Randomness criterion Σ and its applications

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Abstract

The Sigma function, which is the sum of the squares of the number of occurrences of every factor, is a criterion of randomness, measuring specially the uniformity of the block distribution. An infinite word whose prefixes attain asymptotically the smallest possible value of it is called Sigma-random. We prove that the Champernowne word is Sigma-random. We also consider less complex words which have values with asymptotically larger order, Sturmian words and almost 0-words.

1 Introduction

Let $x = x_1x_2\cdots \in \{0,1\}^\infty$ be an infinite word over the binary alphabet and $\xi \in \{0,1\}^+$ be a finite word, where $\{0,1\}^+ = \cup_{k=1}^\infty \{0,1\}^k$. Denote by $|\xi|$ the length of ξ, that is, $|\xi| = k$ if $\xi \in \{0,1\}^k$. Let

$$|x_1x_2\cdots x_n|_\xi := \#\{1 \le i \le n - |\xi| + 1 \ ;\ \xi = x_ix_{i+1}\cdots x_{i+|\xi|-1}\}.$$

We say that ξ is a *factor* of $x_1x_2\cdots x_n$ (and of x) if $|x_1x_2\cdots x_n|_\xi \ge 1$ and denote $\xi \prec x_1x_2\cdots x_n$.

For an infinite word $x = x_1x_2\cdots \in \{0,1\}^\infty$ and integer $n \ge 1$,we define the sigma function by the sum of the number of occurrences of every factors,

$$\Sigma^n(x) := \sum_{\xi\in\{0,1\}^+} |x_1x_2\cdots x_n|_\xi^2,$$

The sigma function Σ^n was first introduced by Kamae and Xue in [6] to serve as a randomness criterion. We also consider Σ^n as a function on finite words of length n. It is known [6] that

$$\liminf_{n\to\infty} \frac{\Sigma^n(x)}{n^2} \ge \frac{3}{2} \quad \text{for any } x \in \{0,1\}^\infty,$$

and with respect to the $(\frac{1}{2}, \frac{1}{2})$ -Bernoulli measure on $\{0,1\}^\infty$,

$$\lim_{n\to\infty} \frac{\Sigma^n(x)}{n^2} = \frac{3}{2} \tag{1.1}$$

holds almost surely.

More precisely, $\Sigma^n(x_1x_2\cdots x_n)$ attains the minimum in $\{0,1\}^n$ if

$$\lfloor (n-|\xi|+1)/2^{|\xi|}\rfloor \le |x_1x_2\cdots x_n|_\xi \le \lceil (n-|\xi|+1)/2^{|\xi|}\rceil$$

holds for any $\xi\in\{0,1\}^+$. We call such $x_1x_2\cdots x_n$ an *equi-distributed* word. This is equivalent to say that

$$||x_1x_2\cdots x_n|_\xi - |x_1x_2\cdots x_n|_\eta| \le 1$$

for any $\xi,\eta\in\{0,1\}^+$ with $|\xi|=|\eta|$. Using the Bruijn graph, we can construct an equi-distributed word of length n for infinitely many n, but we don't know the answer to the following question:

Open problem: For any $n=1,2,\cdots$, does an equi-distributed word of length n exist?

An infinite word $x\in\{0,1\}^\infty$ satisfying (1.1) is called a Σ-*random* word. We call an infinite word $x=(x_1x_2\dots)\in\{0,1\}^\infty$ normal if $\sum_{i=1}^\infty x_i/2^i$ is a normal number, i.e., every factor of length n appear in x with frequency of 2^{-n}. It is known [6] that any Σ-random word is normal, but the converse is not true. In fact, let

$$\Sigma_k^n(x) = \sum_{\xi\in\{0,1\}^k} |x_1x_2\cdots x_n|_\xi^2 \qquad (k=1,2,\cdots)$$

and decompose the limit superior of $\Sigma^n(x)/n^2$ into 3 parts:

$$\limsup_{n\to\infty}\frac{\Sigma^n(x)}{n^2} \le S_B(x)+S_P(x)+S_L(x),$$

where

$$\begin{aligned} S_B(x) &= \lim_{K\to\infty}\limsup_{n\to\infty}\frac{1}{n^2}\sum_{k\le K}\Sigma_k^n(x),\\ S_P(x) &= \lim_{K\to\infty}\limsup_{n\to\infty}\frac{1}{n^2}\sum_{K<k\le n/K}\Sigma_k^n(x),\\ S_L(x) &= \lim_{K\to\infty}\limsup_{n\to\infty}\frac{1}{n^2}\sum_{k>n/K}\Sigma_k^n(x). \end{aligned} \tag{1.2}$$

Then it is shown [6, Theorem 1] that x is Σ-random if and only if

$$S_B(x)=1,\quad S_P(x)=0,\quad S_L(x)=\frac{1}{2}.$$

On the other hand, x is normal if and only if $S_B(x)=1$ [6, Corollary 3]. Let $x=x_1x_2\cdots\in\{0,1\}^\infty$ be normal. Then, for any $m_1<m_2<\cdots$ and $l_1\le l_2\le\cdots$,

$$y := (x_1x_2\cdots x_{m_1})^{l_1}(x_1x_2\cdots x_{m_2})^{l_2}\cdots$$

is normal. By Lemma 1 in [7], for any N, we can construct $y \in \{0,1\}^\infty$ as above such that $S_P(y) > N$ and $S_L(y) > N$. Thus, a normal word is not necessarily Σ-random.

Remark 1. In Definition 2 of the paper [6], the following explanation of the notation $A(I, I')$ was missing:

For any intervals I, I' of $\{1, 2, \cdots, n\}$ with $|I| = |I'|$, define

$$A(I, I') = \{\{i + j, i' + j\};\ j = 0, 1, \cdots, |I| - 1\},$$

where i, i' are the first elements of I and I', respectively.

In this paper, we consider the Champernowne word. For a positive integer l and $j = 0, 1, \cdots, 2^l - 1$, we define $w^l(j)$ the binary word of length l representing j, that is

$$w^l(j) := x_1 x_2 \cdots x_l \in \{0,1\}^l, \quad \text{where} \quad j = \sum_{i=1}^{l} x_i 2^{l-i}.$$

Let γ^l be the concatenation of $w^l(j)$'s for $j = 0, 1, \cdots, 2^l - 1$ in the order, that is

$$\gamma^l := w^l(0) w^l(1) \cdots w^l(2^l - 1) \in \{0,1\}^{l2^l}.$$

For example,

$$\gamma^1 = 01, \quad \gamma^2 = 00011011, \quad \gamma^3 = 000001010011100101110111.$$

We define the *modified Champernowne word* $\beta \in \{0,1\}^\infty$ as their concatenations for $l = 1, 2, \cdots$. That is,

$$\beta = \gamma^1 \gamma^2 \gamma^3 \cdots = 0100011011000001010011100101110111 \cdots \in \{0,1\}^\infty.$$

The Champernowne word, which will be denoted by $\tilde{\beta}$, is a little different from this. It is defined by arranging binary representations of positive integers in the increasing order. That is, with

$$\tilde{\gamma}^l := w^l(2^{l-1}) w^l(2^{l-1} + 1) \cdots w^l(2^l - 1)$$

instead of γ^j, define $\tilde{\beta} = \tilde{\gamma}^1 \tilde{\gamma}^2 \tilde{\gamma}^3 \cdots$. Champernowne[4] constructed his celebrated normal number whose decimal expansion is Champernowne word in the alphabet $\{0, 1, \ldots, 9\}$. The normality of the modified Champernowne word β is immediate by the normality of Champernowne word $\tilde{\beta}$. For some statistical properties of the Champernowne word, consult [2, 9].

The first theorem is to show the Σ-randomness of the modified Champernowne word β:

Theorem 1. *The modified Champernowne word β is Σ-random.*

Corollary 1. *The Champernowne word $\tilde{\beta}$ is Σ-random.*

The r-adic versions of Theorem 1 and Corollary 1 hold as well for any $r \geq 2$. In this case, Σ-random word is defined to be $x \in \{0,1,\cdots,r-1\}^\infty$ such that

$$\lim_{n\to\infty} \frac{\Sigma^n(x)}{n^2} = \frac{r+1}{2(r-1)}$$

since $S_B(x) = 1/(r-1)$ for almost all x with respect to the uniform Bernoulli mesure on $\{0,1,\cdots,r-1\}^\infty$.

Yuval Peres and Benjamin Weiss [12] defined a notion of randomness (*Poisson random*, say) for $x = x_1x_2\cdots \in \{0,1\}^\infty$ so as to satisfy the Poisson law of small number. That is, for each $m = 0,1,2,\cdots$

$$\lim_{k\to\infty} \frac{1}{2^k} \# \left\{ \xi \in \{0,1\}^k;\ |x_1x_2\cdots x_n|_\xi = m \right\} = \frac{e^{-1}}{m!},$$

where we denote $n = 2^k + k - 1$.

Suggested by this notion, we define $x \in \{0,1\}^\infty$ to be *P-random* if

$$\lim_{k\to\infty} \frac{1}{2^k} \sum_{\xi\in\{0,1\}^k} |x_1x_2\cdots x_{2^k+k-1}|_\xi^2 = 2. \tag{1.3}$$

It should be noted that the Poisson randomness implies neither the Σ-randomness nor the P-randomness since the Poisson randomness is free of replacing a subword of length $\lfloor 2^{2k/3} \rfloor$ of $x_{n_k}x_{n_k+1}\cdots x_{n_{k+1}-1}$ for each $k = 1,2,\cdots$, where $n_k = 2^k + k$, while it may change the property to be Σ-random or P-random. In fact, by replacing a "random" subword by a constant word, it increases Σ-value by orders n^2 of the Σ-random word or n of the P-random word, so that the randomness properties are lost.

Yuval Peres and Benjamin Weiss [12] proved the following theorem for the case of Poisson random instead of P-random. The proof of our theorem is based on the same idea.

Theorem 2. *(1) Almost all $x \in \{0,1\}^\infty$ are P-random with respect to the $(\frac{1}{2},\frac{1}{2})$-Bernoulli measure.*
(2) Any P-random word is normal.
(3) Neither the modified Champernowne word β nor the Champernowne word $\tilde{\beta}$ is P-random.

By Theorems 1 and 2, the Σ-randomness does not imply the P-randomness. We don't know whether the P-randomness implies the Σ-randomness or not.

