DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **38**, Number **10**, October **2018**

doi:10.3934/dcds.2018223

pp. **5085–5103**

CHARACTERIZATION OF NONCORRELATED PATTERN SEQUENCES AND CORRELATION DIMENSIONS

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(Communicated by Xiangdong Ye)

Abstract. We consider the correlation functions of binary pattern sequences of degree 3 as well as those with general degrees and special patterns and obtain necessary and sufficient conditions to be noncorrelated. We also obtain the correlation dimensions for those with degree 2.

1. **Introduction.** An infinite sequence over a finite alphabet can be lifted to a shift dynamical system with a shift invariant measure associated with it. The relative frequency of blocks in the sequence corresponds to this measure and the Fourier analysis on the sequence corresponds to the spectral study of this dynamical system.

The correlation function of a sequence taking values in the unit circle in $\mathbb{C}$ is transformed by the Fourier transform into the power spectral measure of the dynamical system. We call the sequence *noncorrelated* if the correlation function takes value 0 at any nonzero place. In this case, the power spectral measure is the Lebesgue measure on $\mathbb{R}/\mathbb{Z} = [0, 1)$ and the sequence is free of any kind of periodicity.

We study infinite sequences $x = x_0x_1 \cdots \in \{1, -1\}^{\mathbb{N}}$, where x_n is defined by counting patterns in a given set appearing in the binary representation of n in the way that $x_n = 1$ or -1 corresponds to this number being even or odd. We call these sequences *pattern sequences*. For example, the Thue-Morse sequence is the pattern sequence with pattern set $\{1\}$, while the Rudin-Shapiro sequence is with pattern set $\{11\}$. In this paper, we consider mainly the pattern sequences of degree 3, that

2010 *Mathematics Subject Classification.* Primary: 47B15; Secondary: 11B50.
Key words and phrases. Fractal geometry, dynamical system, pattern sequence, correlation measure, correlation dimension.
This work was supported by the NSFC [11571127, 11431007].
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is, having an arbitrary nonempty subset of $\{0,1\}^3 \setminus \{000\}$ as the pattern set, where the pattern 000 is always excluded since it is contained infinitely many times in the binary expansions of nonnegative integers, and give a necessary and sufficient condition to be noncorrelated and have the Lebesgue power spectrum (Theorem 1.5).We also obtained a necessary and sufficient condition to be noncorrelated for the pattern sequences of degree 2 (Theorem 1.4) and of arbitrary degree with special pattern sets (Theorem 1.3).

Even if it is not noncorrelated, if the power spectrum is absolutely continuous, it has the highest nonperiodicity in the sense that the long range correlation tends to 0, while if it has a point spectrum, it admits a kind of periodicity, say an *eigenfunction*. In fact, the *correlation dimension* (Section 3) of the sequence represents the degree of nonperiodicity quantitatively and takes values between 0 and 1. The correlation dimension is 1 if it has an absolutely continuous power spectrum measure with the density function belonging to L^2, and is 0 if it has a point spectrum. For the pattern sequences of degree 2, we obtain the correlation dimension.

General pattern sequences with patterns in G^+ instead of $\{1,-1\}^+$, and d-adic representation instead of binary, where G^+ is the set of nonempty finite sequences of elements in a finite abelian group G, are studied by Morton and Mourant ([7] and [6]). They introduced them as a kind of self-similar and self-generating sequences, mentioning that they can be thought of as arithmetic analogues of fractal. The pattern sequences of degree 3 considered here coincide with their notion of $\Gamma_d(G)$ with *conductor* 4, where $G = \{1,-1\}$ (the multiplicative group) and $d = 2$.

The pattern sequences were also discussed by J. Coquet, T. Kamae and M. Mendès France [3] and David W. Boyd, J. H. Cook and P. Morton [2].

Our fundamental notions and the main results are as follows. For a finite abelian group G, let $G^{\mathbb{N}}$ be the set of concatenation sequences $a = a_0a_1a_2\cdots$ over G. For an integer $d > 1$, denote by $\Gamma_d(G)$ the set of all $a \in G^{\mathbb{N}}$ for which there exists a positive integer M such that

$$(a_n)^{-1}a_{nd+i} = (a_m)^{-1}a_{md+i} \ (i = 0,1,\cdots,d-1) \tag{1}$$

holds for any nonnegative integers n, m with $n \equiv m \pmod M$, where the product here is that of the group G. The modulus M for $a \in \Gamma_d(G)$ is called the *conductor* of a. Without loss of generality, we assume that a_0 is the identity in G.

Definition 1.1. For $n \in \mathbb{N}$, let $(n) = (n)_0(n)_1(n)_2\cdots \in \{0,1\}^{\mathbb{N}}$ be the 2-adic representation of n, that is, $n = \sum_{i=0}^{\infty}(n)_i 2^i$. For a positive integer k, let P be a nonempty subset of $\{0,1\}^k$ with $0^k \notin P$. For a sequence $\xi = \xi_1\xi_2\cdots\xi_l \in \{0,1\}^l$ or $b = b_0b_1b_2\cdots \in \{0,1\}^{\mathbb{N}}$, let $\#(P,\xi)$ or $\#(P,b)$ be the number of P in ξ or b, that is,

$$\#(P,\xi) = \#\{i \in \mathbb{N};\ 1 \le i \le l-k+1;\ \xi_i\xi_{i+1}\cdots\xi_{i+k-1} \in P\}$$
$$\#(P,b) = \#\{i \in \mathbb{N};\ b_ib_{i+1}\cdots b_{i+k-1} \in P\}.$$

We will often denote $\#(P,(n))$ simply by $\#(P,n)$. We call $a \in \{1,-1\}^{\mathbb{N}}$ the *binary pattern sequence* with pattern set P if $a_n = (-1)^{\#(P,n)}$ for any $n \in \mathbb{N}$. Note that we always have $a_0 = 1$ for any binary pattern sequence a. We call k the *degree* of the pattern set P.

Definition 1.2. Let $\xi = \xi_1\xi_2\cdots\xi_k \in \{0,1,?\}^k$ be such that $\xi_1 \in \{0,1\}$, $\xi_k \in \{0,1\}$ and $\xi_i = 1$ for at least one i. Let $l = \#\{i;\ \xi_i \in \{0,1\}\}$. Define a pattern set

$$P_\xi = \left\{ \eta = \eta_1\eta_2\cdots\eta_k \in \{0,1\}^k;\ \eta_i = \begin{cases} \xi_i & (\xi_i \in \{0,1\}) \\ 0 \text{ or } 1 & (\xi_i = ?) \end{cases} \right. \\ \left. \text{for any } i = 1,2,\cdots,k \right\}.$$

We call P_ξ a *special pattern set* of degree k and *bound* l. The binary pattern sequence with pattern set P_ξ is called a *special pattern sequence* with special pattern set P_ξ.

