l}$, which is upward or downward depending on whether $\tau_{i_1}\cdots\tau_{i_l}$ is 1 or -1. It is known [2] that

$$\Omega = \bigcap_{l=1}^{\infty} \bigcup_{i_1\cdots i_l\in\{1,2,\cdots,k\}^l} \psi_{i_1\cdots i_l}(([0, 1] \times [0, 1]).$$

We refer to $\psi_{i_1\cdots i_l}$ as a *fundamental map* of level l. Moreover, it is easy to see that $|I_{i_1\cdots i_l}| = |J_{i_1\cdots i_l}|^2$. We refer to the rectangle $I_{i_1\cdots i_l} \times J_{i_1\cdots i_l}$ as a *fundamental rectangle* of level l. It is clear that there exist k^l+1 points

$$0 = s_0^l < s_1^l < \cdots < s_{k^l}^l = 1$$

such that $I_{i_1\cdots i_l} = [s_{j-1}^l, s_j^l]$, where j is such that $i_1\cdots i_l$ is the j-th element from below in the lexicographical order of $\{1,2,\cdots,k\}^l$. The interval $[s_{j-1}^l, s_j^l]$ for $j=1,2,\cdots,k^l$ is referred to as the j-th *fundamental interval* of level l. Let

$$C_0 := \min_{i=1,\cdots,k} |s_i - s_{i-1}|\ ,\quad C_1 := \max_{i=1,\cdots,k} |s_i - s_{i-1}|. \tag{2.1}$$

Then, it is clear that $0 < C_0 \le C_1 < 1$ and $C_0^l \le s_j^l - s_{j-1}^l \le C_1^l$.

The following facts are well known, but we give proofs to ensure that our study is self-contained.

Fact 1. *The* $3/2$*-dimensional Hausdorff measure of* Ω *satisfies* $\mathcal{H}^{3/2}(\Omega) \le 2$, *hence* $\dim_H \Omega \le 3/2$.

Proof Take a sufficiently small $\delta > 0$. Take a positive integer l such that $C_1^l < \delta$. Take the j-th fundamental rectangle from the left, such as $I_{i_1\cdots i_l} \times J_{i_1\cdots i_l}$, of level l and let $\varepsilon_j := |I_{i_1\cdots i_l}|$. It is clear that $\varepsilon_j \le C_1^l < \delta$. Since $|J_{i_1\cdots i_l}| = \varepsilon_j^{1/2}$, $I_{i_1\cdots i_l} \times J_{i_1\cdots i_l}$ is covered by $\lceil \varepsilon_j^{-1/2} \rceil$ squares, such as $U_1^j, \cdots, U^j_{\lceil \varepsilon_j^{-1/2}\rceil}$ where the diameters $|U_h^j|$ are equal to $\sqrt{2}\varepsilon_j$. Hence,

$$\begin{aligned}\sum_{j=1}^{k^l} \sum_{h=1}^{\lceil \varepsilon_j^{-1/2}\rceil} |U_h^j|^{3/2} &= \sum_{j=1}^{k^l} \sum_{h=1}^{\lceil \varepsilon_j^{-1/2}\rceil} (\sqrt{2}\varepsilon_j)^{3/2} \\ &\le 2^{3/4} \sum_{j=1}^{k^l} (\varepsilon_j^{-1/2}+1)\varepsilon_j^{3/2} \le 2\sum_{j=1}^{k^l} \varepsilon_j = 2.\end{aligned}$$

Since

$$\bigcup_{j=1}^{k^l} \bigcup_{h=1}^{\lceil \varepsilon_j^{-1/2}\rceil} U_h^j \supset \Omega,$$

this implies that $\mathcal{H}^{3/2}(\Omega) \le 2$, and hence $\dim_H \Omega \le 3/2$. □

Fact 2. *There exists a Borel function* $f_\Omega : [0,1] \to [0,1]$ *such that* $\Omega_x = \{f_\Omega(x)\}$ *holds for all but countably many* $x \in [0,1]$, *where* $\Omega_x := \{y;\ (x,y) \in \Omega\}$.

Proof Take any $x \in [0,1]$ such that

$$x \notin \bigcup_{l=1}^{\infty} \{s_0^l, s_1^l, \cdots, s_{k^l}^l\}.$$

Then, for any $l = 1, 2, \cdots$, Ω_x is contained in a unique interval $J_{i_1 \cdots i_l}$ for some $i_1 \cdots i_l \in \{1, 2, \cdots, k\}^l$. Since these closed intervals are decreasing in l and $|J_{i_1 \cdots i_l}| \le C_1^{l/2}$, it holds that $\#\Omega_x = 1$. For a general $x \in [0,1]$, we define

$$f_\Omega(x) = \lim_{l \to \infty} \max \bigcup_{i_1 \cdots i_l;\ x \in I_{i_1 \cdots i_l}} J_{i_1 \cdots i_l}.$$

Then, f_Ω is a Borel function such that $\Omega_x = \{f_\Omega(x)\}$ holds for all but countably many $x \in [0,1]$. □

Definition 2. Let λ be the Lebesgue measure on $[0,1]$. We define a probability Borel measure λ_Ω on $[0,1] \times [0,1]$ and a probability Borel measure μ_Ω on $[0,1]$, so for any Borel sets $U \subset [0,1] \times [0,1]$ and $V \subset [0,1]$,

$$\lambda_\Omega(U) = \lambda\{x \in [0,1];\ (x, f_\Omega(x)) \in U\} \text{ and } \mu_\Omega(V) = \lambda(f_\Omega^{-1} V).$$

Then by Fact 2, $\lambda_\Omega(\Omega) = 1$ holds. Moreover, the marginal distribution of λ_Ω is λ to the first coordinate and μ_Ω to the second coordinate.

Definition 3. If μ_Ω is absolutely continuous with respect to λ, the density $\dfrac{d\mu_\Omega}{d\lambda}$ is denoted by ρ_Ω, which is called the *local time* of Ω.