For the other extreme, the maximum value of $\Sigma^n(x)$ is

$$1^2 + 2^2 + \cdots + n^2 = \frac{n^3}{3} + o(n^3),$$

which is attained by the constant words. It is known by Kamae and Kim [7] that the infinite word $x_1x_2\cdots$ is eventually periodic if and only if

$$\lim_{n\to\infty} \Sigma(x_1x_2\cdots x_n)/n^3$$

exists and is positive.

We are also interested in the intermediate cases, that is, $\Sigma^n(x)$ increases faster than n^2, but slower than n^3. Especially, we consider the Sturmian words and almost 0-words.

Recall that $x = x_1x_2\cdots \in \{0,1\}^\infty$ is a *Sturmian word* if it has exactly $k+1$ factors of length k $(k = 1, 2, \cdots)$. It is well known [10] that x is a Sturmian word if and only if there exist an irrational number $\theta \in (0,1)$ (called *slope*) and a real number $\rho \in [0,1)$ (called *intercept*) such that

$$\begin{aligned} &x_i = \lfloor (i+1)\theta + \rho \rfloor - \lfloor i\theta + \rho \rfloor \text{ for all } i = 1, 2, \dots \\ \text{or} \\ &x_i = \lceil (i+1)\theta + \rho \rceil - \lceil i\theta + \rho \rceil \text{ for all } i = 1, 2, \dots . \end{aligned} \tag{1.4}$$

Note that they differs at most two letters. For an irrational slope θ, let $(a_i)_{i=1,2,\cdots}$ and $(p_i/q_i)_{i=1,2,\cdots}$ be the partial quotients and principal convergents of θ, respectively. Then, with $p_{-1} = 1$, $p_0 = 0$, $q_{-1} = 0$, $q_0 = 1$, (p_i) and (q_i) satisfy that

$$p_i = a_ip_{i-1} + p_{i-2}, \qquad q_i = a_iq_{i-1} + q_{i-2} \qquad (i = 1, 2, \cdots),$$

where $(a_i)_{i=1,2,\cdots}$ are the partial quotients of θ.

For A, B which depend on some variable(s), say $n, m, \cdots$, which may satisfy some relations when $n, m, \cdots \to \infty$, we denote

$$\begin{aligned} &A \asymp B && \text{if} \quad 0 < \liminf_{n,m,\cdots\to\infty} \frac{A}{B} \le \limsup_{n,m,\cdots\to\infty} \frac{A}{B} < \infty, \\ &A \sim B && \text{if} \quad \lim_{n,m,\cdots\to\infty} \frac{A}{B} = 1. \end{aligned}$$

For a real number t, we denote $\|t\| = \min_{n\in\mathbb{Z}} |t-n|$ and $t_+ = \max\{t, 0\}$.

Theorem 3. *Let $x = x_1x_2\cdots \in \{0,1\}^\infty$ be a Sturmian word with the slope θ and (a_i), (p_i/q_i) be the partial quotients and principal convergents of θ. For $n = 1, 2, \cdots$, define m by $q_m \le n < q_{m+1}$. Then, we have*

(1) $$\liminf_{n\to\infty} \frac{\Sigma^n(x)}{n^2\log n} \ge 1,$$

(2) $$\frac{n^2}{24000}\sum_{i=1}^{m-9} a_i < \Sigma^n(x) < \frac{4n^2}{3}\sum_{i=1}^{m+1}(a_i + 50).$$

The condition that $\sum_{i=1}^m a_i = O(m)$ appeared in [11, Corollary 1.65] as a low discrepancy condition of the sequence coming from the irrational rotation. Here, we get the same condition to characterize the minimum increasing order of Σ^n.

Corollary 2. *With the same setting as Theorem 3, $\Sigma^n(x) \asymp n^2 \log n$ holds if and only if $(1/m)\sum_{i=1}^m a_i$ is bounded in m, in particular, if $(a_i)_{i=1,2,\cdots}$ is bounded. Moreover, if $(a_i)_{i=1,2,\cdots}$ is bounded, then $\Sigma^n(x) \sim \sum_{K<k\le n/K} \Sigma_k^n(x)$ holds with any constant $K > 0$.*

In Section 4, we discuss the order of Σ^n for $x_1x_2\cdots \in \{0,1\}^\infty$ with $\sum_{i=1}^n x_n = O(n^{1/\alpha})$ for $\alpha > 1$. We denote by $\{k_i\}$ the increasing sequence given by $x_{k_i} = 1$. In particular, we prove in Corollary 3 that if $k_i = i^\alpha + O(1)$ (as $i \to \infty$) with $\alpha \ge 3/2$, then $\Sigma(x_1x_2\cdots x_n) \asymp n^{3-(1/\alpha)}$.

2 Champernowne word

Define $\phi : \{0,1\}^+ \to \{0,1,2,\cdots\}$ so that for any $x_1x_2\cdots x_l \in \{0,1\}^l$, $\phi(x_1x_2\cdots x_l) = \sum_{i=1}^l x_i 2^{l-i}$. For $\xi = x_1x_2\ldots x_l$ and $1 \le i \le j \le l$ we write $\xi_{[i,j]} = x_ix_{i+1}\ldots x_j$.

Lemma 1. *For any $\xi \in \{0,1\}^l$ with $l \ge 15k^2$ $(k = 1,2,\cdots)$, we have $|\gamma^1\gamma^2\cdots\gamma^k|_\xi \le 1$.*

Proof. Since $l \ge 15$, our lemma holds trivially for $k = 1,2$.

We assume $k \ge 3$. Consider any occurrence of ξ in $\gamma^1\gamma^2\cdots\gamma^k$ to see which of γ^j it or a part of it is located in. Since $2|\gamma^j| < |\gamma^{j+1}|$ $(j = 1,2,\cdots,k-1)$, there are j_1 and $\eta \prec \xi$ with $|\eta| \ge l/3 \ge 5k^2$ such that at this occurrence, η is located totally in γ^{j_1}. Since $j_1 \le k$, η contains at least $5k-2$ successive $w^{j_1}(i)$'s, and hence, $4k+1$ successive $w^{j_1}(i)$'s. Suppose that there exists another occurrence of ξ in $\gamma^1\gamma^2\cdots\gamma^k$ after this. Then, η appears again afterward among at most 2 of consecutive γ^j's. Hence one of them, say γ^{j_2} $(j_1 \le j_2 \le k)$ contains at least $2k$ successive $w^{j_1}(i)$'s, say

$$w^{j_1}(i_1)w^{j_1}(i_1+1)\cdots w^{j_1}(i_1+2k-1).$$

This implies that there exists $j_2 \ge j_1$, i_2 and $b \in \{0,1,\cdots,j_2-1\}$ with $i_1 < i_2$ if $j_1 = j_2$ such that

$$w^{j_1}(i_1)w^{j_1}(i_1+1)\cdots w^{j_1}(i_1+2k-1) \prec_b w^{j_2}(i_2)w^{j_2}(i_2+1)w^{j_2}(i_2+2) \cdots w^{j_2}(2^{j_2}-1) \quad (2.1)$$

where we write $A \prec_b B$ if $A = c_{b+1}\cdots c_l$ with $B = c_1c_2\cdots c_{b+1}\cdots c_l\cdots$.

For each $i = i_1, i_1+1, \cdots, i_1+2k-1$, define $\tau(i) \in \{0, 1, \cdots, j_2-1\}$ so that

$$w^{j_1}(i) \prec_{\tau(i)} w^{j_2}(r)w^{j_2}(r+1)$$

for some r in (2.1). Then we have $\tau(i+1) \equiv \tau(i)+j_1 \pmod{j_2}$. Let i_0 be the minimum h such that $\tau(h) = \min\{\tau(i);\ i = i_1, i_1+1, \cdots, i_1+2k-1\}$ and $\tau_0 = \tau(i_0)$. Then we have $\tau_0 + j_1 < j_2$ if $j_1 < j_2$. Moreover, $i_0 - i_1 \leq j_2 - 1$ by the minimality of i_0.

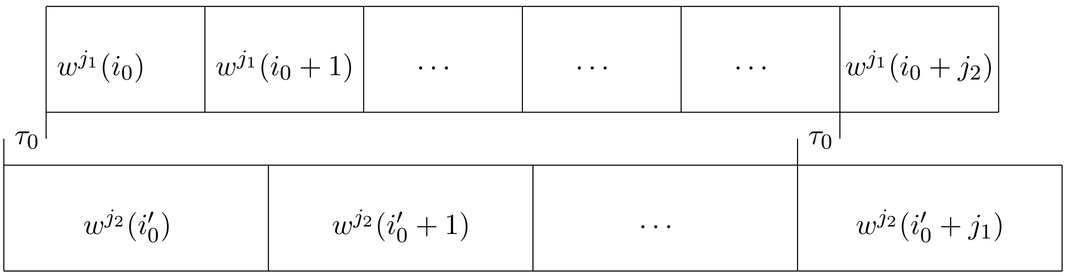

Figure 1: Overlapping between w^{j_1} and w^{j_2}

Consider the case where $j_1 < j_2$. Then, we have $\tau_0 + j_1 < j_2$ and

$$i_0 + j_2 \leq i_1 + j_2 - 1 + j_2 = i_1 + 2j_2 - 1 \leq i_1 + 2k - 1.$$

There exists i'_0 with $0 \leq i'_0 < i'_0 + j_1 \leq 2^{j_2} - 1$ such that

$$w^{j_1}(i_0) \prec_{\tau_0} w^{j_2}(i'_0), \quad w^{j_1}(i_0+j_2) \prec_{\tau_0} w^{j_2}(i'_0+j_1).$$

Note that

$$\phi(w^{j_1}(i_0+j_2)) = \phi(w^{j_1}(i_0)) + j_2.$$

On the other hand, we have

$$\begin{aligned} \phi\left(w^{j_2}(i'_0+j_1)_{[\tau_0+1,\tau_0+j_1]}\right) &\leq \phi\left(w^{j_2}(i'_0)_{[\tau_0+1,\tau_0+j_1]}\right) + \left\lfloor \frac{j_1}{2} \right\rfloor + 1 \\ &\leq \phi\left(w^{j_2}(i'_0)_{[\tau_0+1,\tau_0+j_1]}\right) + j_1 \end{aligned}$$

since $j_1 \geq 1$ and $j_2 - (\tau_0 + j_1) \geq 1$, while

$$w^{j_1}(i_0+j_2) = w^{j_2}(i'_0+j_1)_{[\tau_0+1,\tau_0+j_1]}, \quad w^{j_1}(i_0) = w^{j_2}(i'_0)_{[\tau_0+1,\tau_0+j_1]}.$$

Hence, we have

$$\begin{aligned} j_2 &= \phi\left(w^{j_1}(i_0+j_2)\right) - \phi(w^{j_1}(i_0)) \\ &= \phi\left(w^{j_2}(i'_0+j_1)_{[\tau_0+1,\tau_0+j_1]}\right) - \phi\left(w^{j_2}(i'_0)_{[\tau_0+1,\tau_0+j_1]}\right) \leq j_1, \end{aligned}$$

which is a contradiction since $j_1 < j_2$.

