Spectral measure of the Thue-Morse sequence and the dynamical system and random walk related to it

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Abstract

Let $1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, \cdots$ be the $\{-1, 1\}$-valued Thue-Morse sequence. The correlation dimension of it is D_2 satisfying that

$$\sum_{k=0}^{K-1} |\gamma(k)|^2 \asymp K^{1-D_2}$$

in the sense that the ratio between the left and right sides is bounded away from 0 and ∞ as $K \to \infty$, where γ is the correlation function, and is known ([6]) to be

$$D_2 = 1 - \log \frac{1+\sqrt{17}}{4} / \log 2 = 0.64298\cdots.$$

Under its spectral measure μ on $[0, 1)$, consider the transformation T with $Tx = 2x \pmod 1$. It is shown to be of Kolmogorov type having the entropy at least $D_2 \log 2$. Moreover, T^{-1} define a random walk on $[0, 1)$ with the transition probability

$$P_1((1/2)x + (1/2)j \,|\, x) = (1/2)(1 - \cos(\pi(x + j))) \quad (j = 0, 1).$$

It is proved that this random walk is mixing and μ is the unique stationary measure. Moreover,

$$\lim_{N\to\infty} \int P_N((x-\varepsilon, x+\varepsilon)|x) d\mu(x) \asymp \varepsilon^{D_2} \quad (\text{as } \varepsilon \to 0),$$

where $P_N(\cdot \mid \cdot)$ is the N-step transition probability.

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1 Introduction

Let $\omega = \omega(0)\omega(1)\omega(2)\cdots \in \{-1,1\}^{\mathbb{N}}$ be the Thue-Morse sequence, that is, for any $n \in \mathbb{N} := \{0,1,2,\cdots\}$, $\omega(n) = (-1)^{e_1(n)}$, where $e_1(n)$ is the number of 1 in the 2-adic representation of n. There have been a lot of studies on the Thue-Morse sequence from various point of views, e.g. language theory, ergodic theory, number theory and physics. From the point view of ergodic theory, it is known to be strictly ergodic (Kakutani [2]), and thesymbolic dynamics of the shift on $\{-1,1\}^{\mathbb{N}}$ with respect to the unique shift invariant probability measureon the orbit closure of ω has a partially continuous spectrum and the entropy 0. Let μ be the power spectrum measure of this dynamical system with respect to the coordinate function. It known ([3], for example) to be continuous but singular. This μ is a probability Borel measure on the torus $\mathbb{R}/\mathbb{Z}$, which we identify with $[0,1)$ (sometimes with $[-1/2,1/2)$). It is known (Theorem 3) that μ has a representation as an infinite product converging in the weak sense. That is,

$$d\mu(x) = \prod_{k=0}^{\infty}(1-\cos 2\pi 2^k x)dx. \tag{1.1}$$

It is the Fourier transform of the correlation function $\gamma(k)$ ($k \in \mathbb{Z}$) defined as

$$\gamma(k) = \lim_{N\to\infty}(1/N)\sum_{n=0}^{N-1}\omega(n+k)\omega(n),$$

that is,

$$\int e^{2\pi ikx}d\mu(x) = \gamma(k) \quad (k \in \mathbb{Z}).$$

It is proved by Mahler [1] that

$$\begin{cases} \gamma(0) = 1 \\ \gamma(2k) = \gamma(k) \\ \gamma(2k+1) = (-1/2)(\gamma(k)+\gamma(k+1)) \end{cases} \qquad (k \in \mathbb{N}) \tag{1.2}$$

and by Zaks, Pikovsky and Kurths [6] that

$$\sum_{k=0}^{K-1}|\gamma(k)|^2 \asymp K^{1-D_2} \quad (\text{as } K \to \infty) \tag{1.3}$$

with

$$D_2 = 1 - \log\frac{1+\sqrt{17}}{4}/\log 2 = 0.64298\cdots,$$

which is called the *correlation dimension* (Theorem 1). Note that we use the notation $A(K) \asymp B(K)$ in the sense that

$$0 < \liminf_{K\to\infty}\frac{A(K)}{B(K)} \le \limsup_{K\to\infty}\frac{A(K)}{B(K)} < \infty.$$

By (1.2), μ is a T-invariant probability measure, where T is the transformation on $[0,1)$ such that $Tx = 2x \pmod 1$. It is proved (Theorem 4) that the dynamical system $([0,1), \mu, T)$ is K-system (K for Kolmogrov), that is, it has the trivial tail field. We don't know the exact value of the entropy of the system, but it is at least $D_2 \log 2$ (Theorem 2).