The "if" part of the following theorem was proved by J-P. Allouche and P. Liardet [1].

Theorem 1.3. *A special pattern sequence with a special pattern set of bound l is noncorrelated if and only if $l = 2$.*

Theorem 1.4. *Binary pattern sequences $a \in \{1,-1\}^{\mathbb{N}}$ with pattern sets of degree 2 have the correlation dimension D_2 as follows:*
(1) if $a_1a_2a_3 = -1$, then a is noncorrelated and $D_2 = 1$
(2) if $a_1a_2a_3 = 1$ and $a_2 = -1$, then $D_2 = 3 - \log_2(1+\sqrt{17})$
(3) if $a_1a_2a_3 = 1$ and $a_2 = 1$, then a is a periodic sequence $(1,-1,1,-1,\cdots)$ and $D_2 = 0$.

Theorem 1.5. *Binary pattern sequences $a \in \{1,-1\}^{\mathbb{N}}$ with pattern sets of degree 3 are noncorrelated if and only if one of the following three conditions are satisfied:*
(1) $a_1 = -a_4a_5$ and $a_2a_3 = -a_6a_7$,
(2) $a_1 = a_4a_5 = -1$, $a_2a_3 = a_6a_7 = 1$ and $a_2 = a_5a_6$,
(3) $a_1 = a_4a_5 = 1$, $a_2a_3 = a_6a_7 = -1$ and $a_2 = -a_5a_6$.

We will give the proofs of the above theorems in Section 4, 5 and 6 respectively.

2. **Preliminary lemmas.**

Lemma 2.1. *Let $a \in \Gamma_2(\{1,-1\})$ have conductor $M = 2^h$ (h is a nonnegative integer). Then, a is determined by $a[0,2M) = a_0a_1\cdots a_{2M-1}$. In fact, we have*

$$\begin{cases} a_{2Mm+2i} = a_{Mm+i}a_ia_{2i} \\ a_{2Mm+2i+1} = a_{Mm+i}a_ia_{2i+1} \end{cases} \quad (i = 0,1,\cdots,M-1;\ m = 1,2,\cdots) \tag{2}$$

Conversely, if $a \in \{1,-1\}^{\mathbb{N}}$ is determined by (2) together with arbitrary given $a_0 = 1, a_1, \cdots, a_{2M-1}$, then $a \in \Gamma_2(\{1,-1\})$ with conductor M.

Proof. By (1) with $G = \{1,-1\}$, $d = 2$, $i = 0,1$, $n = 0,1,\cdots,M-1$ and $n+Mm$ for m, we have

$$\begin{cases} a_na_{2n} = a_{Mm+n}a_{2Mm+2n} \\ a_na_{2n+1} = a_{Mm+n}a_{2Mm+2n+1} \end{cases}$$

Hence, we have (2) replacing n by i.

Let the values $a_0, a_1, \cdots, a_{2M-1} \in \{1,-1\}$ be given. We prove that the equation (2) determines all the values $a_n \in \{1,-1\}$ for $n \in \mathbb{N}$. Assume that this value is determined up to 2^kM-1 for some $k = 1,2,\cdots$. Then, the right side of (2) up to $m = 2^k - 1$ is determined. Hence by (2), a_n is determined up to $2^{k+1}M-1$. Thus, (2) determines all the values a_n ($n \in \mathbb{N}$).

We prove that if $a \in \{1,-1\}^{\mathbb{N}}$ satisfies (2), then (1) holds for any $n \equiv m \pmod M$ with $d = 2$. Let n, m be as this. Then, n, m can be written as $n = Mn' + i$ and $m = Mm' + i$ with the same i such that $0 \le i < M$. Hence by (2), we have

$$a_na_{2n} = a_ia_{2i} = a_ma_{2m} \text{ and } a_na_{2n+1} = a_ia_{2i+1} = a_ma_{2m+1},$$

which implies that $a \in \Gamma_2(\{1,-1\})$ with conductor M. □

Lemma 2.2. *Let $a \in \{1,-1\}^{\mathbb{N}}$ be a nonconstant sequence. Then, a is a pattern sequence with a pattern set of degree k if and only if $a \in \Gamma_2(\{1,-1\})$ with conductor $M = 2^{k-1}$.*

Proof. Assume that a is a pattern sequence with a pattern set P of degree k. Note that $P \neq \emptyset$ and $0^k \notin P$. Then, it satisfies that

$$\begin{cases} a_{2n} = a_n(-1)^{1_{0(n)_0(n)_1\cdots(n)_{k-2}\in P}} \\ a_{2n+1} = a_n(-1)^{1_{1(n)_0(n)_1\cdots(n)_{k-2}\in P}} \end{cases} \quad (n \in \mathbb{N}).$$

Let $n \equiv m \pmod{M}$. Then, since $(n)_i = (m)_i$ $(i = 0,1,\cdots,k-2)$, it follows from the above equalities that

$$a_n a_{2n} = a_m a_{2m} \text{ and } a_n a_{2n+1} = a_m a_{2m+1}.$$

Thus, $a \in \Gamma_2(\{1,-1\})$ with conductor $M = 2^{k-1}$.

Conversely, let $a \in \{1,-1\}^{\mathbb{N}}$ be a nonconstant sequence satisfying (2) with $M = 2^{k-1}$. Let

$$\begin{aligned} (\star)\ P = &\{(2i)_0(2i)_1\cdots(2i)_{k-1};\ 0 < i < M \text{ such that } a_i a_{2i} = -1\} \\ &\cup \{(2i+1)_0(2i+1)_1\cdots(2i+1)_{k-1};\ 0 \le i < M \text{ such that } a_i a_{2i+1} = -1\}. \end{aligned}$$

Then, P is nonempty since a is nonconstant. We prove that $a_n = (-1)^{\#(P,n)}$ $(n \in \mathbb{N})$. Since by the above argument, the pattern sequence with a pattern set of degree k satisfies (2), it is sufficient to prove $a_n = (-1)^{\#(P,n)}$ for $n = 0,1,\cdots,M-1$. For $n = 0,1,\cdots,M-1$, let

$$n_0 = n,\ n_1 = \lfloor n/2 \rfloor,\ \cdots,\ n_{k-2} = \lfloor n/2^{k-2} \rfloor,\ n_{k-1} = 0.$$