Consider the case $j_1 = j_2$. If $b = 0$ in (2.1), then we have a contradiction that $w^{j_1}(i_1) = w^{j_2}(i_2)$ for $i_1 < i_2$. Assume that $b \geq 1$. Denote $j = j_1 = j_2$. By (2.1), we have

$$w^j(i_1) = w^j(i_2)_{[b+1,j]} w^j(i_2+1)_{[1,b]},$$
$$w^j(i_1+1) = w^j(i_2+1)_{[b+1,j]} w^j(i_2+2)_{[1,b]}.$$

Note that

$$\phi\left(w^j(i_2)_{[b+1,j]}\right) + 1 \equiv \phi\left(w^j(i_2+1)_{[b+1,j]}\right) \pmod{2^{j-b}}, \tag{2.2}$$
$$\phi\left(w^j(i_1)_{[1,j-b]}\right) \leq \phi\left(w^j(i_1+1)_{[1,j-b]}\right)$$

and

$$w^j(i_2)_{[b+1,j]} = w^j(i_1)_{[1,j-b]}\ ,\ w^j(i_2+1)_{[b+1,j]} = w^j(i_1+1)_{[1,j-b]}.$$

Hence, $\phi\left(w^j(i_2)_{[b+1,j]}\right) \leq \phi\left(w^j(i_2+1)_{[b+1,j]}\right)$. Since (2.2) implies that either

$$\phi\left(w^j(i_2)_{[b+1,j]}\right) + 1 = \phi\left(w^j(i_2+1)_{[b+1,j]}\right)$$

or

$$\phi\left(w^j(i_2)_{[b+1,j]}\right) + 1 = 2^{j-b}, \quad \phi\left(w^j(i_2+1)_{[b+1,j]}\right) = 0,$$

it follows that

$$\phi\left(w^j(i_2)_{[b+1,j]}\right) + 1 = \phi\left(w^j(i_2+1)_{[b+1,j]}\right). \tag{2.3}$$

By exchanging the roles of i_1 and i_2, we have

$$\phi\left(w^j(i_1)_{[j-b+1,j]}\right) + 1 = \phi\left(w^j(i_1+1)_{[j-b+1,j]}\right).$$

Since (2.3) implies

$$\phi\left(w^j(i_1)_{[1,j-b]}\right) + 1 = \phi\left(w^j(i_1+1)_{[1,j-b]}\right),$$

we have

$$\phi\left(w^j(i_1)\right) + 2^b + 1 = \phi\left(w^j(i_1+1)\right),$$

which contradicts with $\phi\left(w^j(i_1)\right) + 1 = \phi\left(w^j(i_1+1)\right)$.

Thus, we have $|\gamma^1\gamma^2\cdots\gamma^k|_\xi \leq 1$. □

Lemma 2. *Let $\beta = \beta_1\beta_2\cdots \in \{0,1\}^\infty$ be the modified Champernowne word. Then, for any $\xi \in \{0,1\}^l$ $(l = 1,2,\cdots)$ and $n = 1,2,\cdots$, we have*

$$|\beta_1\beta_2\cdots\beta_n|_\xi \leq 1$$

if $l \geq 15(\log_2 n)^2$.

Proof. Lemma 2 holds trivially for $n = 1, 2. \cdots, 10$. Assume that $n > 10$. Note that $|\gamma^1\gamma^2\cdots\gamma^k| = (k-1)2^{k+1}+2$. Define $k = 1, 2, \cdots$ by

$$(k-2)2^k + 2 < n \le (k-1)2^{k+1} + 2.$$

Assume that $l \ge 15(\log_2 n)^2$. Since $n > 10$, $k \ge 3$ holds. Hence, $n > (k-2)2^k + 2 > 2^k$ and $k < \log_2 n$ holds. Since $l \ge 15(\log_2 n)^2 > 15k^2$, by Lemma 1, we have

$$|\beta_1\beta_2\cdots\beta_n|_\xi \le |\gamma^1\gamma^2\cdots\gamma^k|_\xi \le 1,$$

which completes the proof. □

Let $x_1x_2\cdots x_n \in \{0,1\}^n$ $(n = 1, 2, \cdots)$ and $\xi \in \{0,1\}^l$ $(l = 1, 2, \cdots)$. For integers $a,\ b$ with $0 \le b < a$, define

$$|x_1x_2\cdots x_n|_\xi^{a,b} := \#\{1 \le j \le n-l+1;\ \xi \prec_j x_1x_2\cdots x_n \text{ with } j \equiv b \text{ (mod a)}\}.$$

Lemma 3. *For any* $k = 1, 2, \cdots$, $\xi \in \{0,1\}^l$ $(l = 1, 2, \cdots)$ *and* $b \in \{0, 1, \cdots, k-1\}$, *we have* $|\gamma^k|_\xi^{k,b} \le 2^{(k-l)_+}$, *and hence,* $|\gamma^k|_\xi \le k2^{(k-l)_+}$.

Proof. Let $b = 1, 2, \cdots, k-1$ and $\xi \in \{0,1\}^k$ be given with $\xi = w^k(h)$. If

$$\xi \prec_b w^k(j)w^k(j+1) \tag{2.4}$$

for some j with $0 \le j \le 2^k - 2$, then we have

$$j+1 = 2^{k-b}\phi\left(w^k(h)_{[k-b+1,k]}\right) + \phi\left(w^k(h)^{+1}_{[1,k-b]}\right),$$

where for $\zeta \in \{0,1\}^+$ with $|\zeta| = l$, $\zeta^{+1} \in \{0,1\}^l$ is defined so that

$$\phi(\zeta^{+1}) \equiv \phi(\zeta) + 1 \pmod{2^l}.$$

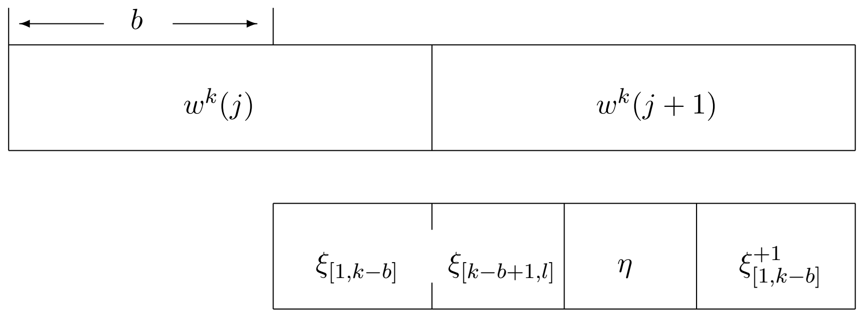

Figure 2: $\eta = w^k(j+1)_{[b+l-k,b]} \in \{0,1\}^{k-l}$

Hence, (2.4) holds at most for one j if $b = 1, 2, \cdots, k-1$ is given. Moreover, $\xi \prec_0 w^k(j)$ holds exactly for one j with $0 \le j \le 2^k - 1$, say $j = h$.

This implies that $|\gamma^k|_\xi^{k,b} \le 1$ if $|\xi| \ge k$ for any $b = 0, 1, \cdots, k-1$. Let $l = |\xi| < k$. If $b + l \le k$, then it is clear that $|\gamma^k|_\xi^{k,b} = 2^{k-l}$. Let $b + l > k$. If (2.4) holds for some j with $0 \le j \le 2^k - 2$, then we have

$$j + 1 = 2^{2k-b-l}\phi\left(\xi_{[k-b+1,l]}\right) + 2^{k-b}\phi\left(w^k(j+1)_{[b+l-k,b]}\right) + \phi\left(\xi^{+1}_{[1,k-b]}\right).$$

Hence, there are exactly 2^{k-l} many of j satisfying (2.4) if $b = 0, 1, \cdots, k-1$ is given, since in the above $w^k(j+1)_{[b+l-k+1,b]}$ can be any element in $\{0,1\}^{k-l}$.

Thus, we have $|\gamma^k|_\xi^{k,b} \le 2^{(k-l)_+}$, and $|\gamma^k|_\xi \le k2^{(k-l)_+}$ for any $\xi \in \{0,1\}^l$ ($l = 1, 2, \cdots$), which completes the proof. □

Lemma 4. *For any $k = 1, 2, \cdots,\ \xi \in \{0,1\}^l$ ($l = 1, 2, \cdots$) and $\eta \in \{0,1\}^+$, we have $|\gamma^k\eta|_\xi - |\eta|_\xi \le k(2^{(k-l)_+} + 1)$.*

Proof. Let $b \in \{1, 2, \cdots, k\}$. Note that $|\gamma^k|$ is a multiple of k. Thus, $|\gamma^k\eta|_\xi^{k,b} - |\eta|_\xi^{k,b}$ is the number of occurrences of ξ in $\gamma^k\eta$ starting at, say j, with $j \equiv b \pmod{k}$ and $1 \le j \le k2^k$. The number of such j with $j \le k(2^k - 1)$ is at most $2^{(k-l)_+}$ as shown in the proof of Lemma 3. Since there can be at most one more occurrence at $j = k(2^k - 1) + b$, we have $|\gamma^k\eta|_\xi^{k,b} - |\eta|_\xi^{k,b} \le 2^{(k-l)_+} + 1$. Thus, $|\gamma^k\eta|_\xi - |\eta|_\xi \le k(2^{(k-l)_+} + 1)$. □