By (1.1), the probability under $d\mu(y)$ that $y = (1/2)x + (1/2)j$ $(j = 0, 1)$ given $Ty = x$ is $(1 - \cos \pi(x + j))/2$. Hence, we have a random walk on $[0,1)$ with the transition probability

$$P_1((1/2)x + (1/2)j \mid x) = (1 - \cos \pi(x + j))/2 \quad (j = 0, 1).$$

Then, we prove (Theorem 5) that μ is the unique stationary measure of the random walk and

$$\lim_{N\to\infty} \int P_N((x - \varepsilon, x + \varepsilon)|x) d\mu(x) \asymp \varepsilon^{D_2} \quad (\text{as } \varepsilon \to 0)$$

holds (Theorem 6), where P_N is the N-step transition probability. For a general reference to the ergodic theory and dynamical systems, we cite [4].

2 Ergodicity of the system

In this section we prove that the dynamical system $([0,1), \mu, T)$ is ergodic.

We review the properties of the correlation function γ obtained in [6]. Let $X = X_0X_1X_2\cdots$ be an i.i.d. process with $P(X_0 = 0) = P(X_0 = 1) = 1/2$, which is also considered as 2-adic integer $\sum_{n=0}^{\infty} X_n 2^n$. For $k \in \mathbb{N}$, consider the addition $X + k$. Note that by the addition, only finitely many digits of X changes almost surely. Hence, the increased number of digit 1 in $X + k$ from X makes sense, which we denote by $e_1(k, X)$. Then we have

$$\gamma(k) := \lim_{N\to\infty} (1/N) \sum_{n=0}^{N-1} \omega(n + k)\omega(n) = E[(-1)^{e_1(k,X)}].$$

The second equality follows since the space of 2-adic integers is strictly ergodic with respect to adding 1 and the sets of n in this space such that $(-1)^{e_1(k,n)} = \pm 1$ have boundaries of measure 0 with respect to the unique invariant measure, that is, the distribution of $X_0X_1X_2\cdots$.

Since $e_1(2k, X_0X_1\cdots) = e_1(k, X_1X_2\cdots)$, we have $\gamma(2k) = \gamma(k)$. Moreover, since

$$e_1(2k + 1, X_0X_1\cdots) = \begin{cases} 1 + e_1(k, X_1X_2\cdots) & (X_0 = 0) \\ -1 + e_1(k + 1, X_1X_2\cdots) & (X_0 = 1) \end{cases},$$

we have $\gamma(2k + 1) = (-1/2)(\gamma(k) + \gamma(k + 1))$.

Definition 1. The spectral measure μ of the Thue-Morse sequence is the unique Borel measure on $[0,1)$ such that $\int e^{2\pi ikx}d\mu(x) = \gamma(k)$ for any $k \in \mathbb{N}$. It is a probability measure since $\int d\mu(x) = \gamma(0) = 1$. Moreover, $\gamma(-k) = \int e^{-2\pi ikx}d\mu(x) = \gamma(k)$ for any $k \in \mathbb{N}$, since $\omega(n)$ is real for any $\omega \in \mathbb{N}$.

Lemma 1. *The probability measure μ is T-invariant.*

Proof For any $k \in \mathbb{N}$, we have

$$\int e^{2\pi ikTx}d\mu(x) = \int e^{2\pi ik2x}d\mu(x) = \gamma(2k) = \gamma(k) = \int e^{2\pi ikx}d\mu(x),$$

which implies that μ is T-invariant. □

Lemma 2. *For any $k, l \in \mathbb{Z}$, we have $\lim_{n\to\infty} \gamma(k + 2^n l) = \gamma(k)\gamma(l)$.*

Proof We only prove the lemma for $k, l \in \mathbb{N}$. Let $k < 2^n$. If there is no carry to the 2^n term in the addition $X + k$, we have

$$e_1(k + 2^n l, X) = e_1(k, X) + e_1(l, X_n X_{n+1} \cdots).$$

Let this event be B_n. Then,

$$\begin{aligned}
&E[(-1)^{e_1(k+2^n l,X)} \mid B_n] = E[(-1)^{e_1(k,X)}(-1)^{e_1(l,X_nX_{n+1}\cdots)} \mid B_n] \\
&= E[(-1)^{e_1(k,X)} \mid B_n]\ E[(-1)^{e_1(l,X)}].
\end{aligned}$$