Let $\Omega = \Omega(I_1, \cdots, I_k; J_1, \cdots, J_k; \tau_1, \cdots, \tau_k)$. In the present study, we prove the following results.

Lemma 1. *The measure μ_Ω is the unique probability Borel measure μ that satisfies*

$$\mu = \sum_{i=1}^{k} |J_i|^2 \mu \circ \varphi_{J_i, \tau_i}^{-1}. \tag{2.2}$$

In particular, μ_Ω depends only on the totality of $(J_1, \tau_1), \cdots, (J_k, \tau_k)$ such that $|J_1|^2 + \cdots + |J_k|^2 = 1$, and it does not depend on their order or $I_1, \cdots, I_k$.

Lemma 2. *Ω has the local time ρ if and only if a Borel function $\rho : [0,1] \to [0, \infty)$ satisfies the following functional equation:*

$$\int \rho d\lambda = 1, \ \textit{and} \tag{2.3}$$

$$\rho(x) = \sum_{i=1}^{k} |J_i| 1_{x \in J_i} \rho(\varphi_{J_i, \tau_i}^{-1} x) \ \textit{for almost all } x \in [0,1]. \tag{2.4}$$

Theorem 1. ([9]) *For any Ω, one of the following three cases holds:*
(1) *μ_Ω is supported by a one-point set,*
(2) *μ_Ω is a continuous and singular measure,*
(3) *Ω has a local time.*

Theorem 2. *If Ω has a bounded local time, then* $\dim_H \Omega = 3/2$. *Moreover,* $0 < \mathcal{H}^{3/2}(\Omega) < \infty$ *holds and* λ_Ω *is absolutely continuous with respect to* $\mathcal{H}^{3/2}$.

Remark 1. It was proved in [9] that $\dim_H \lambda_\Omega = 3/2$ if $\dim_H \mu_\Omega = 1$, where $\dim_H \mu$ was defined as $\inf\{\dim_H S;\ \mu(S) = 1\}$ for a probability measure μ. However, it is not generally true that $\dim_H \lambda_\Omega = \dim_H \Omega$ (Example 3).

3 Jigsaw puzzle

Definition 4. For a set S in $[0,1] \times [0,1]$ and $x \in [0,1]$, denote $S_x = \{y \in [0,1];\ (x,y) \in S\}$. We say that S is *vertically convex* if S is a regular set (in the sense that the closure of the interior of S is S) such that S_x is either the empty set, one-point set, or a closed interval for any $x \in [0,1]$. Let U and V be vertically convex sets in $[0,1] \times [0,1]$. They are said to be *vertically isomorphic* if $|U_x| = |V_x|$ holds for almost all $x \in [0,1]$ (w.r.t. the Lebesgue measure). The vertically isomorphic class that contains U is denoted by $[U]$. In the following figures of the solutions of the jigsaw puzzle, we use elements in $[U]$, which we denote by $[U]$.

Definition 5. Let $\rho : [0,1] \to [0,\infty)$ be a *piecewise continuous* function in the sense that it is either left continuous or right continuous at any point in $[0,1]$, where limits from both sides exist at any point in $(0,1)$, and it is continuous except for finitely many points. For a piecewise continuous function $\rho : [0,1] \to [0,\infty)$ with $\int \rho d\lambda = 1$, denote

$$\Gamma_\rho = \text{the closure of } \{(x,y);\ 0 \le y \le \rho(x)\}.$$

Then, Γ_ρ is vertically convex and $\rho(x) = |(\Gamma_\rho)_x|$ holds for almost all $x \in [0,1]$. Let $J = [a,b]$ be an interval in $[0,1]$. Denote

$$J^1\Gamma_\rho = \{(\varphi_J(x), |J|y);\ (x,y) \in \Gamma_\rho\}\,,\quad J^{-1}\Gamma_\rho = \{(\varphi_{J,-1}(x), |J|y);\ (x,y) \in \Gamma_\rho\}$$

(see Figure 2). Let $\{J_i;\ i = 1, \cdots, k\}$ be a family of intervals in $[0,1]$ such that $\sum_{i=1}^k |J_i|^2 = 1$ and $\tau_1 \cdots \tau_k \in \{-1,1\}^k$. In the above, ρ is called the *solution of the jigsaw puzzle* for $\{J_1, \cdots, J_k; \tau_1, \cdots, \tau_k\}$ if the vertically convex sets $\Lambda_i \in [J^{\tau_i}\Gamma_\rho]$ $(i = 1, \cdots, k)$ exist such that:
(1) $\Gamma_\rho = \cup_{i=1}^k \Lambda_i$, and
(2) the interiors of Λ_i and Λ_j are disjoint for any i, j with $i \ne j$.
In this case, $|(\Gamma_\rho)_x| = \sum_{i=1}^k |\Lambda_x|$ holds for almost all $x \in [0,1]$.

Theorem 3. *Let* $\rho : [0,1] \to [0,\infty)$ *be a piecewise continuous function with* $\int \rho d\lambda = 1$. *For any*

$$\Omega = \Omega(I_1, \cdots, I_k; J_1, \cdots, J_k; \tau_1, \cdots, \tau_k),$$

ρ *is the local time of* Ω *if and only if* ρ *is the solution of the jigsaw puzzle for* $\{J_1, \cdots, J_k; \tau_1, \cdots, \tau_k\}$.