Lemma 5. *Let $\beta = \beta_1\beta_2\cdots \in \{0,1\}^\infty$ be the modified Champernowne word. Then, for any $\xi \in \{0,1\}^l$ ($l = 1, 2, \cdots$) and $n = 1, 2, \cdots$, we have*

$$|\beta_1\beta_2\cdots\beta_n|_\xi \le \frac{4n}{2^l} + (\log_2 n)^2.$$

Proof. Lemma 5 is verified directly for $n = 1, 2, \cdots, 10$. Assume that $n \ge 11$. Let $(k-2)2^k + 2 < n \le (k-1)2^{k+1} + 2$ for some $k \ge 3$. Then, $\beta_1\beta_2\cdots\beta_n$ is a prefix of $\gamma^1\gamma^2\cdots\gamma^k$. By lemmas 3 and 4, we have

$$\begin{aligned}
|\beta_1\beta_2\cdots\beta_n|_\xi \le |\gamma^1\gamma^2\cdots\gamma^k|_\xi &= \sum_{j=1}^{k-1}\left(|\gamma^j\gamma^{j+1}\cdots\gamma^k|_\xi - |\gamma^{j+1}\cdots\gamma^k|_\xi\right) + |\gamma^k|_\xi \\
&\le \sum_{j=1}^{k-1} j(2^{(j-l)_+} + 1) + k2^{(k-l)_+} \\
&\le \begin{cases} k^2 & (k \le l) \\ (k-1)2^{k-l+1} + \dfrac{k^2 + l^2}{2} - \dfrac{k}{2} - \dfrac{3l}{2} + 2 & (k > l) \end{cases} \\
&\le 2(k-1)2^{k-l} + k^2 \le \frac{4n}{2^l} + (\log_2 n)^2.
\end{aligned}$$

□

Proof of Theorem 1: To prove that the modified Champernowne word $\beta = \beta_1\beta_2\cdots \in \{0,1\}^\infty$ is Σ-random, by [6] it is sufficient to prove that

$$\limsup_{n\to\infty} \frac{\Sigma^n(\beta)}{n^2} \le \frac{3}{2}.$$

We have an upper bound of 3 terms which are a little different from (1.2):

$$\limsup_{n\to\infty} \frac{\Sigma^n(\beta)}{n^2} \le S_B + S'_P + S'_L,$$

where

$$S_B = \lim_{K\to\infty} \limsup_{n\to\infty} \frac{1}{n^2} \sum_{k\le K} \Sigma^n_k(\beta),$$

$$S'_P = \lim_{K\to\infty} \limsup_{n\to\infty} \frac{1}{n^2} \sum_{K<k\le 15(\log_2 n)^2} \Sigma^n_k(\beta),$$

$$S'_L = \limsup_{n\to\infty} \frac{1}{n^2} \sum_{k>15(\log_2 n)^2} \Sigma^n_k(\beta).$$

Since β is known [5] to be normal, we have $S_B = 1$ by [6]. By Lemma 2, if $k \ge 15(\log_2 n)^2$, we have

$$\Sigma^n_k(\beta) = \sum_{\xi\in\{0,1\}^k} |\beta_1\beta_2\cdots\beta_n|^2_\xi = \sum_{\xi\in\{0,1\}^k} |\beta_1\beta_2\cdots\beta_n|_\xi = n-k+1.$$

Hence, we have

$$\sum_{k>15(\log_2 n)^2} \Sigma^n_k(\beta) \le \sum_{k=1}^{n}(n-k+1) = \frac{n(n+1)}{2}.$$

Thus, $S'_L \le 1/2$. By Lemma 5, we have

$$\begin{aligned}
S'_P &= \lim_{K\to\infty} \limsup_{n\to\infty} \frac{1}{n^2} \sum_{K<k\le 15(\log_2 n)^2} \Sigma^n_k(\beta) \\
&\le \lim_{K\to\infty} \limsup_{n\to\infty} \frac{1}{n^2} \sum_{K<k\le 15(\log_2 n)^2} \min\{2^k, n\}\cdot 2\left(\left(\frac{4n}{2^k}\right)^2 + (\log_2 n)^4\right) \\
&\le \lim_{K\to\infty} \frac{32}{2^K} + \limsup_{n\to\infty} \frac{30n(\log_2 n)^6}{n^2} = 0
\end{aligned}$$

Thus, we have

$$\limsup_{n\to\infty} \frac{\Sigma^n(\beta)}{n^2} \le 1 + 0 + \frac{1}{2} = \frac{3}{2},$$

which completes the proof. □

Proof of Corollary 1: Lemmas 1, 3 and 4 hold for the Champernowne word $\tilde{\beta} = \tilde{\gamma}^1\tilde{\gamma}^2\tilde{\gamma}^3\cdots$. As for Lemmas 2 and 5, since $|\tilde{\gamma}^i| = |\gamma^i|/2$ for $i = 1, 2, \cdots$, we have the same results for $\tilde{\beta}$ only by replacing $\log_2 n$ by $\log_2 n + 1$. Thus, we establish Corollary 1. □

Proof of Theorem 2:

(1) Let $X_1X_2\cdots$ be independent random variables with $P(X_i = 0) = P(X_i = 1) = 1/2$ $(i = 1, 2, \cdots)$. For $k = 1, 2, \cdots$, let $n = 2^k + k - 1$. Denoting

$$Z_k = \sum_{\xi\in\{0,1\}^k} |X_1X_2\cdots X_n|_\xi^2,$$

we prove that $\mathbb{E}[Z_k] = 2{\cdot}2^k(1+o(1))$ and $\mathbb{V}[Z_k] = O(k2^k)$, so that $(1/2^k)Z_k \to 2$ holds almost surely by the law of large number. We follow the proofs in Section 2 of [6] (note Remark 1). It holds that

$$\begin{aligned}
\mathbb{E}[Z_k] &= \mathbb{E}\left[\sum_{\xi\in\{0,1\}^k}\left(\sum_{i=1}^{2^k} 1_{X_i\cdots X_{i+k-1}=\xi}\right)^2\right] \\
&= \mathbb{E}\left[\sum_{\xi\in\{0,1\}^k}\sum_{i,j=1}^{2^k} 1_{X_i\cdots X_{i+k-1}=\xi}1_{X_j\cdots X_{j+k-1}=\xi}\right] \\
&= \sum_{i,j=1}^{2^k} \mathbb{P}\left[X_i\cdots X_{i+k-1} = X_j\cdots X_{j+k-1}\right] \\
&= 2^k + \sum_{\substack{i,j=1 \\ i\neq j}}^{2^k} \mathbb{P}\left[X_i\cdots X_{i+k-1} = X_j\cdots X_{j+k-1}\right] \\
&= 2^k + 2^k(2^k-1)2^{-k} = 2n(1+o(1)),
\end{aligned}$$

where $\mathbb{P}[X_i\cdots X_{i+k-1} = X_j\cdots X_{j+k-1}] = 2^{-k}$ holds if $i \neq j$ ([6]). Let $\mathbb{I}$ be the set of intervals in $\{1, 2, \cdots, n+k-1\}$ of length k. For $I = \{i, i+1, \cdots, i+k-1\} \in \mathbb{I}$, let $X[I] = X_iX_{i+1}\cdots X_{i+k-1}$. Then, by the same argument of the proof of Lemma 9 of [6] we have

$$\begin{aligned}
\mathbb{V}[Z_k] = &\sum_{\substack{I,I',J,J'\in\mathbb{I} \\ I,I',J,J' \text{ are coupled}}} \mathbb{C}\mathrm{ov}\left(1_{X[I]=X[I']}, 1_{X[J]=X[J']}\right) \\
&+ \sum_{\substack{I,I',J,J'\in\mathbb{I} \\ I,I',J,J' \text{ are connected}}} \mathbb{C}\mathrm{ov}\left(1_{X[I]=X[I']}, 1_{X[J]=X[J']}\right) \\
&\leq n^2k\mathbb{P}[X[I] = X[I']] + O(n) = n^2k2^{-k} + O(n) = O(k2^k).
\end{aligned}$$

Here, we say $I, I', J, J' \in \mathbb{I}$ with $|I| = |I'|$ and $|J| = |J'|$ are coupled if $|I \cap J| = |I' \cap J'| \geq 1$ and $(I \cup J) \cap (I' \cup J') = \emptyset$ or $|I \cap J'| = |I' \cap J| \geq 1$ and $(I \cup J') \cap (I' \cup J) = \emptyset$. The intervals $I, I', J, J' \in \mathbb{I}$ are connected if $I_1 \cap I_2 \neq \emptyset$, $I_2 \cap I_3 \neq \emptyset$ and $I_3 \cap I_4 \neq \emptyset$ for some rearrangement I_1, I_2, I_3, I_4 of I, I', J, J'.

(2) For $k = 1, 2, \cdots$, let $n = 2^k + k - 1$. Let $x = x_1 x_2 \cdots \in \{0,1\}^\infty$ be P-random. Let Y_k^x be the random variable on the probability space $(\{0,1\}^k, P_k)$ with the normalized counting measure P_k on $\{0,1\}^k$ such that

$$Y_k^x(\xi) = |x_1 x_2 \cdots x_{2^k+k-1}|_\xi \text{ for any } \xi \in \{0,1\}^k.$$

Then, since $\mathbb{E}[(Y_k^x)^2] \to 2$ as $k \to \infty$, the random variables Y_k^x for $k = 1, 2, \cdots$ are uniformly integrable.