Since $\lim_{n\to\infty} P(B_n) = 1$, we have

$$\lim_{n\to\infty} E[(-1)^{e_1(k,X)} \mid B_n]\ E[(-1)^{e_1(l,X)}] = E[(-1)^{e_1(k,X)}]\ E[(-1)^{e_1(l,X)}].$$

Thus, we have $\lim_{n\to\infty} \gamma(k + 2^n l) = \gamma(k)\gamma(l)$. □

Lemma 3. *The system $([0,1), \mu, T)$ is ergodic.*

Proof It is sufficient to prove that for any $k \in \mathbb{N}$,

$$I_N := \int \left| (1/N) \sum_{n=0}^{N-1} e^{2\pi i 2^n kx} - \gamma(k) \right|^2 d\mu(x) \to 0$$

as $N \to \infty$. By (1.2) and Definition 1, we have

$$\begin{aligned}
I_N = {}& (1/N^2) \sum_{n,m=0}^{N-1} \int e^{2\pi i(2^n - 2^m)kx} d\mu(x) \\
& - (1/N)\gamma(k) \sum_{n,m=0}^{N-1} \int (e^{2\pi i 2^n kx} + e^{-2\pi i 2^m kx}) d\mu(x) + \gamma(k)^2 \\
= {}& (1/N^2) \sum_{n,m=0}^{N-1} \gamma(2^n k - 2^m k) - 2\gamma(k)^2 + \gamma(k)^2.
\end{aligned}$$

Since by Lemma 2,

$$\lim_{|n-m|\to\infty} \gamma(2^n k - 2^m k) = \gamma(k)^2,$$

we have

$$\lim_{N\to\infty} (1/N^2) \sum_{n,m=0}^{N-1} \gamma(2^n k - 2^m k) = \gamma(k)^2,$$

and hence, $\lim_{N\to\infty} I_N = 0$. □

3 Correlation dimension and the entropy

Let $S_N = \sum_{k=0}^{2^N-1} |\gamma(k)|^2$ and $W_N = \sum_{k=0}^{2^N-1} \gamma(k)\gamma(k+1)$. Then, by (1.2), we have

$$\begin{cases} S_{N+1} = (3/2)S_N + (1/2)W_N - (2/9) \\ W_{N+1} = -S_N - W_N + (4/9) \end{cases} \quad (N = 1, 2, \cdots)$$

with $S_0 = 1$, $W_0 = -1/3$. This linear equation has eigenvalues $\frac{1\pm\sqrt{17}}{4}$. Hence, the following theorem holds.

Theorem 1. (Zaks, Pikovsky and Kurths [6])

$$\sum_{k=0}^{K-1} |\gamma(k)|^2 \asymp K^{1-D_2} \quad \textit{(as } K \to \infty\textit{)}$$

holds with

$$D_2 = 1 - \log \frac{1+\sqrt{17}}{4} / \log 2 = 0.64298\cdots,$$

which is called the correlation dimension.

Since the system $([0,1), \mu, T)$ is ergodic, by the Shannon-McMillan-Breiman theorem

$$\lim_{n\to\infty} \frac{-\log \mu(\Xi_n(x))}{n} = h_\mu(T) \tag{3.1}$$

holds μ-almost surely, where we denote

$$\Xi_n = \{[k2^{-n}, (k+1)2^{-n});\ k = 0, 1, \cdots, 2^n - 1\}$$

and $\Xi_n(x)$ denotes the interval in Ξ_n containing x. On the other hand, by Theorem 1, we have

$$\lim_{n\to\infty} \frac{\log \sum_{n=0}^{N-1} \left| \int e^{2\pi i n x} d\mu(x) \right|^2}{\log N} = 1 - D_2. \tag{3.2}$$

We'll show that $h_\mu(T) \geq D_2 \log 2$. We often consider $[-1/2, 1/2)$ instead of $[0,1)$ for the domain of the following Poisson kernel. For $x \in [-1/2, 1/2)$, let

$$\mathcal{P}_r(x) = \frac{1-r^2}{1+r^2-2r\cos 2\pi x}$$

be the Poisson kernel, where we always assume that $2/3 < r < 1$.