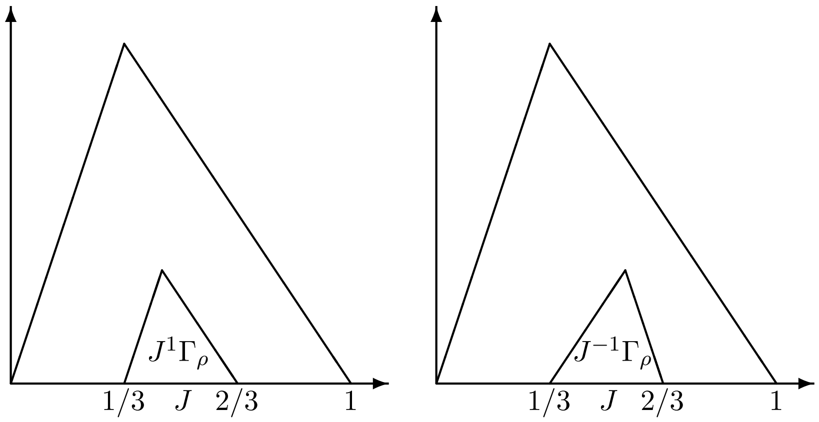

Figure 2: $J^1\Gamma_\rho$ and $J^{-1}\Gamma_\rho$ for $J = [1/3, 2/3]$ and ρ with $\rho(x) = 6x$ $(0 \le x \le 1/3)$, $= 3 - 3x$ $(1/3 < x \le 1)$

Example 1. If

$$J_1 = [0, 2/3],\ J_2 = [1/3, 2/3],\ J_3 = [1/3, 1],$$

then the local time $\rho = \rho_\Omega$ exists and it satisfies

$$\rho(x) = \begin{cases} 4x & 0 \le x \le 1/2 \\ 4 - 4x & 1/2 < x \le 1 \end{cases}$$

for any $\tau_1, \tau_2, \tau_3 \in \{1, -1\}$ since it is symmetric and the solution of the jigsaw puzzle is as shown in Figure 3. In a special case where

$$\Omega = \Omega([0, 4/9], [4/9, 5/9], [5/9, 1]; [0, 2/3], [1/3, 2/3], [1/3, 1]; 1, -1, 1),$$

$\Omega = \{(t, f_\Omega(t));\ t \in [0, 1]\}$ holds with a continuous function f_Ω, which is called a deterministic Brownian path and this was studied in [7] (Figure 4).

Example 2. If $J_1 = J_2 = [0, 1/2],\ J_3 = J_4 = [1/2, 1]$, and $\tau_1, \tau_2, \tau_3, \tau_4 \in \{1, -1\}$ are arbitrary, then the local time $\rho = \rho_\Omega$ exists and it satisfies $\rho(x) = 1$ for almost all $x \in [0, 1]$ since it is the solution to the jigsaw puzzle for $\{J_1, J_2, J_3, J_4, 1, 1, 1, 1\}$, as shown in Figure 5.

Example 3. If $J_1 = [0, 1/2],\ J_2 = J_3 = J_4 = [1/2, 1]$ and $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 1$, then μ_Ω is a singular measure such that

$$\mu_\Omega([m/2^n, (m+1)/2^n)) = \prod_{i=0}^{n-1} \theta(m_i)$$

for any $n = 1, 2, \cdots$ and $m = 0, 1, \cdots, 2^n - 1$, where $m = \sum_{i=0}^{n-1} m_i 2^i$ with $m_i \in \{0, 1\}$ and $\theta(0) = 1/4,\ \theta(1) = 3/4$. It was proved in [3] that $\dim_H \Omega = \log_2(1 + \sqrt{3})$, while it was proved in [9] that $\dim_H \lambda_\Omega = 2 - (3/8)\log_2 3$.

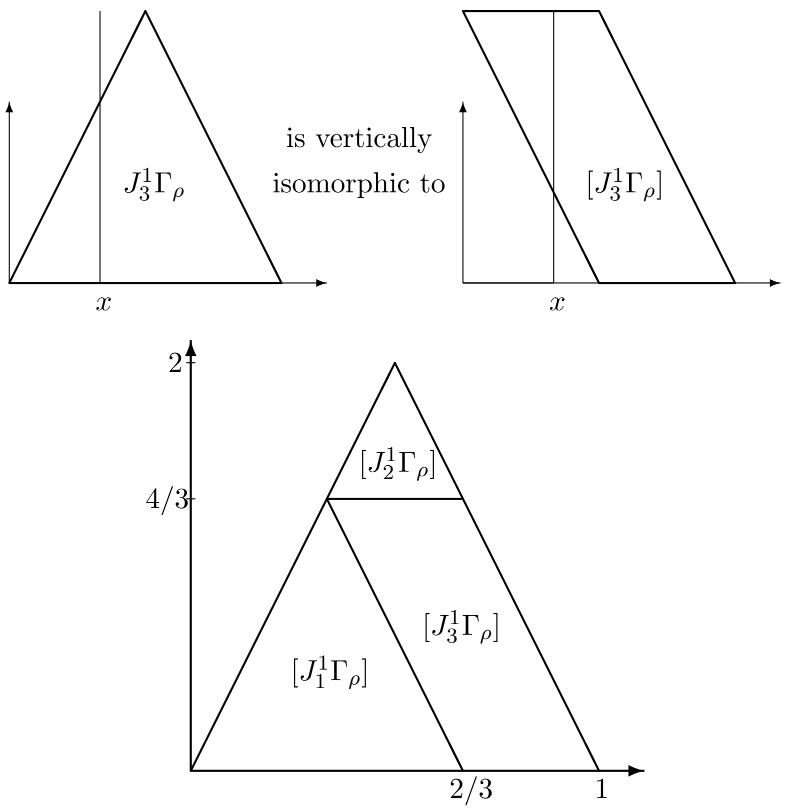

Figure 3: The solution ρ of the jigsaw puzzle for $\{J_1, J_2, J_3; 1, 1, 1\}$ with $J_1 = [0, 2/3]$, $J_2 = [1/3, 2/3]$, $J_3 = [1/3, 1]$

Figure 4: $\Omega([0, 4/9], [4/9, 5/9], [5/9, 1]; [0, 2/3], [1/3, 2/3], [1/3, 1]; 1, -1, 1)$

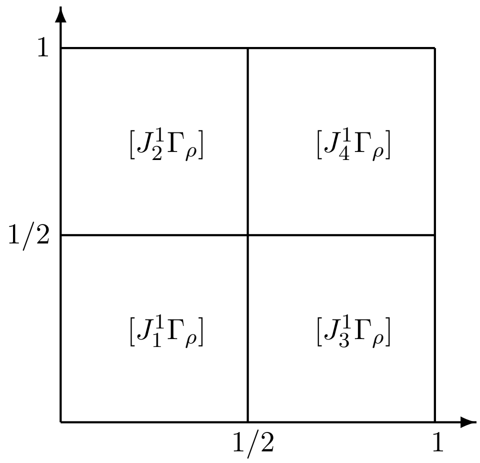

Figure 5: Solution of the jigsaw puzzle in Example 2.