Let

$$\mathrm{Nor}(\varepsilon, h, k) := \left\{ \eta \in \{0,1\}^k \, ; \left| \frac{|\eta|_\xi}{k - h + 1} - \frac{1}{2^h} \right| < \varepsilon \text{ for any } \xi \in \{0,1\}^h \right\}$$

be the set of (ε, k)-normal words. Then for any $\varepsilon, \delta > 0$ and h, there exists k_0 such that for any $k \geq k_0$,

$$\#\mathrm{Nor}(\varepsilon, h, k) > (1 - \delta) 2^k.$$

Take any $S \subset \{0,1\}^k$. Then for any m_0, we have

$$\begin{aligned}
&\frac{1}{2^k} \#\{1 \leq i \leq 2^k ;\ x_i x_{i+1} \cdots x_{i+k-1} \in S\} \\
&= \frac{1}{2^k} \sum_{\xi \in S} |x_1 x_2 \cdots x_n|_\xi = \sum_{m=1}^{\infty} m P_k(S \cap \{Y_k^x = m\}) \\
&\leq m_0 P_k(S) + \sum_{m=m_0+1}^{\infty} m P_k(Y_k^x = m).
\end{aligned}$$

Since the random variables Y_k^x $(k = 1, 2, \cdots)$ are uniformly integrable, $\sum_{m=m_0+1}^{\infty} m P_k(Y_k^x = m) \to 0$ as $m_0 \to \infty$ uniformly in k. Take any $\varepsilon > 0$ and h. Choose m_0 so that

$$\sum_{m=m_0+1}^{\infty} m P_k(Y_k^x = m) < \frac{\varepsilon}{3} \quad (k = 1, 2, \cdots).$$

Choose $\delta > 0$ so that $m_0 \delta < \varepsilon/3$. Let $k \geq \max\{k_0, 3h/\varepsilon\}$, where k_0 is determined as above corresponding to h, ε, δ. Then, $P_k(\mathrm{Nor}(\varepsilon, h, k)) > 1 - \delta$ holds. Hence,

$$\begin{aligned}
&(1/2^k)\#\{1 \leq i \leq 2^k ;\ x_i x_{i+1} \cdots x_{i+k-1} \notin \mathrm{Nor}(\varepsilon, h, k)\} \\
&\qquad\qquad \leq m_0 \delta + \sum_{m=m_0+1}^{\infty} m P_k(Y_k^x = m) < \frac{2\varepsilon}{3}.
\end{aligned}$$

Thus, $x_1x_2\cdots x_n \in \mathrm{Nor}(\varepsilon, h, n)$ for $n = 2^k + k - 1$ and $x_1x_2\cdots x_n \in \mathrm{Nor}(2\varepsilon, h, n)$ for any sufficiently large n not necessarily related to some k. Since this holds for arbitrary $\varepsilon > 0$ and any sufficiently large n corresponding to $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{|x_1x_2\cdots x_{n-h+1}|_\xi}{n-h+1} = 2^{-h}$$

for any $\xi \in \{0,1\}^h$. Since h is arbitrary, this implies that x is normal.

(3) Let $\beta = \beta_1\beta_2\cdots \in \{0,1\}^\infty$ be the modified Champernowne word, that is, $\beta = \gamma_1\gamma_2\cdots$. For k, let $n = 2^k + k - 1$ and l be the integer such that

$$(l-1)2^{l+1} + 2 \le n < l2^{l+2} + 2.$$

Then, $\beta_1\beta_2\cdots\beta_n$ contains γ_l and $l \sim k$ holds. Let $L = k - l$. Then, $0 < L \sim \log_2 k$ holds. Take any $\xi \in \{0,1\}^k$ of the form $\xi = \eta 0\zeta\eta$ with $\eta \in \{0,1\}^L$. Then, we have $|\gamma_l|_\xi \ge k - 2L - 1$. This is because for any decomposition $\zeta = \zeta'\zeta''$ with ζ', $\zeta'' \in \{0,1\}^*$, we have $\xi \prec_{|\zeta''|} w^l(j)w^l(j+1)$, where $j = \phi(\zeta''\eta 0\zeta')$ and $j+1 = \phi(\zeta''\eta(0\zeta')^{+1})$ ($(0\zeta')^{+1}$ is the word in $\{0,1\}^{|\zeta'|+1}$ such that $\phi((0\zeta')^{+1})) = \phi(0\zeta') + 1$). The number of $\xi \in \{0,1\}^k$ of the above form is 2^{k-L-1} since we can determine $\zeta\eta$ arbitrary. These ξ satisfy $|\beta_1\beta_2\cdots\beta_n|_\xi \ge k - 2L - 1$. Hence we have

$$\Sigma_k^n(\beta_1\beta_2\cdots\beta_n) \ge (k-2L-1)^2 2^{k-L-1} = k^{1+o(1)}n.$$

Thus, $\lim_{k\to\infty}(1/n)\Sigma_k^n(\beta_1\beta_2\cdots\beta_n) = 2$ does not hold, and β is not P-random.

The proof is same for the Champerwowne word $\tilde{\beta}$. □

3 Sturmian words

Proof of (1) of Theorem 3: Let $x = x_1x_2\cdots \in \{0,1\}^\infty$ be a Sturmian word. Since the number of factors of x of length k is $k+1$, by Jensen's inequality

$$\Sigma_k^n(x) = \sum_{\xi\in\{0,1\}^k} |x_1x_2\cdots x_n|_\xi^2 \ge (k+1)\left(\frac{n-k+1}{k+1}\right)^2 = \frac{(n-k+1)^2}{k+1}$$

holds for $k = 1, 2, \cdots$. Thus,

$$\Sigma^n(x) \ge \sum_{k=1}^n \frac{(n-k+1)^2}{k+1} = \sum_{k=1}^n \left(\frac{(n+2)^2}{k+1} - 2(n+2) + (k+1)\right)$$
$$\ge (n+2)^2 \log\left(\frac{n+2}{2}\right) - \frac{3}{2}n^2 - \frac{5}{2}n \ge n^2\log n - (4+\log 2)n^2.$$

Thus, we have

$$\liminf_{n\to\infty} \frac{\Sigma^n(x)}{n^2\log n} \ge 1.$$

□

Let $x = x_1x_2\cdots$ be a Sturmian word with slope θ and intercept ρ (see (1.4)). Assume that $x_i = \lfloor (i+1)\theta + \rho \rfloor - \lfloor i\theta + \rho \rfloor$. Then, $x_i = 0$ or 1 according to $i\theta + \rho \in [0, 1-\theta)$ or $[1-\theta, 0)$, where $\mathcal{P} = \{[0, 1-\theta), [1-\theta, 0)\}$ is the partition of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. For each positive integer k, the partition of $\mathbb{T}$ divided by $(k+1)$ points $0, -\theta, -2\theta, \ldots, -(k-1)\theta, -k\theta$ is the least common refinement of $\mathcal{P}, \mathcal{P}-\theta, \cdots, \mathcal{P}-k\theta$, and hence each element of the partition determine a subword (factor) of x of length k.

The three distance theorem[14, 13] states that that the partition has intervals of three length: For each $q_i < k \le q_{i+1}$, the $k+1$ intervals in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ divided by the points $\{-j\theta;\ 0 \le j \le k\}$ consists of $k - q_i + 1$ intervals of length $\|q_i\theta\|$, $q_i - \ell - 1$ intervals of length $\|q_{i-1}\theta\| - (c-1)\|q_i\theta\|$ and $\ell + 1$ intervals of length $\|q_{i-1}\theta\| - c\|q_i\theta\|$. where $c = \lfloor \frac{k - q_{i-1}}{q_i} \rfloor$ and $\ell = k - cq_i - q_{i-1}$. Each interval determine a subword of x of length k. See [1] for the detail.

As for the number of visit to the subwords corresponding to these intervals, we have the following estimates.

Lemma 6. *Let d be an integer with $0 \le d < a_{j+1}$. Let J be an interval in $\mathbb{T}$ with $|J| = \|q_{j-1}\theta\| - d\|q_j\theta\|$. The number of visits t (say) to J of an orbit of the rotation by θ among the times $1, 2, \cdots, n$ satisfies*

$$|t - n|J|| \le (dq_j + q_{j-1})|J| + 1.$$

Proof. Let T be the rotation by θ on $\mathbb{T}$. Consider the induced map T_J of T on J. Then T_J is also an irrational rotation. The return time on J is q_j on a subinterval of length $\|q_{j-1}\theta\| - (d+1)\|q_j\theta\|$ and $(d+1)q_j + q_{j-1}$ on the remaining subinterval of length $\|q_j\theta\|$.

Note that T_J with respect to the partition corresponding to the return time produces a balanced word, and hence, the total time until k-th return deviates at most $dq_j + q_{j-1}$ for any $k = 1, 2, \cdots$. Let $u \in J$. Then, n with $T^n u \in J$ such that just t number of $Tu,\ T^2u,\ \cdots,\ T^n u$ are in J are in between $(t/|J|) + dq_j + q_{j-1}$ and $(t/|J|) - dq_j - q_{j-1}$, since $t/|J|$ is the average and $dq_j + q_{j-1}$ is the deviation bound. Therefore, in the general case that $\{u, T^n u\} \subset J$ does not necessarily hold, if t number of $u,\ Tu,\ T^2u,\ \cdots,\ T^n u$ are in J, then there exist $0 \le n_1 < n_2 \le n$ with $T^{n_1} \in J$ and $T^{n_2} \in J$ such that $t-1$ number of $T^{n_1+1}u,\ T^{n_1+2}u,\ \cdots,\ T^{n_2}u$ are in J. Therefore,

$$n \ge n_2 - n_1 \ge (t-1)/|J| - dq_j - q_{j-1}.$$

Moreover, there exist $n_1' \le 0 < n \le n_2'$ with $T^{n_1'} \in J$ and $T^{n_2'} \in J$ such that $t+1$ number of $T^{n_1'+1}u,\ T^{n_1'+2}u,\ \cdots,\ T^{n_2'}u$ are in J. Hence,

$$n \le n_2' - n_1' \le (t+1)/|J| + dq_j + q_{j-1}.$$

Therefore for a given n, the number of $t = 0, 1, \cdots, n$ such that $T^t u \in J$ is estimated as

$$(n - dq_j - q_{j-1})|J| - 1 \le t \le (n + dq_j + q_{j-1})|J| + 1.$$

$\square$

By Lemma 6, we deduce the following Lemma.

Lemma 7. *Let $q_i < k \le q_{i+1}$ and $k \le n$. Then, the factors ξ of $x_1x_2\cdots x_n$ with $|\xi| = k$ satisfy one of the following conditions:*
$k - q_i + 1$ many of them satisfy

$$||x_1x_2\cdots x_n|_\xi - (n-k+1)\|q_i\theta\|| \le q_i\|q_i\theta\| + 1,$$

$q_i - \ell - 1$ many of them satisfy

$$\begin{aligned}||x_1x_2\cdots x_n|_\xi &- (n-k+1)\left(\|q_{i-1}\theta\| - (c-1)\|q_i\theta\|\right)| \\ &\le ((c-1)q_i + q_{i-1})\left(\|q_{i-1}\theta\| - (c-1)\|q_i\theta\|\right) + 1,\end{aligned}$$

$\ell + 1$ many of them satisfy

$$\begin{aligned}||x_1x_2\cdots x_n|_\xi &- (n-k+1)\left(\|q_{i-1}\theta\| - c\|q_i\theta\|\right)| \\ &\le (cq_i + q_{i-1})\left(\|q_{i-1}\theta\| - c\|q_i\theta\|\right) + 1,\end{aligned}$$

where $c = \lfloor \frac{k-q_{i-1}}{q_i} \rfloor$ and $\ell = k - cq_i - q_{i-1}$.