Lemma 4. *It holds that*

$$\int \mathcal{P}_r(x-y)d\mu(x)d\mu(y) \asymp ((1-r)^{-1})^{1-D_2} \quad (\textit{as } r \to 1).$$

Proof Let $p = 1 - D_2$. Then, there exists a constant $0 < C_1 \leq C_2 < \infty$ such that

$$C_1 N^p \leq S_N = \sum_{n=0}^{N-1} \left| \int e^{2\pi i n x} d\mu(x) \right|^2 \leq C_2 N^p$$

as $N \to \infty$. Therefore,

$$\begin{aligned}
&\sum_{n=0}^{\infty} r^n \left| \int e^{2\pi i n x} d\mu(x) \right|^2 = \sum_{n=0}^{\infty} r^n (S_{n+1} - S_n) \\
&= \sum_{n=1}^{\infty} (r^{n-1} - r^n) S_n = \frac{1-r}{r} \sum_{n=1}^{\infty} r^n S_n \leq \frac{1-r}{r} \sum_{n=1}^{\infty} r^n C_2 n^p \\
&= C_2 \frac{1-r}{r} (\log(1/r))^{-1-p} \sum_{n=1}^{\infty} e^{-n\log(1/r)} (n \log(1/r))^p \log(1/r) \\
&\leq 2C_2 (1-r)^{-p} \int_0^\infty e^{-t} t^p dt = 2C_2(1-r)^{-p}\Gamma(p+1).
\end{aligned}$$

In the same way,

$$\sum_{n=0}^{\infty} r^n \left| \int e^{2\pi i n x} d\mu(x) \right|^2 \geq (1/2) C_1 (1-r)^{-p} \Gamma(p+1).$$

Thus, we have

$$\sum_{n=0}^{\infty} r^n \left| \int e^{2\pi i n x} d\mu(x) \right|^2 \asymp ((1-r)^{-1})^{1-D_2}$$

On the other hand, since

$$\sum_{n=0}^{\infty} r^n \left| \int e^{2\pi i n x} d\mu(x) \right|^2 = (1/2)\left(\int \mathcal{P}_r(x-y) d\mu(x) d\mu(y) + 1\right),$$

we completes the proof. $\square$

Lemma 5. *For any x with $|x| \leq 1-r$, it holds that*

$$\mathcal{P}_r(x) \geq (1/40)(1-r)^{-1}.$$

Proof Since $\cos 2\pi x \geq 1 - 2\pi^2 x^2$ holds for any x, we have

$$\mathcal{P}_r(x) \geq \frac{1-r^2}{1+r^2-2r(1-2\pi^2x^2)} = \frac{(1+r)(1-r)}{(1-r)^2+4r\pi^2x^2} \geq \frac{1-r}{(1-r)^2+39x^2}.$$

Hence, if $|x| \leq 1-r$, then $\mathcal{P}_r(x) \geq (1/40)(1-r)^{-1}$. □

Theorem 2. *It holds that $h_\mu(T) \geq D_2 \log 2$.*

Proof Assume (3.1)(3.2). Let $\alpha = h_\mu(T)/\log 2$. Then for any ε with $0 < \varepsilon < 1/2$, there exists n_0 such that for any $n \geq n_0$,

$$\mu(\{x;\ (\alpha-\varepsilon)n\log 2 \leq -\log\mu(\Xi_n(x)) \leq (\alpha+\varepsilon)n\log 2\}) \geq 1-\varepsilon.$$

Hence, there exists $S \subset \{0, 1, \cdots, 2^n-1\}$ such that

$$(2^{-n})^{\alpha+\varepsilon} \leq \mu([k2^{-n}, (k+1)2^{-n})) \leq (2^{-n})^{\alpha-\varepsilon}$$

for any $k \in S$ and

$$\mu(\cup_{k\in S}[k2^{-n}, (k+1)2^{-n})) \geq 1-\varepsilon.$$

Moreover, since $\#S \cdot (2^{-n})^{\alpha-\varepsilon} \geq 1/2$, we have $\#S \geq (1/2)(2^n)^{\alpha-\varepsilon}$. Hence,

$$\begin{aligned}(\mu\times\mu)(\Lambda_n) &\geq \sum_{k\in S}\mu([k2^{-n},(k+1)2^{-n}))^2 \\ &\geq \sum_{k\in S}(2^{-n})^{2\alpha+2\varepsilon} \geq (1/2)(2^n)^{\alpha-\varepsilon}(2^{-n})^{2\alpha+2\varepsilon} = (1/2)(2^{-n})^{\alpha+3\varepsilon},\end{aligned}$$

where

$$\Lambda_n = \bigcup_{k\in S}\ [k2^{-n},(k+1)2^{-n}) \times [k2^{-n},(k+1)2^{-n}).$$