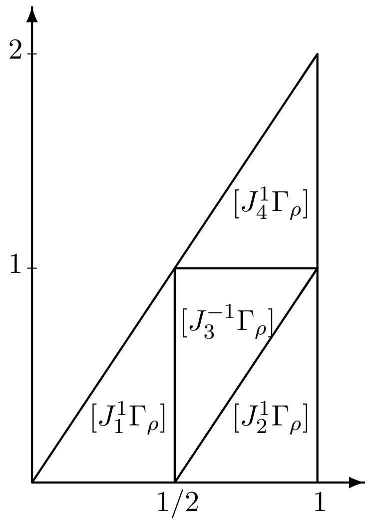

Figure 6: Solution of the jigsaw puzzle for $\{J_1, J_2, J_3, J_4; 1, 1, -1, 1\}$ with $J_1 = [0, 1/2],\ J_2 = J_3 = J_4 = [1/2, 1]$.

Example 4. If

$$J_1 = J_2 = J_3 = J_4 = [1/2, 1] \text{ and } \tau_1 = \tau_2 = \tau_3 = \tau_4 = 1,$$

then μ_Ω is supported by the one-point set $\{1\}$, since $\Omega = [0, 1] \times \{1\}$.

Example 5. Let $J_1 = [0, 1/2],\ J_2 = J_3 = J_4 = [1/2, 1]$, as in Example 3, but let $\tau_1 = \tau_2 = 1,\ \tau_3 = -1,\ \tau_4 = 1$. Then, μ_Ω is an absolutely continuous measure since $\rho(x) = 2x$ $(x \in [0, 1])$ is the solution of the jigsaw puzzle shown in Figure 6. In this case, f_Ω is a continuous function, which is related to the Rudin-Shapiro sequence [4].

4 Proof of the results

Proof of Lemma 1

By (1.1) and Fact 2, it holds that $\varphi_{J_i,\tau_i}(f_\Omega(x)) = f_\Omega(\varphi_{I_i}(x))$ for any $i = 1, 2, \cdots, k$ and for all but countably many $x \in [0, 1]$. Moreover, since $1_{I_i}\lambda = |I_i|\lambda \circ \varphi_{I_i}^{-1}$ for any $i = 1, 2, \cdots, k$, we have $\lambda = \sum_{i=1}^{k} |I_i|\lambda \circ \varphi_{I_i}^{-1}$. Hence, we have

$$\begin{aligned}\mu_\Omega = \lambda \circ f_\Omega^{-1} &= \sum_{i=1}^{k} |I_i|\lambda \circ \varphi_{I_i}^{-1} \circ f_\Omega^{-1} \\ &= \sum_{i=1}^{k} |I_i|\lambda \circ f_\Omega^{-1} \circ \varphi_{J_i,\tau_i}^{-1} = \sum_{i=1}^{k} |J_i|^2 \mu_\Omega \circ \varphi_{J_i,\tau_i}^{-1}\end{aligned}$$

Therefore, μ_Ω satisfies (2.2).

If a probability Borel measure μ satisfies (2.2), then by applying (2.2) l times, we have

$$\mu = \sum_{i_1,\cdots,i_l=1}^{k} |J_{i_1\cdots i_l}|^2 \mu \circ \varphi_{J_{i_1\cdots i_l},\tau_{i_1}\cdots\tau_{i_l}}^{-1}, \tag{4.1}$$

Let $g : [0, 1] \to [0, 1]$ be any continuous function such that $|g(x) - g(y)| \le C|x-y|$ for any $x, y \in [0, 1]$ with a constant C. Then, since $\mu \circ \varphi_{J_{i_1\cdots i_l},\tau_{i_1}\cdots\tau_{i_l}}^{-1}$ is supported by the interval $J_{i_1\cdots i_l}$ with a length of at most $C_1^{l/2}$, we have

$$|\int g d\mu \circ \varphi_{J_{i_1\cdots i_l},\tau_{i_1}\cdots\tau_{i_l}}^{-1} - \int g d\nu \circ \varphi_{J_{i_1\cdots i_l},\tau_{i_1}\cdots\tau_{i_l}}^{-1}| \le C C_1^{l/2}$$

for any probability Borel measures μ and ν. Therefore, if both μ and ν satisfy (2.2), and hence (4.1), then

$$\begin{aligned}&|\int g d\mu - \int g d\nu| \\ &\le \sum_{i_1,\cdots,i_l=1}^{k} |J_{i_1\cdots i_l}|^2 \; |\int g d\mu \circ \varphi_{J_{i_1\cdots i_l},\tau_{i_1}\cdots\tau_{i_l}}^{-1} - \int g d\nu \circ \varphi_{J_{i_1\cdots i_l},\tau_{i_1}\cdots\tau_{i_l}}^{-1}| \\ &\le C C_1^{l/2} \sum_{i_1,\cdots,i_l=1}^{k} |J_{i_1\cdots i_l}|^2 = C C_1^{l/2}.\end{aligned}$$

Thus, letting $l \to \infty$, we have $\int g d\mu = \int g d\nu$. Since this holds for any Hölder continuous function g, we have $\mu = \nu$. Thus, the probability Borel measure μ that satisfies (2.2) is unique. □

Proof of Lemma 2

If Ω has the local time ρ, then by (2.2), we have

$$\rho = \frac{d\mu_\Omega}{d\lambda} = \sum_{i=1}^{k} |J_i|^2 \frac{d\mu_\Omega \circ \varphi_{J_i,\tau_i}^{-1}}{d\lambda}$$
$$= \sum_{i=1}^{k} |J_i| \cdot 1_{J_i} \frac{d\mu_\Omega \circ \varphi_{J_i,\tau_i}^{-1}}{d\lambda \circ \varphi_{J_i,\tau_i}^{-1}} = \sum_{i=1}^{k} |J_i| \cdot 1_{J_i}\ \rho \circ \varphi_{J_i,\tau_i}^{-1}.$$

Conversely, if ρ satisfies (2.3) and (2.4), then the probability Borel measure μ on $[0,1]$ with $d\mu(x) = \rho(x)d\lambda(x)$ satisfies (2.2). Thus, ρ is the local time of Ω. □

Assume that Ω has a bounded local time ρ_Ω such that $\rho_\Omega(x) \leq C_2$ for any $x \in [0,1]$.