For $n = 1, 2, \cdots$, define $m = 1, 2, \cdots$ by

$$q_m \le n < q_{m+1}. \tag{3.1}$$

Lemma 8. *We have*

$$\Sigma^n(x_1x_2\cdots x_n) < \frac{4n^2}{3}\sum_{i=1}^{m+1}(a_i + 50).$$

Proof. Let $q_i < k \le q_{i+1}$ and $k \le n$. By Lemma 7, we have

$$\Sigma_k^n(x) \le (k - q_i + 1)A_1^2 + (q_i - \ell - 1)A_2^2 + (\ell + 1)A_3^2,$$

with

$$\begin{aligned}A_1 &= (n-k+1+q_i)\|q_i\theta\| + 1,\\ A_2 &= (n-k+1+(c-1)q_i + q_{i-1})\left(\|q_{i-1}\theta\| - (c-1)\|q_i\theta\|\right) + 1,\\ A_3 &= (n-k+1+cq_i + q_{i-1})\left(\|q_{i-1}\theta\| - c\|q_i\theta\|\right) + 1,\end{aligned}$$

where $c = \lfloor (k - q_{i-1})/q_i \rfloor$.
Since $1 - q_{i+1}\|q_i\theta\| = q_i\|q_{i+1}\theta\| < q_i/q_{i+2}$, we have

$$\frac{1}{q_{i+1}}\left(1 - \frac{q_i}{q_{i+2}}\right) < \|q_i\theta\| < \frac{1}{q_{i+1}}. \tag{3.2}$$

Thus, we have

$$\begin{aligned}
\|q_{i-1}\theta\| - (c-1)\|q_i\theta\| &< \frac{1}{q_i} - \left(\frac{k-q_{i-1}}{q_i} - 2\right)\frac{1}{q_{i+1}}\left(1 - \frac{q_i}{q_{i+2}}\right) \\
&= \frac{1}{q_i}\left(1 - \frac{k}{q_{i+1}}\right) + \frac{k}{q_{i+1}q_{i+2}} \\
&\quad + \left(\frac{q_{i-1}}{q_i} + 2\right)\frac{1}{q_{i+1}}\left(1 - \frac{q_i}{q_{i+2}}\right) \\
&< \frac{1}{q_i}\left(1 - \frac{k}{q_{i+1}}\right) + \frac{4}{q_{i+1}}.
\end{aligned}$$

Assume that $n \geq q_{i+1}$. Using these inequalities, we have

$$A_1 < \frac{n-k+q_i+1}{q_{i+1}} + 1 \leq \frac{n}{q_{i+1}} + 1,$$

and

$$\begin{aligned}
\max\{A_2, A_3\} &\leq (n-k+1+cq_i+q_{i-1})\left(\|q_{i-1}\theta\| - (c-1)\|q_i\theta\|\right) + 1 \\
&< (n+1)\left(\frac{1}{q_i}\left(1 - \frac{k}{q_{i+1}}\right) + \frac{4}{q_{i+1}}\right) + 1 \leq \frac{2n}{q_i}\left(1 - \frac{k}{q_{i+1}}\right) + \frac{7n}{q_{i+1}}.
\end{aligned}$$

Combining them, we have

$$\Sigma_k^n(x) < (k-q_i+1)\left(\frac{n}{q_{i+1}} + 1\right)^2 + q_i\left(\frac{2n}{q_i}\left(1 - \frac{k}{q_{i+1}}\right) + \frac{7n}{q_{i+1}}\right)^2.$$

Therefore, for $i = 0, 1, \cdots, m-1$, we have

$$\begin{aligned}
\sum_{k=q_i+1}^{q_{i+1}} \Sigma_k^n(x) &< \frac{q_{i+1}^2}{2}\left(\frac{n}{q_{i+1}} + 1\right)^2 + \frac{4n^2 q_{i+1}}{3q_i} + 14n^2 + \frac{49n^2 q_i}{q_{i+1}} \\
&\leq \frac{4n^2 q_{i+1}}{3q_i} + 65n^2 < \frac{4n^2(a_{i+1}+1)}{3} + 65n^2 < \frac{4n^2(a_{i+1}+49)}{3}.
\end{aligned}$$

Now assume that $n < q_{i+1}$. This is the case when $i = m$ and $k > q_m$. Then, we have

$$\begin{aligned}
\sum_{k=q_m+1}^{n} \Sigma_k^n(x) &< \sum_{k=q_m+1}^{n} (k-q_m+1)\left(\frac{n}{q_{m+1}} + 1\right)^2 \\
&\quad + \sum_{k=q_m+1}^{n} q_m\left(\frac{2n}{q_m}\left(1 - \frac{k}{q_{m+1}}\right) + \frac{7n}{q_{m+1}}\right)^2 \\
&\leq \frac{n^2}{2}\left(\frac{n}{q_{m+1}} + 1\right)^2 + \frac{4n^2 q_{m+1}}{3q_m} + 14n^2 + \frac{49n^2 q_m}{q_{m+1}} \\
&\leq \frac{4n^2 q_{m+1}}{3q_m} + 65n^2 < \frac{4n^2(a_{m+1}+1)}{3} + 65n^2 \\
&< \frac{4n^2(a_{m+1}+49)}{3}.
\end{aligned}$$

Thus, adding the above terms, we get

$$\Sigma^n(x) = \Sigma_1^n(x) + \sum_{k=2}^{n} \Sigma_k^n(x) = \Sigma_1^n(x) + \sum_{i=0}^{m-1} \sum_{k=q_i+1}^{q_{i+1}} \Sigma_k^n(x) + \sum_{k=q_m+1}^{n} \Sigma_k^n(x)$$
$$< n^2 + \frac{4n^2}{3} \sum_{i=0}^{m} (a_{i+1} + 49) \le \frac{4n^2}{3} \sum_{i=1}^{m+1} (a_i + 50),$$

which completes the proof. □

Lemma 9. *Under the same setting as Theorem 3, we have*

$$\Sigma^n(x) > \frac{n^2}{24000} \sum_{i=1}^{m-9} a_i.$$

Proof. Let $q_i \le k < q_{i+1}$ and ξ be a factor of x of length k. Let $k \le n$. By Lemma 7, we have

$$\Sigma_k^n(x) \ge (q_i - \ell - 1)(B_2)_+^2 + (\ell + 1)(B_3)_+^2,$$

with

$$B_2 = (n + k - 1 - (c-1)q_i - q_{i-1}) (\|q_{i-1}\theta\| - (c-1)\|q_i\theta\|) - 1,$$
$$B_3 = (n - k + 1 - cq_i - q_{i-1}) (\|q_{i-1}\theta\| - c\|q_i\theta\|) - 1,$$

where $c = \lfloor \frac{k - q_{i-1}}{q_i} \rfloor$. Then, we have

$$\min\{B_2, B_3\} \ge (n - k + 1 - cq_i - q_{i-1}) (\|q_{i-1}\theta\| - c\|q_i\theta\|) - 1$$
$$> (n - 2k)\frac{1}{q_i}\left(1 - \frac{k}{q_{i+1}}\right) - 1 > \frac{n}{q_i}\left(1 - \frac{k}{q_{i+1}}\right) - \frac{3k}{q_i},$$

where we used (3.2) as follows:

$$\|q_{i-1}\theta\| - c\|q_i\theta\| > \frac{1}{q_i}\left(1 - \frac{q_{i-1}}{q_{i+1}}\right) - \frac{k - q_{i-1}}{q_i}\frac{1}{q_{i+1}} = \frac{1}{q_i}\left(1 - \frac{k}{q_{i+1}}\right).$$

Since $q_{i+2} \ge 2q_i$ holds for $i = 0, 1, 2, \cdots$, either

$$q_{i+1} \ge \sqrt{2}q_i \text{ or } q_{i+2} \ge \sqrt{2}q_{i+1}$$

holds. Assume that

$$q_{i+1} \ge \sqrt{2}q_i,\ n \ge 16q_i \text{ and } n \ge q_{i+1}$$

Then, since

$$n - \frac{n + 3q_{i+1}}{q_{i+1}}\ q_i \ge n - \frac{1}{\sqrt{2}}n - 3q_i \ge n - \frac{1}{\sqrt{2}}n - \frac{3}{16}n > \frac{n}{10},$$

we have

$$\sum_{k=q_i}^{q_{i+1}-1} \Sigma_k^n(x) > \sum_{k=q_i}^{q_{i+1}-1} q_i \left(\frac{n}{q_i} \left(1 - \frac{k}{q_{i+1}} \right) - \frac{3k}{q_i} \right)_+^2$$

$$= \frac{1}{q_i} \sum_{k=q_i}^{q_{i+1}-1} \left(n - \frac{n + 3q_{i+1}}{q_{i+1}} k \right)_+^2 \geq \frac{1}{q_i} \int_{q_i}^{u} \left(n - \frac{n + 3q_{i+1}}{q_{i+1}} k \right)^2 dk,$$

where $u = nq_{i+1}/(n + 3q_{i+1})$. Hence we have

$$\sum_{k=q_i}^{q_{i+1}-1} \Sigma_k^n(x) > \frac{1}{3q_i} \frac{q_{i+1}}{n + 3q_{i+1}} \left(n - \frac{n + 3q_{i+1}}{q_{i+1}} q_i \right)^3$$

$$\geq \frac{a_{i+1}}{3} \frac{1}{n + 3q_{i+1}} \left(n - \frac{n + 3q_{i+1}}{q_{i+1}} q_i \right)^3 \geq \frac{a_{i+1}}{12n} \left(\frac{n}{10} \right)^3 = \frac{a_{i+1}}{12000} n^2.$$

We conclude that

$$\sum_{k=q_i}^{q_{i+2}-1} \Sigma_k^n(x) > \frac{a_{i+1} + a_{i+2}}{24000} n^2$$

for any $i = 0, 1, 2, \cdots$ if both $n \geq 16q_{i+1}$ and $n \geq q_{i+2}$ are satisfied, and hence, if $n \geq 16q_{i+2}$ is satisfied. This is because, if $q_{i+1} \geq \sqrt{2} q_i$ is not satisfied, then $a_{i+1} = 1$ and $q_{i+2} \geq \sqrt{2} q_{i+1}$ is satisfied. Hence,

$$\sum_{k=q_i}^{q_{i+2}-1} \Sigma_k^n(x) \geq \sum_{k=q_{i+1}}^{q_{i+2}-1} \Sigma_k^n(x) \geq \frac{a_{i+2}}{12000} n^2 \geq \frac{1 + a_{i+2}}{24000} n^2 = \frac{a_{i+1} + a_{i+2}}{24000} n^2.$$

The proof is same for the other case that $q_{i+2} \geq \sqrt{2} q_{i+1}$ is not satisfied.