Then, we have

$$\begin{aligned} a_n =& a_n a_{n_1}^2 \cdots a_{n_{k-2}}^2 a_0 = \prod_{j=0}^{k-2} (a_{n_j} a_{n_{j+1}}) \\ =& \prod_{j=0}^{k-2} (-1)^{1_{(n_j)_0(n_j)_1\cdots(n_j)_{k-1}\in P}} = (-1)^{\#(P,n)}. \end{aligned}$$

Now assume that $a_n = (-1)^{\#(P,n)}$ holds for any $n \le 2^h - 1$ with $h \ge k-1$. Take any n with $2^h \le n \le 2^{h+1} - 1$. Let $\tilde{n} = \lfloor n/2 \rfloor$. Then, we have

$$(-1)^{\#(P,n)} = (-1)^{\#(P,\tilde{n})}(-1)^{1_{(n)_0(n)_1\cdots(n)_{k-1}\in P}}.$$

Let $i = (\tilde{n})_0 + (\tilde{n})_1 2 + \cdots + (\tilde{n})_{k-2} 2^{k-2}$. Then,

$$j := (n)_0 + (n)_1 2 + \cdots + (n)_{k-1} 2^{k-1} = \begin{cases} 2i & (n : \text{even}) \\ 2i+1 & (n : \text{odd}) \end{cases}$$

By the definition of P, $(n)_0(n)_1\cdots(n)_{k-1} \in P$ if and only if $a_i a_j = -1$. Moreover by (2) with $m = \frac{\tilde{n}-i}{M}$, we have $a_n = a_{\tilde{n}} a_i a_j$. Hence,

$$(-1)^{\#(P,n)} = (-1)^{\#(P,\tilde{n})} a_i a_j = (-1)^{\#(P,\tilde{n})} a_{\tilde{n}} a_n.$$

Since $\tilde{n} \le 2^h - 1$, we have $a_{\tilde{n}} = (-1)^{\#(P,\tilde{n})}$ by the assumption, which implies that $a_n = (-1)^{\#(P,n)}$. Thus by the induction, we complete the proof that a is a pattern sequence. □

3. **Correlation dimension.** Let $a \in \{1,-1\}^{\mathbb{N}}$ be a pattern sequence with a pattern set P of degree k. By Lemma 2.2, a is in $\Gamma_2(\{1,-1\})$ with conductor 2^{k-1}. The following *correlation function* is proved to exist.

$$\gamma_a(r) = \lim_{N\to\infty} (1/N) \sum_{0\le n<N} a_n a_{n+r} \ (r \in \mathbb{N}). \tag{3}$$

Let $X = X_0X_1X_2\cdots$ be a sequence of i.i.d. random variables on $\{0,1\}$ such that $\mathbb{P}(X_0 = 0) = \mathbb{P}(X_0 = 1) = 1/2$. We consider X a random variable on $\mathbb{Z}_2$ (the group of 2-adic integers) so that the addition $X + r \in \mathbb{Z}_2$ is defined for $r \in \mathbb{N}$, that is,

$$\sum_{0\le i<N} (X+r)_i 2^i \equiv \sum_{0\le i<N} X_i 2^i + \sum_{0\le i<N} (r)_i 2^i \pmod{2^N}$$

for any N. We can define $\#(P, X + r) - \#(P, X)$ almost surely as

$$\#(P,X+r) - \#(P,X) = \lim_{N\to\infty} (\#(P,(X+r)[0,N)) - \#(P,X[0,N))).$$

Let $S_r^L = \{X \in \mathbb{Z}_2;\ (X)_i = (X + r)_i$ holds for any $i \ge L\}$. If $X \in S_r^L$, then $(-1)^{\#(P,X+r)-\#(P,X)}$ is determined by $X_0, X_1, \cdots, X_{L+k-1}$. Also, if $n \in S_r^L$, then $a_n a_{n+r}$ is determined by $(n)_0, (n)_1, \cdots (n)_{L+k-1}$, and hence, is a periodic function of n along these suffixes. This implies that

$$\lim_{N\to\infty} (1/N) \sum_{\substack{0\le n<N \\ n\in S_r^L}} a_n a_{n+r} = \mathbb{E}[(-1)^{\#(P,X+r)-\#(P,X)} 1_{X\in S_r^L}],$$

where $\mathbb{E}$ is the expectation of random variables. Since both the density of $\{n \in \mathbb{N};\ n \in S_r^L\}$ and the probability $\mathbb{P}(S_r^L)$ tend to 1 as $L \to \infty$, it is known that $\lim_{N\to\infty}(1/N)\sum_{0\le n<N} a_n a_{n+r}$ exists and is equal to $\mathbb{E}[(-1)^{\#(P,X+r)-\#(P,X)}]$.

It determines the *power spectral measure* μ_a on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ so that

$$\int e^{i2\pi rx} d\mu_a(x) = \gamma_a(r) \ (\forall r \in \mathbb{N}). \tag{4}$$

Definition 3.1. We call μ_a the *spectral measure* of the sequence a. We say that a has a *Lebesgue spectrum* if μ_a is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}$. In particular, we say a is *noncorrelated* if $\gamma_a(r) = 0$ for any $r \ne 0$. In this case, μ_a is the Lebesque measure itself. We say that a has a *point spectrum* if μ_a has a discrete part. We say that a has a *singular spectrum* if μ_a has neither a absolutely continuous part nor a discrete part.

It is known (K. Petersen [10], for example) that μ_a is absolutely continuous if $\sum_{r\in\mathbb{N}} |\gamma_a(r)|^2 < \infty$ and μ_a has a point spectrum if and only if

$$\lim_{T\to\infty} (1/T) \sum_{0\le r<T} |\gamma_a(r)|^2 > 0.$$

Moreover, if μ_a is absolutely continuous, then $\lim_{r\to\infty} \gamma_a(r) = 0$.

For a sequence $a \in \{1,-1\}^{\mathbb{N}}$, if there exists D_2 such that

$$(1/T) \sum_{0\le r<T} |\gamma_a(r)|^2 \asymp T^{-D_2} \ (\text{as } T \to \infty),$$

then D_2 is called the *correlation dimension* of a, where $A \asymp B$ as $T \to \infty$ means

$$0 < \liminf_{T\to\infty} \frac{A}{B} \le \limsup_{T\to\infty} \frac{A}{B} < \infty.$$

Godreche and Luck [5] introduce the notion of correlation dimension. M. A. Zaks, A. S. Pikovsky and J. Kurths [12] calculate the exact value of correlation dimension of the Thue-Morse sequence as $D_2 = 3 - \log_2(1+\sqrt{17})$ as well as that of m-tuplings sequences with $m = 3$ or 4, as $3 - \log_3 11$ or $(5/2) - \log_4(3+\sqrt{41})$, respectively. M. Niu and Z. X. Wen [8] get the values for general m-tuplings sequences. The correlation dimension of the Thue-Morse sequence is given another meaning by L. Peng and T. Kamae [9] as the decay rate of the return probability of the random walk related to the spectral measure. We know that the spectral measure of Rudin-Shapiro sequence is absolutely continuous and that of Fibonacci sequence is discrete by M. Queffélec [11] and N. P. Fogg [4]. So the correlation dimensions of them are 1 and 0 respectively.