Lemma 3. *Let I be a fundamental interval and J be any closed interval. Then, it holds that $\lambda_\Omega(I \times J) \leq C_2|I|^{1/2}|J|$.*

Proof Let $I \times J'$ be the fundamental rectangle that corresponds to I. To estimate $\lambda_\Omega(I \times J)$ from the above, we may assume that $J \subset J'$ since $\lambda_\Omega(I \times ([0,1] \setminus J')) = 0$. Let $\varphi_I \times \varphi_{J',\tau}$ be the fundamental mapping where $\tau = 1$ or -1 correspond to whether J' is upward or downward. Since $(\varphi_I \times \varphi_{J',\tau})^{-1}$ maps $I \times J$ to a rectangle $[0,1] \times [a,b]$ with $b - a = |J||I|^{-1/2}$ and

$$\lambda_\Omega(I \times J) = |I|\lambda_\Omega((\varphi_I \times \varphi_{J',\tau})^{-1}(I \times J)) = |I|\lambda_\Omega([0,1] \times [a,b]),$$

we have

$$\lambda_\Omega(I \times J) = |I|\mu_\Omega([a,b]) \leq |I|C_2|J||I|^{-1/2} = C_2|I|^{1/2}|J|.$$

□

Lemma 4. *For any rectangle U, which is a product of the intervals on the x-axis and the y-axis, it holds that $\lambda_\Omega(U) \leq 2C_0^{-1/2}C_2|U|^{3/2}$, where $|U|$ is the diameter of U.*

Proof We may take $U' = [a,b] \times [c,d] \supset U$ such that $d - c \leq b - a \leq |U|$. Take at most two fundamental intervals I^i $(i = 1,2)$ (one of them can be the empty set) such that $[a,b]$ is contained in $I^1 \cup I^2$ and $|I^i| \leq C_0^{-1}(b-a)$ $(i = 1,2)$. Since $U \subset (I^1 \times [c,d]) \cup (I^2 \times [c,d])$, by Lemma 3, we have

$$\lambda_\Omega(U) \leq \lambda_\Omega(I^1 \times [c,d]) + \lambda_\Omega(I^2 \times [c,d])$$
$$\leq 2C_2(C_0^{-1}(b-a))^{1/2}(b-a) \leq 2C_0^{-1/2}C_2|U|^{3/2}.$$

□

<u>Proof of Theorem 2</u>

For any $\delta > 0$ and rectangles $U_i \in \mathcal{R}$ $(i = 1, 2, \cdots)$, which are products of the intervals on the x-axis and y-axis such that $|U_i| < \delta$ $(i = 1, 2, \cdots)$ and $\cup_{i=1} U_i \supset \Omega$, we have

$$\sum_{i=1}^{\infty} |U_i|^{3/2} \geq (1/2) C_0^{1/2} C_2^{-1} \sum_{i=1}^{\infty} \lambda_\Omega(U_i) \geq (1/2) C_0^{1/2} C_2^{-1} \lambda_\Omega(\Omega) = (1/2) C_0^{1/2} C_2^{-1}$$

by Lemma 4. Hence, $\mathcal{H}^{3/2}(\Omega) \geq (1/2) C_0^{1/2} C_2^{-1}$. Since $\mathcal{H}^{3/2}(\Omega) \leq 2$ was proved in Fact 1, we have $0 < \mathcal{H}^{3/2}(\Omega) < \infty$ and $\dim_H \Omega = 3/2$. By Lemma 4, Λ_Ω is absolutely continuous with respect to $\mathcal{H}^{3/2}$.

Assume that a function $\rho : [0, 1] \to [0, \infty)$ is the solution of the jigsaw puzzle for $\{J_1, \cdots, J_k; \tau_1, \cdots, \tau\}$. Thus, the vertically convex sets Λ_i $(i = 1, 2, \cdots, k)$ exist such that for each $i = 1, 2, \cdots, k$, Λ_i is vertically isomorphic to $J_i^{\tau_i} \Gamma_\rho$ and (1)(2) in Definition 5 hold. Then, we have

$$|(\Gamma_\rho)_x| = \sum_{i=1}^{k} |(\Lambda_i)_x| = \sum_{i=1}^{k} |(J_i^{\tau_i} \Gamma_\rho)_x|$$

for almost all $x \in [0, 1]$. Therefore, we have

$$\rho(x) = \sum_{i=1}^{k} |J_i| 1_{x \in J_i} \rho(\varphi_{J_i, \tau_i}^{-1} x)$$

for almost all $x \in [0, 1]$. Hence, by Lemma 2, ρ is the local time of Ω.