Hence, we have

$$\Sigma^n(x) > \frac{n^2}{24000} \sum_{i=0}^{\infty} a_{i+1} \mathbf{1}_{16q_{i+2} \leq n}.$$

Since $q_{i+2} \geq 2q_i$ holds for any $i = 0, 1, 2, \cdots$ and $q_m \leq n$, we have $16q_{i+2} \leq n$ for any $i \leq m - 10$. Thus, we have

$$\Sigma^n(x) > \frac{n^2}{24000} \sum_{i=0}^{m-10} a_{i+1},$$

which completes the proof. □

Proof of (2) of Theorem 3: Clear from Lemmas 8 and 9. □

Definition 1. *For $x_1x_2\cdots x_n \in \{0,1\}^n$, define*

$$\Lambda(x_1x_2\cdots x_n) = \max\{|\eta|^2(\ell+1)^3 : \eta^\ell \prec x_1x_2\cdots x_n\}$$

Lemma 10. ([7]) *For any $x_1x_2\cdots x_n \in \{0,1\}^n$, it holds that*

$$\Sigma(x_1x_2\cdots x_n) \geq \frac{\Lambda(x_1x_2\cdots x_n)}{48}.$$

Let $x = x_1x_2\cdots \in \{0,1\}^\infty$ be a Sturmian word with the slope θ and $(a_i)_{i=1,\cdots}$ and $(p_i/q_i)_{i=1,\cdots}$ be the partial quotients and principal convergents of θ. Put $p_0 = 0, q_0 = 1$. For each $n = 1, 2, \cdots$, let m be satisfying $q_m \leq n < q_{m+1}$. There is a sequence of subwords $w_{-1} = 0$, $w_0 = 1$, $w_1 = (w_0)^{a_1-1}w_{-1}$ and $w_{i+1} = (w_i)^{a_{i+1}}w_{i-1}$ $(i = 1, 2, \cdots)$. Note that $|w_i| = q_i$. For any $i = 1, 2, \cdots$, it holds that x is a one sided infinite word contained in a bi-infinite word which is obtained by concatenating w_i and w_{i-1} in a way that w_i-blocks are either $w_i^{a_{i+1}}$ or $w_i^{a_{i+1}+1}$ which are separated by isolated w_{i-1}, that is, something like

$$\cdots w_{i-1}w_i^{a_{i+1}}w_{i-1}w_i^{a_{i+1}+1}w_{i-1}w_i^{a_{i+1}}w_{i-1}w_i^{a_{i+1}}w_{i-1}w_i^{a_{i+1}+1}w_{i-1}\cdots.$$

Lemma 11. (1) *If $\sum_{i=1}^{m+1} a_i \asymp \log n$, then $\log q_{m+1} = O(\log q_m)$ holds.*
(2) *Assume that $\Sigma^n(x) \asymp n^2\log n$. Then, we have $a_{m+1} = O(\log n)$. It also holds that $a_{m+1-i} = O(\log n)$ for any fixed $i = 0, 1, 2, \cdots$.*

Proof. (1) Assume that $\sum_{i=1}^{m+1} a_i \asymp \log n$. Then we have $\log(q_{m+1}-1) \asymp \log q_m$, since both of $n = q_{m+1}-1$ and $n = q_m$ correspond to the same m. Hence, $\log q_{m+1} = O(\log q_m)$.

(2) Assume that $\Sigma^n(x) \asymp n^2\log n$. If $n = q_{m+1}-1$, then we have $a_{m+1}q_m \leq n < (a_{m+1}+1)q_m$. We may assume that a_{m+1} is sufficiently large, since otherwise, we have nothing to prove. Since $x_1x_2\cdots x_n$ contains w_m^l with $l = \lfloor a_{m+1}/2 \rfloor$, by Lemma 11, we have

$$\Sigma^n(x) \geq \frac{q_m^2 l^3}{48} \geq \frac{q_m^2(a_{m+1}+1)^3}{100} \geq \frac{n^2 a_{m+1}}{100}.$$

Since $\Sigma^n(x) \asymp n^2\log n$, we have $a_{m+1} = O(\log n)$ for this special $n = q_{m+1}-1$, say n_1.

The smallest possible n corresponding to m is q_m. Let $n_2 = q_m$. Then, since

$$n_2 = q_m \geq \frac{q_{m+1}}{a_{m+1}+1} = \frac{n_1+1}{a_{m+1}+1}$$

and $a_{m+1} = O(\log n_1)$, we have

$$\log n_2 \geq \log(n_1+1) - \log(a_{m+1}+1) \geq \log n_1 - \log\log n_1 + C$$

with some constant C. Hence, $\log n_1 = O(\log n_2)$ and $a_{m+1} = O(\log n_2)$. Thus, $a_{m+1} = O(\log n)$ for general n with $q_m \leq n < q_{m+1}$.

The last part is clear since $a_{m+1-i} = O(\log q_{m+1-i})$ and $q_{m+1-i} \leq q_{m+1}$. $\square$

Proof of Corollary 2: Consider the following statements:

$$\sum_{i=1}^{m} a_i = O(m), \tag{3.3}$$

$$\Sigma^n(x) \asymp n^2 \log n, \tag{3.4}$$

$$\sum_{i=1}^{m+1} a_i \asymp \log n. \tag{3.5}$$

$$\sum_{i=1}^{m} a_i = O\left(\sum_{i=1}^{m} \log(a_i+1)\right). \tag{3.6}$$

To prove Corollary 2, we'll show that (3.3) implies (3.4), (3.4) implies (3.5), (3.5) implies (3.6), and (3.6) implies (3.3).

(3.3)⇒(3.4): Assume (3.3). By Theorem 3 (1), it is sufficient to prove that $\Sigma^n(x) = O(n^2 \log n)$.

Let $m-1 \leq 2l \leq m$. Then we have

$$n \geq q_{2l} \geq \prod_{i=1}^{l}(a_{2i-1}a_{2i}+1) \geq \left(\prod_{i=1}^{l}(a_{2i-1}+1)(a_{2i}+1)\right)^{\frac{1}{2}}.$$

Hence by (3.3) and Theorem 3 (2), we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{\Sigma^n(x)}{n^2\log n} &\leq \limsup_{n\to\infty} \frac{(4/3)n^2\sum_{i=1}^{m+1}(a_i+50)}{n^2(1/2)\log\prod_{i=1}^{l}(a_{2i-1}+1)(a_{2i}+1)} \\ &\leq C\limsup_{n\to\infty}\frac{\sum_{i=1}^{m+1}a_i}{\sum_{i=1}^{m-1}\log(a_i+1)} \leq C\limsup_{n\to\infty}\frac{\sum_{i=1}^{m+1}a_i}{(m-1)\log 2} < \infty\end{aligned}$$

with some positive constant C. Thus, $\Sigma^n(x) = O(n^2\log n)$.

(3.4)⇒(3.5): Assume (3.4). Then, by Theorem 3 (2), we have

$$\infty > \limsup_{n\to\infty}\frac{n^2\sum_{i=1}^{m-9}a_i}{24000\Sigma^n(x)} \geq C\limsup_{n\to\infty}\frac{n^2\sum_{i=1}^{m-9}a_i}{n^2\log n} = C\limsup_{n\to\infty}\frac{\sum_{i=1}^{m-9}a_i}{\log n}$$

with some constant $C > 0$. Hence, $\sum_{i=1}^{m-9} a_i = O(\log n)$. Moreover, by Lemma 12, $\sum_{i=m-8}^{m+1} a_i = O(\log n)$. Thus, we have $\sum_{i=1}^{m+1} a_i = O(\log n)$.

On the other hand, by Theorem 3 (2), we have

$$
\begin{aligned}
\infty &> \limsup_{n\to\infty} \frac{\Sigma^n(x)}{(4/3)n^2\sum_{i=0}^{m}(a_{i+1}+50)} \\
&\geq C \limsup_{n\to\infty} \frac{n^2 \log n}{n^2 \sum_{i=1}^{m+1} a_i} = C \limsup_{n\to\infty} \frac{\log n}{\sum_{i=1}^{m+1} a_i}
\end{aligned}
$$

with some constant $C > 0$. Thus, $\log n = O(\sum_{i=1}^{m+1} a_i)$, and together with $\sum_{i=1}^{m+1} a_i = O(\log n)$, we have $\sum_{i=1}^{m+1} a_i \asymp \log n$.

(3.5)⇒(3.6): Assume (3.5). Then since

$$
\log n < \log q_{m+1} < \log\left(\prod_{i=1}^{m+1}(a_i+1)\right) = \sum_{i=1}^{m+1}\log(a_i+1).
$$

we get

$$
\sum_{i=1}^{m+1} a_i = O\left(\sum_{i=1}^{m+1}\log(a_i+1)\right).
$$

(3.6)⇒(3.3): Suppose that (3.3) does not hold, so that we have

$$
\limsup_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} a_m = \infty.
$$

By applying Jensen's inequality to $f(x) := \log(x+1)$, we have

$$
\log\left(\frac{1}{m}\sum_{i=1}^{m} a_i + 1\right) \geq \frac{1}{m}\sum_{i=1}^{m}\log(a_i+1).
$$

It follows from this that

$$
\limsup_{m\to\infty} \frac{\sum_{i=1}^{m} a_i}{\sum_{i=1}^{m}\log(a_i+1)} \geq \limsup_{m\to\infty} \frac{\frac{1}{m}\sum_{i=1}^{m} a_i}{\log\left(\frac{1}{m}\sum_{i=1}^{m} a_i + 1\right)} = \infty,
$$

since $x/\log(x+1) \to \infty$ as $x \to \infty$. Thus, (3.6) does not hold.