If $2^{-n} \leq 1-r < 2^{-n+1}$, then $|x-y| \leq 1-r$ if $(x,y) \in \Xi_n$. Hence, $\mathcal{P}_r(x-y) \geq (1/40)(1-r)^{-1}$ by Lemma 5. Therefore,

$$\begin{aligned}&\int \mathcal{P}_r(x-y)d\mu(x)d\mu(y) \geq \int_{\Lambda_n}\mathcal{P}_r(x-y)d\mu(x)d\mu(y) \\ &\geq \int_{\Lambda_n}(1/40)(1-r)^{-1}d\mu(x)d\mu(y) \geq (1/40)(1-r)^{-1}(\mu\times\mu)(\Lambda_n) \\ &\geq (1/40)(1-r)^{-1}(1/2)(2^{-n})^{\alpha+3\varepsilon} \geq (1/80)(1-r)^{-1}((1/2)(1-r))^{\alpha+3\varepsilon}.\end{aligned}$$

Thus,

$$1 - D_2 = \lim_{r\to 1}\frac{\log\int\mathcal{P}_r(x-y)d\mu(x)d\mu(y)}{-\log(1-r)} \geq 1-\alpha-3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $h_\mu(T)/\log 2 = \alpha \geq D_2$. □

Remark 1. The relation between the local dimension and $\int\mathcal{P}_r(x-y)d\mu(y)$ is discussed in a general framework by Wen and Zhang [7] or Cao, Xi and Zhang [8]. Though Theorem 2 might follow from them, we give an independent proof for to be self-contained.

4 Product form of μ and K-property

Properties of the spectral measure μ of the Thue-Morse sequence ω is discussed in [3] and [5] in a general setting. We recall some of them.

Lemma 6. *The measure μ_N defined as*

$$d\mu_N(x) = (1/N)\left|\sum_{n=0}^{N-1}\omega(n)e^{2\pi inx}\right|^2 dx$$

converges in the weak sense to μ as $N \to \infty$.

Proof For any $k \in \mathbb{Z}$, we have

$$\int e^{2\pi ikx}d\mu_N(x) = (1/N)\sum_{n,m=0}^{N-1}\omega(n)\omega(m)\int e^{2\pi i(n-m+k)x}dx$$

$$= (1/N)\sum_{\substack{n=0 \\ 0\le n+k<N}}^{N-1}\omega(n)\omega(n+k).$$

Hence, $\lim_{N\to\infty}\int e^{2\pi ikx}d\mu_N(x) = \int e^{2\pi ikx}d\mu$, which completes the proof. □

The following Theorem was proved by M. Keane (see [5]) for the first time.

Theorem 3. *It holds that*

$$d\mu(x) = \prod_{k=0}^{\infty}(1-\cos 2\pi 2^k x)dx,$$

where the infinite product converges in the weak sense. (See Figure 1.)

Proof For $n \in \mathbb{N}$ with $n < 2^N$, let $n = \sum_{k=0}^{N-1} n_k 2^k$ be the 2-adic representation of n. Then, $\omega(n) = \prod_{k=0}^{N-1}(-1)^{n_k}$. Therefore,

$$d\mu(x) = \text{w-}\lim_{N\to\infty} 2^{-N}\left|\sum_{n=0}^{2^N-1}\omega(n)e^{2\pi inx}\right|^2 dx$$

$$= \text{w-}\lim_{N\to\infty} 2^{-N}\left|\sum_{n=0}^{2^N-1}\prod_{k=0}^{N-1}(-1)^{n_k}e^{2\pi in_k2^kx}\right|^2 dx$$

$$= \text{w-}\lim_{N\to\infty} 2^{-N}\left|\prod_{k=0}^{N-1}\sum_{n_k=0,1}(-1)^{n_k}e^{2\pi in_k2^kx}\right|^2 dx$$

$$= \text{w-}\lim_{N\to\infty}\prod_{k=0}^{N-1}2^{-1}\left|1-e^{2\pi i2^kx}\right|^2 dx$$

$$= \prod_{k=0}^{\infty}2^{-1}\left|1-e^{2\pi i2^kx}\right|^2 dx = \prod_{k=0}^{\infty}(1-\cos 2\pi 2^k x)dx$$

$\square$

Theorem 4. *The system* $([0,1),\mu,T)$ *is of Kolmogorov type. That is, it has the trivial tail field.*

Proof Let $\mathcal{B}$ be the Borel field of $[0,1)$. For $n\in\mathbb{N}$, let $\mathcal{B}_n=\{T^{-n}B;\ B\in\mathcal{B}\}$. Note that $\mathcal{B}_n$ is a Borel field such that $\mathcal{B}=\mathcal{B}_0\supset\mathcal{B}_1\supset\mathcal{B}_2\supset\cdots$. The tail field of the system $([0,1),\mu,T)$ is defined to be $\cap_{n=0}^{\infty}\mathcal{B}_n$.