In the following sections, we will denote the correlation function γ_a of the sequence $a \in \{1,-1\}^{\mathbb{N}}$ simply by γ.

4. **Proof of Theorem 1.3.** Let P_ξ be a special pattern set of degree k and bound l. Let $a \in \{1,-1\}^{\mathbb{N}}$ be the special pattern sequence with spacial pattern set P_ξ. We fix this ξ, k, l and a throughout this section.

Case 1. $l = 2$. In this case, $\xi = \xi_1\xi_2\cdots\xi_k \in \{0,1,?\}^k$ satisfies that $\xi_1 \in \{0,1\}$, $\xi_k \in \{0,1\}$, $1 \in \{\xi_1,\xi_k\}$ and $\xi_i = ?$ $(i = 2,\cdots,k-1)$.

Definition 4.1. For $n \in \mathbb{N}$ and a positive integer r, let i_0 be the maximum i such that $(n+r)_i \neq (n)_i$. Define $n^{\sim r}$ to be $m \in \mathbb{N}$ such that $(n)_{i_0+k-1} \neq (m)_{i_0+k-1}$ and $(n)_i = (m)_i$ $(\forall i \neq i_0+k-1)$.

Lemma 4.2. *For any $n \in \mathbb{N}$ and a positive integer r, we have*
(1) $(n^{\sim r})^{\sim r} = n$
(2) $a(n)a(n+r)a(n^{\sim r})a(n^{\sim r}+r) = -1$
(3) $a(n)a(n+r) + a(n^{\sim r})a(n^{\sim r}+r) = 0$,
where we denote a_i as $a(i)$.

Proof. (1) is clear from the definition.

To prove (2), it is sufficient to prove that the sum of the numbers of appearances of P_ξ in the binary representations of $n, n+r, n^{\sim r}$ and $n^{\sim r}+r$ is odd. Let $m = n^{\sim r}$ and $i_0 = \max\{i;\ (n+r)_i \neq (n)_i\}$. For $l \in \mathbb{N}$ and $p < q$, denote

$$(l)[p,q) = (l)_p(l)_{p+1}\cdots(l)_{q-1} \in \{0,1\}^{q-p}$$
$$(l)[p,\infty) = (l)_p(l)_{p+1}\cdots \in \{0,1\}^{\mathbb{N}}.$$

Then, since

$$(n)[0,i_0+k-1) = (m)[0,i_0+k-1)$$
$$(n+r)[0,i_0+k-1) = (m+r)[0,i_0+k-1)$$
$$(n)[i_0+1,\infty) = (n+r)[i_0+1,\infty)$$
$$(m)[i_0+1,\infty) = (m+r)[i_0+1,\infty),$$

we have

$$\begin{aligned}
&\#(P_\xi,n) + \#(P_\xi,n+r) + \#(P_\xi,m) + \#(P_\xi,m+r)\\
=&2\#(P,(n)[0,i_0+k-1)) + 2\#(P,(n+r)[0,i_0+k-1))\\
&+ 2\#(P,(n)[i_0+1,\infty)) + 2\#(P,(m)[i_0+1,\infty))\\
&+ 1_{(n)[i_0,i_0+k)\in P_\xi} + 1_{(n+r)[i_0,i_0+k)\in P_\xi} + 1_{(m)[i_0,i_0+k)\in P_\xi} + 1_{(m+r)[i_0,i_0+k)\in P_\xi}.
\end{aligned}$$

Note that the totality of

$$(n)_{i_0}(n)_{i_0+k-1},\ (n+r)_{i_0}(n+r)_{i_0+k-1},\ (m)_{i_0}(m)_{i_0+k-1},\ (m+r)_{i_0}(m+r)_{i_0+k-1}$$

are $00, 01, 10, 11$, and hence, exactly one of them coincides with $\xi_1\xi_k$. Therefore,

$$1_{(n)[i_0,i_0+k)\in P_\xi} + 1_{(n+r)[i_0,i_0+k)\in P_\xi} + 1_{(m)[i_0,i_0+k)\in P_\xi} + 1_{(m+r)[i_0,i_0+k)\in P_\xi} = 1,$$

and hence,

$$\#(P_\xi, n) + \#(P_\xi, n+r) + \#(P_\xi, m) + \#(P_\xi, m+r)$$

is odd, which implies (2).

(3) follows from (2). $\square$

Corollary 1. *Let γ be the correlation function of a. Then, we have $\gamma(r) = 0$ for any $r > 0$. Thus, a is noncorrelated.*

Proof. $\gamma(0) = 1$ is clear. Assume that $r > 0$. Note that for $n \in [0, N)$, if there exists i with $r \le 2^i \cancel{\le N - 2^k}$ such that $(n)_i = 0$, then $n^{\sim r} \in [0, N)$ holds. Therefore, at most $\cancel{4rN/(N-2^k)}$ number of $n \in [0, N)$ do not satisfy $n^{\sim r} \in [0, N)$. Hence, by (3)

$$\left|\sum_{0\le n<N} a(n)a(n+r)\right| \le \left|\sum_{\substack{0\le n<N;\\ 0\le n^{\sim r}<N}} a(n)a(n+r)\right| + \cancel{(4rN)/(N-2^k)}$$

$$= \left|(1/2)\sum_{\substack{0\le n<N;\\ 0\le n^{\sim r}<N}} (a(n)a(n+r) + a(n^{\sim r})a(n^{\sim r}+r))\right| + \cancel{(4rN)/(N-2^k)}$$

$$\cancel{= (4rN)/(N-2^k)}.$$

Thus, $\gamma(r) = \lim_{N\to\infty}(1/N)\sum_{0\le n<N} a(n)a(n+r) = 0$. $\square$

Case 2. $l = 1$. In this case, $\xi = 1$ and a is the Thue-Morse sequence. Hence, the correlation dimension is $3 - \log_2(+\sqrt{17})$ ([12]) and μ_a is a singular measure.