Assume that $(a_i)_{i=1,2,\cdots}$ is bounded. Recall that in the proof of Lemma 9, we have

$$
\sum_{k=q_i+1}^{q_{i+1}} \Sigma_k^n(x) < \frac{4n^2(a_{i+1}+28)}{3} \quad (i = 0, 1, \cdots, m).
$$

Take an arbitrary $K > 0$. Since $q_{i+2} \geq 2q_i$ holds for any $i = 0, 1, 2, \cdots$, the number of $i = 0, 1, \cdots, m+1$ such that $q_i < K$ or $q_{i+1} > n/K$ is bounded by a constant independent of n. Therefore, $\sum_{k\leq K}\Sigma_k^n(x) + \sum_{k>n/K}\Sigma_k^n(x)$ is bounded from above by a bounded number of sum of the terms $\sum_{k=q_i+1}^{q_{i+1}} \Sigma_k^n(x)$ or $\sum_{k=q_m+1}^{n} \Sigma_k^n(x)$. Hence, by the above inequalities, we have

$$\sum_{k\le K} \Sigma_k^n(x) + \sum_{k>n/K}^{n} \Sigma_k^n(x) = O(n^2).$$

By (1) of Theorem 3, this implies that $\sum_{K<k\le n/K} \Sigma_k^n(x) \sim \Sigma^n(x)$, which completes the proof. □

4 Almost 0-words

Through out this section, we fix $x = x_1x_2\cdots \in \{0,1\}^\infty$ with $\sum_{i=1}^\infty x_i = \infty$. For $n = 1, 2, \cdots$, denote $N = N(x,n) = |x_1x_2\cdots x_n|_1$ as a function of n. Let $\{k_1 < k_2 < \cdots\} = \{i \ge 1;\ x_i = 1\}$. Let $F_n^x : [0,\infty) \to [0,1]$ be the probability distribution function of number of 0's between two consecutive 1's in $x_1x_2\ldots x_n$, i.e.,

$$F_n^x(t) := \frac{\#\{0 \le i \le N-1\ ;\ k_{i+1} - k_i - 1 \le t\} + 1_{n-k_N\le t}}{N+1},$$

where we put $k_0 = 0$. Denoting $\delta F_n^x(t) = F_n^x(t) - F_n^x(t-0)$, we have

Theorem 4. *Assume that $n \asymp N^\alpha$,*

$$\sup_{\delta F_n^x(t)>0} t = \max\{\max\{k_{i+1} - k_i;\ 0 \le i \le N-1\},\ n - k_N\} \asymp N^{\alpha-1}$$

and

$$\sup_t (N+1)\delta F_n^x(t) = \max\{\#\{i;\ k_{i+1} - k_i = s\} + 1_{n-k_N=s};\ s > 0\} \asymp N^\beta$$

with α, β such that $\alpha \ge \beta + 1 \ge 1$. Then, we have $\Sigma^n(x) \asymp n^{3-(1/\alpha)}$.

Corollary 3. *Assume that $k_i = i^\alpha + O(1)$ for some $\alpha \ge 3/2$ as $i \to \infty$. Then, we have $\Sigma^n(x) \asymp n^{3-(1/\alpha)}$.*

Define

$$\Xi_0^n(x) = \sum_{\substack{\xi\in\{0,1\}^+\\ |\xi|_1=0}} |x_1x_2\cdots x_n|_\xi^2\ ,\quad \Xi_1^n(x) = \sum_{\substack{\xi\in\{0,1\}^+\\ |\xi|_1\ge1}} |x_1x_2\cdots x_n|_\xi^2\ .$$

Lemma 12. *We have*

$$\Xi_0^n(x) = \sum_{s=1}^\infty \left(\int_0^\infty (t-s+1)_+(N+1)dF_n^x(t)\right)^2.$$

Proof. Consider $\xi = 0^s$ for $s = 1, 2, \cdots$. Then, $|0^t|_\xi = (t - s + 1)_+$ for $t = 0, 1, 2, \cdots$. Therefore,

$$|x_1 x_2 \cdots x_n|_\xi = \sum_{i=0}^{N-1} |0^{k_{i+1}-k_i-1}|_\xi + |0^{n-k_N}|_\xi = \int_0^\infty (t-s+1)_+ (N+1) dF_n^x(t).$$

Hence, we have

$$\sum_{\substack{\xi \in \{0,1\}^+ \\ |\xi|_1 = 0}} |x_1 x_2 \cdots x_n|_\xi^2 = \sum_{s=1}^\infty \left(\int_0^\infty (t - s + 1)_+ (N + 1) dF_n^x(t) \right)^2 .$$

□

Lemma 13. *Assume that* $n \asymp N^\alpha$ *and* $\sup_{\delta F_n^x(t) > 0} t \asymp N^{\alpha - 1}$ *for some* $\alpha \geq 1$. *Assume also that*

$$\sup_{t \in [0,\infty)} (N + 1) \delta F_n^x(t) = O(N^\beta)$$

for some $0 \leq \beta \leq 1$. *Then, we have* $\Xi_1^n(x) = O(n^{2+(\beta/\alpha)})$.

Proof. Assume the conditions in our lemma. Since $|x_1 x_2 \cdots x_n|_\xi \leq N$ for any $\xi \in \{0, 1\}^+$ with $|\xi|_1 = 1$, we have

$$\begin{aligned} \sum_{\substack{\xi \in \{0,1\}^+ \\ |\xi|_1 = 1}} |x_1 x_2 \cdots x_n|_\xi^2 &\leq N^2 \#\{\xi \in \{0, 1\}^+ ;\ |\xi|_1 = 1,\ \xi \prec x_1 x_2 \cdots x_n\} \\ &\leq N^2 (\sup_{\delta F_n^x(t) > 0} t + 1)^2 \\ &\asymp N^2 (N^{(\alpha - 1)})^2 = N^{2\alpha} \asymp n^2. \end{aligned}$$

Let $\xi \in \{0, 1\}^+$ satisfy $|\xi|_1 \geq 2$. Then, ξ can be written uniquely as $\xi = 0^a 1 0^b 1 *$. Since each occurrence of such ξ in $x_1 x_2 \cdots x_n$ corresponds to different occurrence of 10^b1 in $x_1 x_2 \cdots x_n$. Hence,

$$|x_1 x_2 \cdots x_n|_\xi \leq (N + 1) \delta F_n^x(b) \leq \sup_{t \in [0,\infty)} (N + 1) \delta F_n^x(t).$$

Therefore,

$$\begin{aligned} \sum_{\substack{\xi \in \{0,1\}^+ \\ |\xi|_1 \geq 2}} |x_1 x_2 \cdots x_n|_\xi^2 &\leq \sup_{t \in [0,\infty)} (N + 1) \delta F_n^x(t) \sum_{\xi \in \{0,1\}^+} |x_1 x_2 \cdots x_n|_\xi \\ &\leq \sup_{t \in [0,\infty)} (N + 1) \delta F_n^x(t) \cdot n^2 = O(N^\beta) n^2 = O(n^{2+(\beta/\alpha)}). \end{aligned}$$

Thus, we have

$$\Xi_1^n(x) = \sum_{\substack{\xi\in\{0,1\}^+\\ |\xi|_1=1}} |x_1x_2\cdots x_n|_\xi^2 + \sum_{\substack{\xi\in\{0,1\}^+\\ |\xi|_1\geq 2}} |x_1x_2\cdots x_n|_\xi^2 = O(n^{2+(\beta/\alpha)}).$$

□

Proof of Theorem 4: Assume that $x \in \{0,1\}^\infty$ satisfies the conditions in Theorem 4. By Lemma 13, we have

$$\Xi_0^n(x) = \sum_{s=1}^{\infty} \left(\int_0^\infty (t-s+1)_+ (N+1) dF_n^x(t) \right)^2 .$$

Under the conditions

$$\int (N+1)dF_n^x(t) = N+1, \quad \int t(N+1)dF_n^x(t) + N = n$$

for the probability distribution $F_n^x(t)$ on $[0,\ t_0]$, where

$$t_0 := \sup_{\delta F_n^x(t)>0} t \asymp N^{\alpha-1},$$

we'll show that an upper and a lower estimates of $\Xi_0^n(x)$ coincide in the asymptotical order.

That is,

$$\begin{aligned}
\Xi_0^n(x) &= \sum_{s=1}^{\infty} \left(\int_0^\infty (t-s+1)_+ (N+1) dF_n^x(t) \right)^2 \\
&\leq \sum_{s=1}^{t_0} \left(\int_0^{t_0} (t_0-s+1)_+ (N+1) dF_n^x(t) \right)^2 \\
&= \sum_{s=1}^{t_0} (t_0-s+1)^2 (N+1)^2 \asymp t_0^3 N^2 \asymp N^{3\alpha-1} \asymp n^{3-(1/\alpha)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Xi_0^n(x) &= \sum_{s=1}^{\infty} \left(\int_0^\infty (t-s+1)_+ (N+1) dF_n^x(t) \right)^2 \\
&\geq \sum_{s=1}^{\infty} \left(\int_0^\infty t(N+1)dF_n^x(t) - \int_0^\infty (s-1)(N+1)dF_n^x(t) \right)_+^2 \\
&= \sum_{s=1}^{\infty} (n-N-(s-1)(N+1))_+^2 \asymp \sum_{s=0}^{N^{\alpha-1}} (N^\alpha - sN)^2 \\
&\asymp N^2 \sum_{s=0}^{N^{\alpha-1}} (N^{\alpha-1}-s)^2 \asymp N^2 N^{3(\alpha-1)} = N^{3\alpha-1} \asymp n^{3-(1/\alpha)}.
\end{aligned}$$

Thus, we have $\Xi_0^n(x) \asymp n^{3-(1/\alpha)}$.

We already proved that $\Sigma_1^n(x) = O(n^{2+(\beta/\alpha)})$. Since $\alpha \geq \beta + 1$, we have

$$2 + (\beta/\alpha) \leq 2 + ((\alpha - 1)/\alpha) = 3 - (1/\alpha).$$

Thus,

$$\Sigma^n(x) = \Xi_0^n(x) + \Sigma_1^n(x) \asymp n^{3-(1/\alpha)}.$$

□

Proof of Corollary 3: Let $x \in \{0,1\}^\infty$ satisfy the condition in Corollary 3. Then, it satisfies the conditions in Theorem 4 with the same α and $\beta = (2-\alpha)_+$. Since $\alpha \geq 3/2$, we have $\alpha \geq \beta + 1$. Hence by Theorem 5, $\Sigma^n(x) \asymp n^{3-(1/\alpha)}$. □

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