To prove that $([0,1),\mu,T)$ has a trivial tail field, it is sufficient to prove that

$$\lim_{K\to\infty}E[e^{2\pi ilx}|\mathcal{B}_K](x)=E[e^{2\pi ilx}]=\gamma(l)$$

holds for any $l\in\mathbb{N}$ and μ-almost all $x\in[0,1)$. Take a large K and N of the form $N=(2N'+1)2^K$ with $N'\in\mathbb{N}$. By Lemma 6, we have

$$E[e^{2\pi ilx}|\mathcal{B}_K](x)=\lim_{N\to\infty}\sum_{j=0}^{2^K-1}e^{2\pi il(x+j2^{-K})}d\mu_N(x+j2^{-K})/\sum_{j=0}^{2^K-1}d\mu_N(x+j2^{-K})$$

$$=\lim_{N\to\infty}\sum_{j=0}^{2^K-1}e^{2\pi il(x+j2^{-K})}\left|\sum_{n=0}^{N-1}\omega(n)e^{2\pi in(x+j2^{-K})}\right|^2\Big/\sum_{j=0}^{2^K-1}\left|\sum_{n=0}^{N-1}\omega(n)e^{2\pi in(x+j2^{-K})}\right|^2$$

$$=\lim_{N\to\infty}\sum_{\substack{n,m=0\\ m\equiv n+l \pmod{2^K}}}^{N-1}\omega(n)\omega(m)\Big/\sum_{\substack{n,m=0\\ m\equiv n \pmod{2^K}}}^{N-1}\omega(n)\omega(m).$$

Let $n\equiv m \pmod{2^K}$ and $n=n_1+n_22^K$ and $m=n_1+m_22^K$ with $0\le n_1<2^K$. Then, in the addition $n+l$, the carry goes up to the 2^K term only for a small portion of n_1, say $l/2^K$. In the other case, we have $n+l=n_1+l+n_22^K$ with $0\le n_1+l<2^K$, and hence,

$$\omega(n)\omega(m+l)=\omega(n_1)\omega(n_1+l)\omega(n_2)\omega(m_2).$$

In the other case, we can write

$$\omega(n)\omega(m+l)=\xi\omega(n_1)\omega(n_1+l)\omega(n_2)\omega(m_2)$$

with $\xi\in\{-1,1\}$ depending on n and m. Therefore, we can write

$$\sum_{\substack{n,m=0\\ m\equiv n+l \pmod{2^K}}}^{N-1}\omega(n)\omega(m)=2^K(\gamma(l)+o(1))\sum_{n_2,m_2}\omega(n_2)\omega(m_2)$$

with $o(1)$ which tends to 0 as $K\to\infty$. Therefore,

$$\lim_{N\to\infty}\sum_{\substack{n,m=0\\ m\equiv n+l \pmod{2^K}}}^{N-1}\omega(n)\omega(m)\Big/\sum_{\substack{n,m=0\\ m\equiv n \pmod{2^K}}}^{N-1}\omega(n)\omega(m)$$

$$=\lim_{N\to\infty}\frac{2^K(\gamma(l)+o(1))\sum_{n_2,m_2}\omega(n_2)\omega(m_2)}{2^K\sum_{n_2,m_2}\omega(n_2)\omega(m_2)}=\gamma(l),$$

since $\sum_{n_2,m_2} \omega(n_2)\omega(m_2) = (\sum_{n_2=0}^{2N'} \omega(n_2))^2 = 1$. $\square$

5 Random walk

Let $X_0, X_1, X_2, \cdots$ be the random walk on the open interval $[0,1)$ such that the transition probability satisfies that

$$P(X_{n+1} = y \,|\, X_n = x) = \begin{cases} (1-\cos \pi x)/2 & (y = x/2) \\ (1-\cos \pi(x+1))/2 & (y = (x+1)/2) \\ 0 & (\text{otherwise}) \end{cases} \tag{5.1}$$

for any $n = 0, 1, 2, \cdots$. For $k = 1, 2, \cdots$, denote the k-step transition probability by $P_k(y|x)$. That is,

$$P_k(y|x) = P(X_{n+k} = y \,|\, X_n = x) \quad (n = 0, 1, 2, \cdots).$$

Theorem 5. *The random walk $\{X_0, X_1, X_2, \cdots\}$ has the unique stationary measure μ. Thus, it is mixing.* (See Figure 2.)