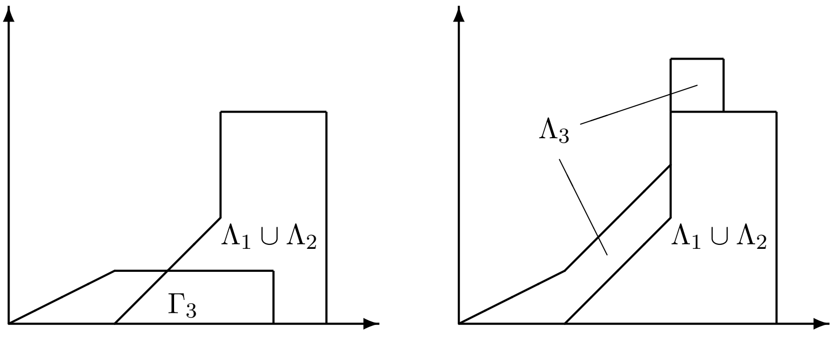

Figure 7: Piling up Γ_3 above $\Lambda_1 \cup \Lambda_2$

.

Conversely, if ρ is the local time of Ω, then we have

$$\rho(x) = \sum_{i=1}^{k} |J_i| 1_{x \in J_i} \rho(\varphi_{J_i, \tau_i}^{-1} x)$$

for almost all $x \in [0,1]$. Since ρ is piecewise continuous, the functions $|J_i| 1_{x\in J_i} \rho(\varphi^{-1}_{J_i,\tau_i} x)$ of x for $i = 1, 2, \cdots, k$ are also piecewise continuous. Moreover, we have $|(\Gamma_\rho)_x| = \sum_{i=1}^{k} |(J_i^{\tau_i}\Gamma_\rho)_x|$ for almost all $x \in [0,1]$.

We denote $\Gamma_i = J_i^{\tau_i}\Gamma_\rho$. We define the vertically convex and mutually disjoint (in their interiors) subsets $\Lambda_1, \cdots, \Lambda_k$ of Γ_ρ so Λ_i is vertically isomorphic to Γ_i for $i = 1, 2, \cdots, k$. Let $\Lambda_1 = \Gamma_1$. We pile up the amount of Γ_2 above Λ_1 and define Λ_2 as the piled part. Next, we pile up the amount of Γ_3 above $\Lambda_1 \cup \Lambda_2$ and define Λ_3 as the piled part (Figure 7). In this manner, we can define a partition $\Lambda_1, \cdots, \Lambda_k$ of Γ_ρ, as in Definition 5, with respect to $\{J_1, \cdots, J_k; \tau_1, \cdots, \tau\}$. Hence, ρ is the solution of the jigsaw puzzle for $\{J_1, \cdots, J_k; \tau_1, \cdots, \tau\}$. □

5 Further examples

Example 6. (Dai, Feng, and Wang [8]) Let $J_1 = [0, \sqrt{2}/2]$, $J_2 = [(2 - \sqrt{2})/2, 1]$ and $\tau_1, \tau_2 \in \{-1, 1\}$ be arbitrary. Then,

$$\rho(x) = \begin{cases} \frac{4+3\sqrt{2}}{2}\, x & 0 \le x \le \sqrt{2} - 1 \\ \frac{2+\sqrt{2}}{2} & \sqrt{2} - 1 < x \le 2 - \sqrt{2} \\ \frac{4+3\sqrt{2}}{2}\,(1-x) & 2 - \sqrt{2} < x \le 1 \end{cases}$$

is the solution of the jigsaw puzzle shown in Figure 8.

Example 7. For any $0 \le a \le 1$, let

$$J_1 = [0, 1/2],\ J_2 = J_3 = [a, a + (1/2)],\ J_4 = [1/2, 1]$$

and either $\tau_2 = -1$ and $\tau_1 = \tau_3 = \tau_4 = 1$ or $\tau_3 = -1$ and $\tau_1 = \tau_2 = \tau_4 = 1$. Then,

$$\rho(x) = \begin{cases} 2x/a & (0 \le x \le a) \\ 2(1-x)/(1-a) & (a < x \le 1) \end{cases},$$

where $\rho(0) = 2$ if $a = 0$ and $\rho(1) = 2$ if $a = 1$ is the solution of the jigsaw puzzle shown in Figure 9. If $a = 1$, this is the same as Example 5.

Example 8. If

$$J_1 = [0, 1/2],\ J_2 = [1/4, 3/4],\ J_3 = J_4 = [1/2, 1]$$

and $\tau_1 = -1$, $\tau_2 = \tau_3 = \tau_4 = 1$, then the local time $\rho := \rho_\Omega$ exists and it satisfies

$$\rho(x) = \begin{cases} 2/3 & (0 \le x \le 1/2) \\ 4/3 & (1/2 < x \le 1) \end{cases}.$$

This is the solution of the jigsaw puzzle for $\{J_1, J_2, J_3, J_4; 1, 1, 1, -1\}$, which is shown in Figure 10.

Example 9. If

$$J_1 = J_2 = J_3 = J_4 = [1/2, 1]$$

and only two of $\tau_1, \tau_2, \tau_3, \tau_4$ are -1 and the others are 1, then the local time $\rho := \rho_\Omega$ exists and it satisfies

$$\rho(x) = \begin{cases} 0 & (0 \leq x \leq 1/2) \\ 2 & (1/2 < x \leq 1) \end{cases}$$

This is the solution of the jigsaw puzzle for $\{J_1, J_2, J_3, J_4; \tau_1, \tau_2, \tau_3, \tau_4\}$ (Figure 11).

Example 10. Let

$$\begin{aligned} &J_1 = [0, 1/3],\ J_2 = [1/9, 4/9],\ J_3 = [1/3, 2/3],\ J_4 = J_5 = [4/9, 7/9], \\ &J_6 = [5/9, 8/9],\ J_7 = J_8 = J_9 = [2/3, 1],\ \text{and} \\ &\tau_1 = -1,\ \tau_1 = 1,\ \tau_3 = -1,\ \tau_4 = -1,\ \tau_5 = 1,\ \tau_6 = 1,\ \tau_7 = \tau_8 = \tau_9 = 1 \\ &(\tau_4 = 1,\ \tau_5 = -1 \text{ is also possible}) \end{aligned}$$

Then, the local time $\rho := \rho_\Omega$ exists and it satisfies

$$\rho(x) = \begin{cases} 1/2 & (0 \leq x \leq 1/3) \\ 1 & (1/3 < x \leq 2/3) \\ 3/2 & (2/3 < x \leq 1) \end{cases}$$

since it is the solution of the jigsaw puzzle for $\{J_1, J_2, J_3, J_4, J_5, J_6, J_7, J_8, J_9; \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8, \tau_9\}$, as it is shown in Figure 12.