Case 3. $l \ge 3$. In this case, $\xi = \xi_1\xi_2\cdots\xi_k \in \{0, 1, ?\}^k$ satisfies that $\xi_1 \in \{0, 1\}$, $\xi_k \in \{0, 1\}$, $1 \in \{\xi_1, \xi_k\}$ and $\xi_i \in \{0, 1\}$ for at least one $i = 2, \cdots, k-1$.

Let $X = X_0X_1X_2\cdots$ be a sequence of i.i.d. random variables on $\{0, 1\}$ such that $\mathbb{P}(X_0 = 0) = \mathbb{P}(X_0 = 1) = 1/2$. We consider X a random variable on $\mathbb{Z}_2$ so that the addition $X + r \in \mathbb{Z}_2$ is defined for $r \in \mathbb{N}$, that is,

$$\sum_{0\le i<N} (X+r)_i 2^i \equiv \sum_{0\le i<N} X_i 2^i + \sum_{0\le i<N} (r)_i 2^i \pmod{2^N}$$

for any N. We can define $\#(P, X+r) - \#(P, X)$ almost surely as

$$\#(P, X+r) - \#(P, X) = \lim_{N\to\infty}(\#(P, (X+r)[0, N)) - \#(P, X[0, N))).$$

Using the notation, we have

$$\gamma(r) = \mathbb{E}\left[(-1)^{\#(P,X+r)-\#(P,X)}\right],$$

where $\mathbb{E}$ is the expectation of random variables. Then, it holds that

$$\begin{aligned}\gamma(1) =&\mathbb{E}\left[(-1)^{\#(P,X+1)-\#(P,X)}\right]\\ =&(1/2)\mathbb{E}\left[(-1)^{\#(P,X+1)-\#(P,X)}|X_0=0\right]\\ &+(1/2)\mathbb{E}\left[(-1)^{\#(P,X+1)-\#(P,X)}|X_0=1\right]\\ =:&(1/2)A+(1/2)B.\end{aligned}$$

In the case $X_0=0$, $(-1)^{\#(P,X+1)-\#(P,X)}=-1$ holds if and only if

$$\xi_1X_1X_2\cdots X_{k-1}\in P_\xi,$$

and hence,

$$\begin{aligned}A=&1-\mathbb{P}(\xi_1X_1X_2\cdots X_{k-1}\in P_\xi)-\mathbb{P}(\xi_1X_1X_2\cdots X_{k-1}\in P_\xi)\\ =&1-2\cdot 2^{-(l-1)}=1-2^{2-l}.\end{aligned}$$

Let T be the shift on $\mathbb{Z}_2=\{0,1\}^{\mathbb{N}}$. Then, we have in the case $X_0=1$ that

$$(-1)^{\#(P,X+1)-\#(P,X)}=(-1)^{\#(P,TX+1)-\#(P,TX)}C$$

with $C\in\{1,-1\}$ and $C=-1$ if and only if either

$$1X_1X_2\cdots X_k\in P_\xi \quad\text{or}\quad 0(X+1)_1(X+1)_2\cdots(X+1)_{k-1}\in P_\xi.$$

Since $\mathbb{P}(C=-1)=2^{-(l-1)}$, we have

$$\begin{aligned}B=&\mathbb{E}\left[(-1)^{\#(P,TX+1)-\#(P,TX)}C\right]\\ =&\mathbb{E}\left[(-1)^{\#(P,TX+1)-\#(P,TX)}\right]+\mathbb{E}\left[(-1)^{\#(P,TX+1)-\#(P,TX)}(C-1)\right]\\ =&\gamma(1)+D\end{aligned}$$

with

$$|D|<\mathbb{E}[|C-1|]=2\cdot 2^{-(l-1)}=2^{2-l},$$

where the strict inequality "$<$" holds since

$$0<\mathbb{P}(\#(P,TX+1)-\#(P,TX)\text{ is even}|C=-1)<1.$$

Therefore,

$$\begin{aligned}\gamma(1)=&(1/2)(1-2^{2-l})+(1/2)(\gamma(1)+D)\\ >&(1/2)(1-2^{2-l})+(1/2)(\gamma(1)-2^{2-l})\geq(1/2)\gamma(1),\end{aligned}$$

and hence, $\gamma(1)>0$.

5. **Proof of Theorem 1.4.** In this section we consider binary pattern sequences $a=(a_n)_{n\geq0}\in\{1,-1\}^{\mathbb{N}}$ with pattern sets of degree 2. That is, by Lemma 2.2 and (1),

$$a_na_{2n+i}=a_ma_{2m+i}\ (i=0,1) \tag{5}$$

holds for any nonnegative integers n,m with $n\equiv m\pmod 2$. Hence we have

$$\begin{aligned}a_{4n}&=a_{2n}\ \text{(by putting } 2n,0,0 \text{ for } n,m,i)\\ a_{4n+1}&=a_1a_{2n}\ \text{(by putting } 2n,0,1 \text{ for } n,m,i)\\ a_{4n+2}&=a_1a_2a_{2n+1}\ \text{(by putting } 2n+1,1,0 \text{ for } n,m,i)\\ a_{4n+3}&=a_1a_3a_{2n+1}\ \text{(by putting } 2n+1,1,1 \text{ for } n,m,i).\end{aligned} \tag{6}$$

That is, the binary pattern sequences of degree 2 are decided by their first 3 terms a_1, a_2, a_3 (recall that $a_0 = 1$) completely.

Denote

$$\gamma_e(h) = \lim_{N\to\infty}(2/N)\sum_{2n<N} a_{2n+h}a_{2n},$$
$$\gamma_o(h) = \lim_{N\to\infty}(2/N)\sum_{2n+1<N} a_{2n+1+h}a_{2n+1}.$$

These limits exist as is shown in Section 3.