Proof Let the distribution of X_0 be μ and the distribution of X_1 be ν. Then, we have

$$d\nu(y) = \begin{cases} ((1-\cos \pi 2y)/2) d\mu(2y) & (y < 1/2) \\ ((1-\cos \pi 2y)/2) d\mu(2y-1) & (y \geq 1/2) \end{cases}.$$

Since

$$d\mu(2y) = d\mu(2y-1) = \prod_{k=0}^{\infty} (1-\cos 2\pi 2^k 2y) d(2y) = 2\prod_{k=1}^{\infty}(1-\cos 2\pi 2^k y)dy,$$

we have

$$d\nu(y) = \prod_{k=0}^{\infty}(1-\cos 2\pi 2^k y)dy = d\mu(y).$$

Hence, μ is a stationary measure of the random walk.

Take an arbitrary $x_0 \in [0,1)$ and consider the random walk $X_0, X_1, X_2, \cdots$ starting at $X_0 = x_0$. We prove that the distribution of X_K, denoted as $\mathcal{L}(X_K|x_0)$ converges weakly to μ as $K \to \infty$. This implies that the random walk is mixing and μ is the unique stationary measure of the random walk.

We prove that

$$\lim_{K\to\infty} E_K[e^{2\pi i l x}] = \int e^{2\pi i l x} d\mu(x) = \gamma(l)$$

holds for any $l \in \mathbb{N}$, where E_K is the expectation with respect to $\mathcal{L}(X_K|x_0)$. Since

$$\mathcal{L}(X_K|x_0) = \sum_{j=0}^{2^K-1} \prod_{k=0}^{K-1} (1-\cos 2\pi 2^k((x_0+j)2^{-K}) \cdot 2^{-K} \delta_{(x_0+j)2^{-K}}$$

and

$$\prod_{k=0}^{K-1}(1-\cos 2\pi 2^k x) = (1/2^K)\left|\sum_{n=0}^{2^K-1}\omega(n)e^{2\pi inx}\right|^2,$$

we have

$$\begin{aligned}
&\lim_{K\to\infty} E_K[e^{2\pi ilx}] = (1/2^{2K})\sum_{j=0}^{2^K-1} e^{2\pi il(x_0+j2^{-K})}\left|\sum_{n=0}^{2^K-1}\omega(n)e^{2\pi in(x_0+j2^{-K})}\right|^2 \\
&= \lim_{K\to\infty}(1/2^K)\sum_{n,m=0}^{2^K-1}\omega(n)\omega(m)\cdot(1/2^K)\sum_{j=0}^{2^K-1} e^{2\pi i(n-m+l)(x_0+j2^{-K})} \\
&= \lim_{K\to\infty}(1/2^K)\sum_{\substack{n,m=0\\ m=n+l}}^{2^K-1}\omega(n)\omega(m) = \gamma(l),
\end{aligned}$$

which completes the proof. □

Lemma 7. *It holds for any $\delta > 0$, $j = 1, 2, \cdots$ and $z \in [0, 1)$ that*

$$\int 1_{|x-y+z|\le j\delta}\ d\mu(x)d\mu(y) \le 6j\int 1_{|x-y|\le\delta}\ d\mu(x)d\mu(y)$$

Proof Take n such that $2^{-n} < \delta \le 2^{-n+1}$. Then, the set $\{(x, y) \in [0, 1) \times [0, 1);\ |x - y + z| \le j\delta\}$ is covered by at most $6j$ number of sets of the following type

$$\bigcup_{k=0}^{2^n-1}[k2^{-n}, (k+1)2^{-n}) \times [(k+h)2^{-n}, (k+h+1)2^{-n}).$$

where $[-1/2, 1/2)$ is identified with $\mathbb{R}/\mathbb{Z}$ and the intervals are considered in the modulo 1 sense. Moreover, since

$$\begin{aligned}
&\sum_{k=0,1,\cdots,2^n-1}\mu([k2^{-n},(k+1)2^{-n}))^2 \\
&\ge \sum_{k=0,1,\cdots,2^n-1}\mu([k2^{-n},(k+1)2^{-n}))\mu([(k+h)2^{-n},(k+h+1)2^{-n})),
\end{aligned}$$