Example 11. If

$$J_1 = [0, 4/7],\ J_2 = [2/7, 4/7],\ J_3 = [2/7, 5/7],\ J_4 = [3/7, 5/7],\ J_5 = [3/7, 1],$$

then the local time $\rho := \rho_\Omega$ exists and it satisfies

$$\rho(x) = \begin{cases} 4x & 0 \leq x \leq 1/2 \\ 4 - 4x & 1/2 < x \leq 1 \end{cases}$$

for any $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5 \in \{1, -1\}$, since ρ is the solution of the jigsaw puzzle for $\{J_1, J_2, J_3.J_4.J_5; \tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$, as it is shown in Figure 13.

6 Open problems

Problem 1: Is there a self-affine set of Brownian motion type with an unbounded local time?
Problem 2: Is there a self-affine set of Brownian motion type with a bounded local time that is not piecewise linear?

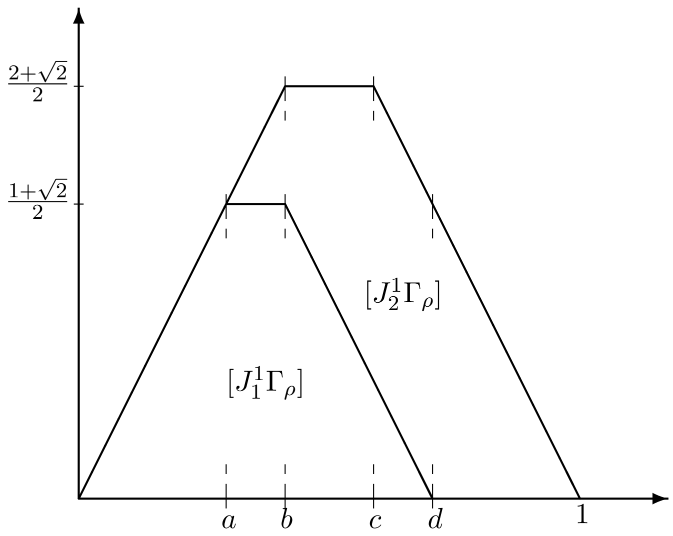

Figure 8: $a = \frac{2-\sqrt{2}}{2},\ b = \sqrt{2}-1,\ c = 2-\sqrt{2},\ d = \frac{\sqrt{2}}{2}$ and $J_1 = [0, d],\ J_2 = [a, 1]$

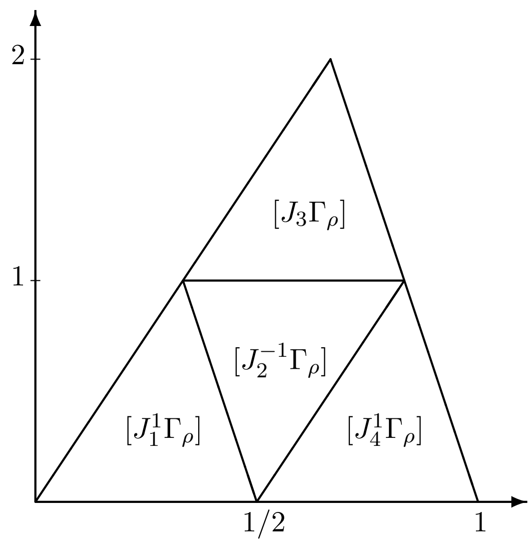

Figure 9: Solution of the jigsaw puzzle for $\{J_1, J_2, J_3, J_4; 1, -1, 1, 1\}$ with $J_1 = [0, 1/2],\ J_2 = J_3 = [a, a + (1/2)],\ J_4 = [1/2, 1]$
.

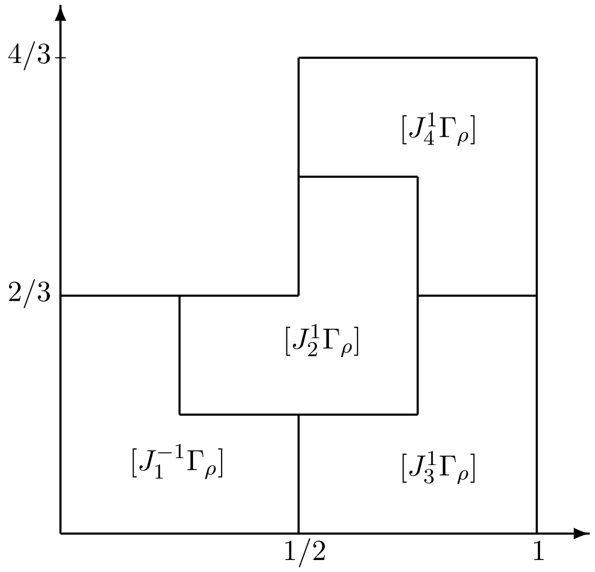

Figure 10: Solution of the jigsaw puzzle for $\{J_1, J_2, J_3, J_4; -1, 1, 1, 1\}$ with $J_1 = [0, 1/2]$, $J_2 = [1/4, 3/4]$, $J_3 = J_4 = [1/2, 1]$.

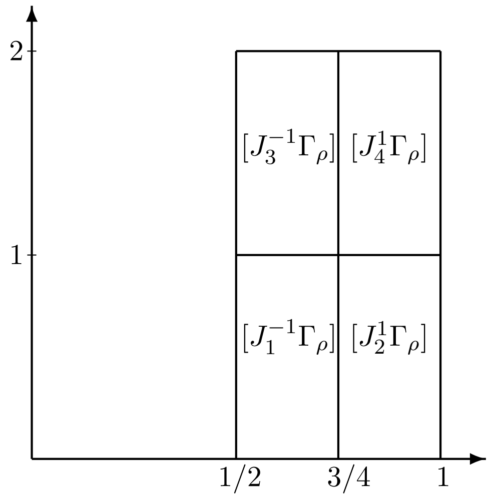

Figure 11: Solution of the jigsaw puzzle for $\{J_1, J_2, J_3, J_4; -1, -1, 1, 1\}$ with $J_1 = J_2 = J_3 = J_4 = [1/2, 1]$.