By (6), we have for any N which is a multiple of 4,

$$\begin{aligned}
\sum_{n<N/2} a_{2n}a_{2n+4k} &= \sum_{n<N/4} a_{4n}a_{4n+4k} + \sum_{n<N/4} a_{4n+2}a_{4n+2+4k}\\
&= \sum_{n<N/4} a_{2n}a_{2n+2k} + \sum_{n<N/4} (a_2a_1)^2 a_{2n+1}a_{2n+1+2k}\\
&= \sum_{n<N/4} a_{2n}a_{2n+2k} + \sum_{n<N/4} a_{2n+1}a_{2n+1+2k}\\
\sum_{n<N/2} a_{2n+1}a_{2n+1+4k} &= \sum_{n<N/4} a_{4n+1}a_{4n+4k+1} + \sum_{n<N/4} a_{4n+3}a_{4n+3+4k}\\
&= \sum_{n<N/4} (a_0a_1)^2 a_{2n}a_{2n+2k} + \sum_{n<N/4} (a_3a_1)^2 a_{2n+1}a_{2n+1+2k}\\
&= \sum_{n<N/4} a_{2n}a_{2n+2k} + \sum_{n<N/4} a_{2n+1}a_{2n+1+2k}.
\end{aligned}$$

Dividing the above equalities by $N/2$ and taking the limits $N \to \infty$, we get

$$\gamma_e(4k) = (1/2)[\gamma_e(2k) + \gamma_o(2k)] = \gamma(2k)$$
$$\gamma_o(4k) = (1/2)[\gamma_e(2k) + \gamma_o(2k)] = \gamma(2k),$$

and hence, $\gamma(4k) = \gamma(2k)$ $(k = 1, 2, \cdots)$.

In the same way, we have

$$\begin{aligned}
(2/N)\sum_{n<N/2} a_{2n}a_{2n+4k+1} =&(2/N)\sum_{n<N/4} a_1a_{2n}a_{2n+2k}\\
&+ (2/N)\sum_{n<N/4} (a_2a_3)a_{2n+1}a_{2n+2k+1}
\end{aligned}$$

$$\begin{aligned}
(2/N)\sum_{n<N/2} a_{2n+1}a_{2n+4k+2} =&(2/N)\sum_{n<N/4} a_2a_{2n}a_{2n+2k+1}\\
&+ (2/N)\sum_{n<N/4} (a_1a_3)a_{2n+1}a_{2n+2k+2}
\end{aligned}$$

$$\begin{aligned}
(2/N)\sum_{n<N/2} a_{2n}a_{2n+4k+2} =&(2/N)(a_1a_2)[\sum_{n<N/4} a_{2n}a_{2n+2k+1}\\
&+ \sum_{n<N/4} a_{2n+1}a_{2n+2k+2}]
\end{aligned}$$

$$\begin{aligned}(2/N)\sum_{n<N/2} a_{2n+1}a_{2n+4k+3} =&(2/N)\ a_3\ [\sum_{n<N/4} a_{2n}a_{2n+2k+1}\\ &+\sum_{n<N/4} a_{2n+1}a_{2n+2k+2}]\\ (2/N)\sum_{n<N/2} a_{2n}a_{2n+4k+3} =&(2/N)[(a_1a_3)\sum_{n<N/4} a_{2n}a_{2n+2k+1}\\ &+a_2\sum_{n<N/4} a_{2n+1}a_{2n+2k+2}]\\ (2/N)\sum_{n<N/2} a_{2n+1}a_{2n+4k+4} =&(2/N)[\ a_1\sum_{n<N/4} a_{2n}a_{2n+2k+2}\\ &+(a_2a_3)\sum_{n<N/4} a_{2n+1}a_{2n+2k+3}].\end{aligned}$$

Letting $N\to\infty$, we get

$$\begin{aligned}\gamma_e(4k+1)&=(1/2)[a_1\gamma_e(2k)+a_2a_3\gamma_o(2k)],\\ \gamma_o(4k+1)&=(1/2)[a_2\gamma_e(2k+1)+a_1a_3\gamma_o(2k+1)],\\ \gamma_e(4k+2)&=(1/2)a_1a_2[\gamma_e(2k+1)+\gamma_o(2k+1)],\\ \gamma_o(4k+2)&=(1/2)a_3[\gamma_e(2k+1)+\gamma_o(2k+1)],\\ \gamma_e(4k+3)&=(1/2)[a_1a_3\gamma_e(2k+1)+a_2\gamma_o(2k+1)],\\ \gamma_o(4k+3)&=(1/2)[a_1\gamma_e(2k+2)+a_2a_3\gamma_o(2k+2)].\end{aligned}$$

If $a_1a_2a_3=-1$, then by the above equalities, we have

$$\begin{aligned}\gamma_e(4k+1)=\gamma_o(4k+1)&=0\\ \gamma_e(4k+2)+\gamma_o(4k+2)&=0\\ \gamma_e(4k+3)=\gamma_o(4k+3)&=0,\end{aligned}$$

and hence, $\gamma(4k+1)=\gamma(4k+2)=\gamma(4k+3)=0$ for any $k=0,1,2,\cdots$. Together with $\gamma(4k)=\gamma(2k)$ $(k=1,2,\cdots)$, it follows that $\gamma(k)=0$ whenever $k\neq 0$. Since $\gamma(0)=1$, a has the Lebesgue spectrum and $D_2=1$. Thus, we have (1) of Theorem 1.4.

Consider the case where $a_1a_2a_3=1$ and $a_2=1$. Since $a_1=a_2=a_3=1$ implies that a is a constant sequence $(1,1,1,1,\cdots)$ and is not a pattern sequence having the empty pattern set, we should have $a_1=a_3=-1$ and $a_2=1$. Then, by $(\star)$, the pattern set of a is $\{10,01\}$. It follows from this that a is a periodic sequence $(1,-1,1,-1,\cdots)$ and $D_2=0$. Thus, (3) of Theorem 1.4 follows.

Consider the case where $a_1a_2a_3=1$ and $a_2=-1$. We have 2 cases: either $a_1=a_2=-1$ and $a_3=1$ or $a_1=1$ and $a_2=a_3=-1$. It follows from the above equalities that

$$\begin{aligned}\gamma(4k)&=\gamma(2k),\\ \gamma(4k+1)&=(1/2)a_1\gamma(2k)-(1/2)\gamma(2k+1),\\ \gamma(4k+2)&=-a_1\gamma(2k+1),\\ \gamma(4k+3)&=-(1/2)\gamma(2k+1)+(1/2)a_1\gamma(2k+2).\end{aligned}\tag{7}$$

It follows from (7) with $k=0,1,2$ that

$$\gamma(4)=-1/3,\ \gamma(5)=0,\ \gamma(6)=1/3,\ \gamma(7)=0,\ \gamma(8)=-1/3\tag{8}$$

in both subcases.