we have

$$\begin{aligned}
&6j\int 1_{|x-y|\le 2^{-n}}\ d\mu(x)d\mu(y) \\
&\ge 6j\sum_{k=0,1,\cdots,2^n-1}\mu([k2^{-n},(k+1)2^{-n}))^2 \\
&\ge \sum_{i=1,\cdots,6j}\ \sum_{k=0,1,\cdots,2^n-1}\mu([k2^{-n},(k+1)2^{-n}))\mu([(k+h_i)2^{-n},(k+h_i+1)2^{-n})) \\
&\ge \int 1_{|x-y+z|\le j\delta}\ d\mu(x)d\mu(y).
\end{aligned}$$

□

Lemma 8. *Let $r < 1$ be sufficiently close to 1. Then, for $\varepsilon = 2^{-n_0}$ such that $2^{-n_0} \le 1 - r < 2 \cdot 2^{-n_0}$, we have*
(1) $(1-r)^{-1} 1_{|x| \le \varepsilon} \le 40 \mathcal{P}_r(x)$ *for any $x \in [-1/2, 1/2)$, and*
(2) $\int \mathcal{P}_r(x-y) d\mu(x) d\mu(y) \le 8(1-r)^{-1} \int 1_{|x-y|<\varepsilon} d\mu(x) d\mu(y)$.

Proof (1) follows from Lemma 5.

For $n = 1, 2, \cdots, n_0$, let $b(n) = \mathcal{P}_r(2^{-n})$. Since

$$\mathcal{P}_r(x) \le 2(1-r)^{-1} \cdot 1_{|x| \le \varepsilon} + \sum_{n=1}^{n_0 - 1} b(n+1) 1_{2^{-n-1} < |x| \le 2^{-n}}$$

and by Lemma 7,

$$\int 1_{2^{-n-1} < |x-y| \le 2^{-n}} d\mu(x) d\mu(y) = 2 \int 1_{|x-y-(3/2)2^{-n-1}| \le 2^{-n-2}} d\mu(x) d\mu(y)$$
$$\le 12 \cdot 2^{n_0 - n - 2} \int 1_{|x-y| \le 2^{-n_0}} d\mu(x) d\mu(y),$$

we have

$$\begin{aligned} & \int \mathcal{P}_r(x-y) d\mu(x) d\mu(y) \\ & \le \left(2(1-r)^{-1} + \sum_{n=1}^{n_0-1} 12 \cdot 2^{n_0-n-2} \right) \int 1_{|x-y| \le \varepsilon} d\mu(x) d\mu(y) \\ & \le (2(1-r)^{-1} + 3 \cdot 2^{n_0}) \int 1_{|x-y| \le \varepsilon} d\mu(x) d\mu(y) \\ & \le 8(1-r)^{-1} \int 1_{|x-y| \le \varepsilon} d\mu(x) d\mu(y), \end{aligned}$$

which completes the proof. □

Theorem 6. *It holds that*

$$\lim_{N \to \infty} \int P_N((x-\varepsilon, x+\varepsilon)|x) d\mu(x) \asymp \varepsilon^{D_2} \quad (\text{as } \varepsilon \to 0),$$

where $P_N(\cdot \mid \cdot)$ is the N-step transition probability of the above random walk.

Proof Since the random walk is mixing, we have

$$\lim_{N \to \infty} \int P_N((x-\varepsilon, x+\varepsilon)|x) d\mu(x) = \int 1_{|x-y|<\varepsilon} d\mu(x) d\mu(y).$$

Hence, by Lemmas 4 and 8 with $\varepsilon \le 1 - r < 2\varepsilon$,

$$\lim_{N \to \infty} \int P_N((x-\varepsilon, x+\varepsilon)|x) d\mu(x) = \int 1_{|x-y|<\varepsilon} d\mu(x) d\mu(y)$$
$$\asymp (1-r) \int \mathcal{P}_r(x-y) d\mu(x) d\mu(y) \asymp (1-r)^{D_2} \asymp \varepsilon^{D_2} \quad (\text{as } \varepsilon \to 0),$$

which completes the proof. □

Figure 1: Approximation of the measure μ as $\prod_{k=0}^{200}(1-\cos(2\pi 2^k x))dx$

Figure 2: Time average of the random walk X_n ($n = 0$ to 10000)

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