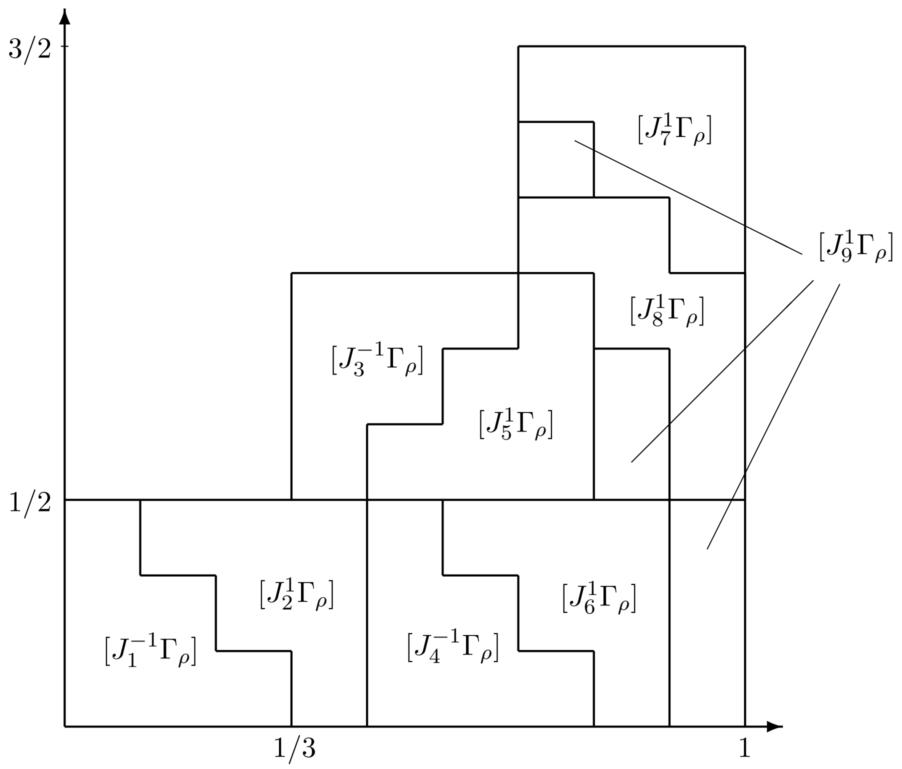

Figure 12: Solution of the jigsaw puzzle in Example 10.

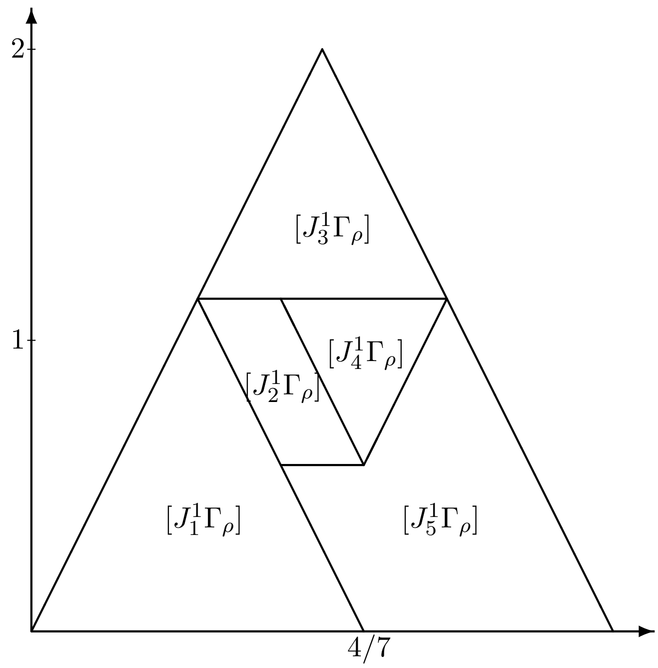

Figure 13: Solution of the jigsaw puzzle for $\{J_1, J_2, J_3, J_4, J_5; 1, 1, 1, 1, 1\}$ with $J_1 = [0, 4/7]$, $J_2 = [2/7, 4/7]$, $J_3 = [2/7, 5/7]$, $J_4 = [3/7, 5/7]$, $J_5 = [3/7, 1]$.

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References

[1] K.J. Falconer, The dimension of self-affine fractals II, Math. Proc. Camb. Phil. Soc. 111 (1992), pp. 169-179

[2] J.E. Hutchinson, Fractal and self-similarity, Indiana Univ. Math. J. 30 (1981), pp. 713-747

[3] C. McMullen, The Hausdorff dimension of general Sierpiński carpets. Nagoya Math. J. 96 (1984), pp. 1-9

[4] T. Kamae and M. Keane, A class of deterministic self-affine processes, Japan J. Applied Math. 7-2 (1990), pp. 183-195

[5] T. Bedford and T. Kamae, Stieltjes integration and stochastic calculus with respect to self-affine functions, Japan J. Industrial & Applied Math. 8-3 (1991), pp. 445-459

[6] T. Kamae, A characterization of self-affine functions, Japan J. Applied Math. 3-2 (1986), pp. 271-280

[7] T.K amae, Stochastic analysis based on deterministic Brownian motion, Israel J. Math. 125 (2001), pp. 317-346

[8] X-R. Dai, D-J. Feng, and Y. Wang, Classification of refinable splines, Constr. Approx. 24 (2006), pp. 187-200

[9] D-J. Feng and H. Hu, Dimension theory of iterated function systems, Communications on Pure and Applied Mathematics, Vol. LXII (2009), pp. 1435-1500