Denoting $S_n = \sum_{t=n+1}^{2n} \gamma^2(t)$, $\pi_n = \sum_{t=n}^{2n-1} \gamma(t)\gamma(t+1)$, and taking $n = 8m$, we can calculate using the above results that

$$\begin{aligned}
S_{8m} =& \sum_{t=8m+1}^{16m} \gamma^2(t) \\
=& \sum_{t=2m+1}^{4m} \gamma^2(4t) + \sum_{t=2m}^{4m-1} \gamma^2(4t+1) + \sum_{t=2m}^{4m-1} \gamma^2(4t+2) + \sum_{t=2m}^{4m-1} \gamma^2(4t+3) \\
=&(3/2) \sum_{t=2m+1}^{4m} \gamma^2(2t) + (3/2) \sum_{t=2m}^{4m-1} \gamma^2(2t+1) \\
&- (1/2)a_1 \sum_{t=2m}^{4m-1} [\gamma(2t) + \gamma(2t+2)]\gamma(2t+1) \\
=&(3/2) \sum_{t=4m+1}^{8m} \gamma^2(t) - (1/2)a_1 \sum_{t=4m}^{8m-1} \gamma(t)\gamma(t+1) \\
=&(3/2)S_{4m} - (1/2)a_1\pi_{4m}
\end{aligned}$$

$$\begin{aligned}
\pi_{8m} =& \sum_{t=8m}^{16m-1} \gamma(t)\gamma(t+1) \\
=& \sum_{t=2m}^{4m-1} [\gamma(4t)\gamma(4t+1) + \gamma(4t+1)\gamma(4t+2) \\
&+ \gamma(4t+2)\gamma(4t+3) + \gamma(4t+3)\gamma(4t+4)] \\
=& \sum_{t=2m}^{4m-1} [(1/2)a_1\gamma^2(2t) - (1/2)\gamma(2t)\gamma(2t+1) - (1/2)\gamma(2t)\gamma(2t+1) \\
&+ (1/2)a_1\gamma^2(2t+1) - (1/2)\gamma(2t+1)\gamma(2t+2) + (1/2)a_1\gamma^2(2t+1) \\
&+ (1/2)a_1\gamma^2(2t+2) - (1/2)\gamma(2t+1)\gamma(2t+2)] \\
=&(1/2)a_1 \sum_{t=2m}^{4m-1} \gamma^2(2t) - \sum_{t=2m}^{4m-1} [\gamma(2t) + \gamma(2t+2)]\gamma(2t+1) \\
&+ (1/2)a_1 \sum_{t=2m}^{4m-1} \gamma^2(2t+2) + a_1 \sum_{t=2m}^{4m-1} \gamma^2(2t+1) \\
=&a_1 \sum_{t=2m+1}^{4m} \gamma^2(2t) + a_1 \sum_{t=2m}^{4m-1} \gamma^2(2t+1) - \sum_{t=2m}^{4m-1} [\gamma(2t) + \gamma(2t+2)]\gamma(2t+1) \\
=&a_1 \sum_{t=4m+1}^{8m} \gamma^2(t) - \sum_{t=4m}^{8m-1} \gamma(t)\gamma(t+1) \\
=&a_1 S_{4m} - \pi_{4m}
\end{aligned}$$

Hence, we have

$$S_{8m} = (3/2)S_{4m} - (1/2)a_1\pi_{4m}\ ,\ \pi_{8m} = a_1 S_{4m} - \pi_{4m},$$

that is,

$$\begin{pmatrix} S_{8m} \\ \pi_{8m} \end{pmatrix} = M \begin{pmatrix} S_{4m} \\ \pi_{4m} \end{pmatrix}.$$

with

$$M = \begin{pmatrix} 3/2 & -(1/2)a_1 \\ a_1 & -1 \end{pmatrix}.$$

If $a_1a_2a_3 = 1$ and $a_2 = -1$, then by (8),

$$\begin{pmatrix} S_4 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} 2/9 \\ 0 \end{pmatrix}.$$

The eigenvalues of M are $(1 \pm \sqrt{17})/4$, and the vector $\begin{pmatrix} S_4 \\ \pi_4 \end{pmatrix}$ has a nonzero component of the eigenvector corresponding to the positive eigenvalue. Hence it behaves asymptotically as the positive eigenvector. Hence, it holds that

$$S_4 + S_8 + \cdots + S_{2^m} = \sum_{5 \le r < 2^{m+1}} |\gamma(r)|^2 \asymp \left((1+\sqrt{17})/4\right)^m \text{ (as } m \to \infty).$$

Therefore,

$$(1/T) \sum_{0 \le r < T} |\gamma(r)|^2 \asymp T^{-D_2} \text{ (as } T \to \infty)$$

with $D_2 = 3 - \log_2(1+\sqrt{17})$ which proves (2) of Theorem 1.4.

Thus, the binary pattern sequences of degree 2 can be summarized as Table 1:

Table 1.

	(a_1, a_2, a_3)	pattern set	spectrum
$a_1a_2a_3 = -1$	$(-1, 1, 1)$	$\{01, 10, 11\}$	noncorrelated sequence $D_2 = 1$
	$(1, -1, 1)$	$\{01\}$	
	$(1, 1, -1)$	$\{11\}$	
	$(-1, -1, -1)$	$\{10\}$	
$a_1a_2a_3 = 1$ $a_2 = -1$	$(1, -1, -1)$	$\{01, 11\}$	singular spectrum $D_2 = 3 - \log_2(1+\sqrt{17})$
	$(-1, -1, 1)$	$\{10, 11\}$	
$a_1a_2a_3 = 1$ $a_2 = 1$	$(-1, 1, -1)$	$\{01, 10\}$	periodic sequence $D_2 = 0$

6. **Proof of Theorem 1.5.** Let $a \in \{1, -1\}^{\mathbb{N}}$ be the binary pattern sequence with pattern sets of degree 3. By Lemma 2.2, $a \in \Gamma_2(\{1, -1\})$ with conductor $2^2 = 4$. By (1) we have

$$a_n a_{2n+i} = a_m a_{2m+i} \ (i = 0, 1) \tag{9}$$

holds for any nonnegative integers n, m with $n \equiv m \pmod 4$. Taking $4n + j$, j, i ($j = 0, 1, 2, 3$; $i = 0, 1$) for n, m, i in (9), we have the following formulas.

$$a_{8n} = a_{4n},$$
$$a_{8n+1} = a_1 \ a_{4n},$$
$$a_{8n+2} = a_1a_2 \ a_{4n+1},$$
$$a_{8n+3} = a_1a_3 \ a_{4n+1},$$

$$
\begin{aligned}
a_{8n+4} &= a_2a_4\ a_{4n+2},\\
a_{8n+5} &= a_2a_5\ a_{4n+2},\\
a_{8n+6} &= a_3a_6\ a_{4n+3},\\
a_{8n+7} &= a_3a_7\ a_{4n+3}.
\end{aligned} \tag{10}
$$

Hence, the sequence a is completely determined by its first seven terms . That is, a binary pattern sequence is identified with $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$.

Note that

$$(1/N)\sum_{n<N} a_n a_{n+h} = (1/N)\sum_{i=0}^{3}\sum_{4n+i<N} a_{4n+i}a_{4n+i+h}.$$

We will consider the following four limits:

$$
\begin{aligned}
\gamma_0(h) &= \lim_{N\to\infty}(4/N)\sum_{4n<N} a_{4n}a_{4n+h},\\
\gamma_1(h) &= \lim_{N\to\infty}(4/N)\sum_{4n+1<N} a_{4n+1}a_{4n+1+h},\\
\gamma_2(h) &= \lim_{N\to\infty}(4/N)\sum_{4n+2<N} a_{4n+2}a_{4n+2+h},\\
\gamma_3(h) &= \lim_{N\to\infty}(4/N)\sum_{4n+3<N} a_{4n+3}a_{4n+3+h}.
\end{aligned}
$$

Considering $h = 8k$ ($k \in \mathbb{N}$), we obtain (N being a multiple of 8)

$$
\begin{aligned}
(4/N)\sum_{n<N/4} a_{4n}a_{4n+8k} &= (4/N)\Big(\sum_{n<N/8} a_{8n}a_{8n+8k} + \sum_{n<N/8} a_{8n+4}a_{8n+8k+4}\Big)\\
&= (4/N)\Big(\sum_{n<N/8} a_{4n}a_{4n+4k} + \sum_{n<N/8} a_{4n+2}a_{4n+4k+2}\Big),\\
(4/N)\sum_{n<N/4} a_{4n+1}a_{4n+8k+1} &= (4/N)\Big(\sum_{n<N/8} a_{8n+1}a_{8n+8k+1} + \sum_{n<N/8} a_{8n+5}a_{8n+8k+5}\Big)\\
&= (4/N)\Big(\sum_{n<N/8} a_{4n}a_{4n+4k} + \sum_{n<N/8} a_{4n+2}a_{4n+4k+2}\Big),\\
(4/N)\sum_{n<N/4} a_{4n+2}a_{4n+8k+2} &= (4/N)\Big(\sum_{n<N/8} a_{8n+2}a_{8n+8k+2} + \sum_{n<N/8} a_{8n+6}a_{8n+8k+6}\Big)\\
&= (4/N)\Big(\sum_{n<N/8} a_{4n+1}a_{4n+4k+1} + \sum_{n<N/8} a_{4n+3}a_{4n+4k+3}\Big),\\
(4/N)\sum_{n<N/4} a_{4n+3}a_{4n+8k+3} &= (4/N)\Big(\sum_{n<N/8} a_{8n+3}a_{8n+8k+3} + \sum_{n<N/8} a_{8n+7}a_{8n+8k+7}\Big)\\
&= (4/N)\Big(\sum_{n<N/8} a_{4n+1}a_{4n+4k+1} + \sum_{n<N/8} a_{4n+3}a_{4n+4k+3}\Big).
\end{aligned}
$$

Letting $N \to \infty$, we get the following recursive formulas:

$$\gamma_0(8k) = (1/2)\ [\gamma_0(4k) +\ \gamma_2(4k)], \tag{11}$$

$$\gamma_1(8k) = (1/2)\ [\gamma_0(4k) +\ \gamma_2(4k)], \tag{12}$$

$$\gamma_2(8k) = (1/2)\ [\gamma_1(4k) +\ \gamma_3(4k)], \tag{13}$$

$$\gamma_3(8k) = (1/2)\ [\gamma_1(4k) +\ \gamma_3(4k)]. \tag{14}$$

The formulas for $\gamma_i(h)$ $(i \in \{0,1,2,3\})$ are completed by putting $h = 8k + j$ $(j \in \{1,2,3,4,5,6,7\})$ and using the same argument as above. That is,

$$\gamma_0(8k+1) = (1/2)\ [a_1\ \gamma_0(4k) + a_4a_5\ \gamma_2(4k)], \tag{15}$$
$$\gamma_1(8k+1) = (1/2)\ [a_2\ \gamma_0(4k+1) + a_2a_3a_5a_6\ \gamma_2(4k+1)], \tag{16}$$
$$\gamma_2(8k+1) = (1/2)\ [a_2a_3\ \gamma_1(4k) + a_6a_7\ \gamma_3(4k)], \tag{17}$$
$$\gamma_3(8k+1) = (1/2)\ [a_1a_2a_3a_4\ \gamma_1(4k+1) + a_3a_7\ \gamma_3(4k+1)]. \tag{18}$$

$$\gamma_0(8k+2) = (1/2)\ [a_1a_2\ \gamma_0(4k+1) + a_2a_3a_4a_6\ \gamma_2(4k+1)], \tag{19}$$
$$\gamma_1(8k+2) = (1/2)\ [a_3\ \gamma_0(4k+1) + a_2a_3a_5a_7\ \gamma_2(4k+1)], \tag{20}$$
$$\gamma_2(8k+2) = (1/2)\ [a_1a_4\ \gamma_1(4k+1) + a_3a_6\ \gamma_3(4k+1)], \tag{21}$$
$$\gamma_3(8k+2) = (1/2)\ [a_1a_2a_3a_5\ \gamma_1(4k+1) + a_1a_3a_7\ \gamma_3(4k+1)], \tag{22}$$

$$\gamma_0(8k+3) = (1/2)\ [a_1a_3\ \gamma_0(4k+1) + a_2a_3a_4a_7\ \gamma_2(4k+1)], \tag{23}$$
$$\gamma_1(8k+3) = (1/2)\ [a_1a_2a_4\ \gamma_0(4k+2) + a_2a_5\ \gamma_2(4k+2)], \tag{24}$$
$$\gamma_2(8k+3) = (1/2)\ [a_1a_5\ \gamma_1(4k+1) + a_1a_3a_6\ \gamma_3(4k+1)], \tag{25}$$
$$\gamma_3(8k+3) = (1/2)\ [a_1a_6\ \gamma_1(4k+2) + a_1a_2a_3a_7\ \gamma_3(4k+2)], \tag{26}$$

$$\gamma_0(8k+4) = (1/2)\ [a_2a_4\ \gamma_0(4k+2) + a_2a_4\ \gamma_2(4k+2)], \tag{27}$$
$$\gamma_1(8k+4) = (1/2)\ [a_1a_2a_5\ \gamma_0(4k+2) + a_1a_2a_5\ \gamma_2(4k+2)], \tag{28}$$
$$\gamma_2(8k+4) = (1/2)\ [a_1a_2a_3a_6\ \gamma_1(4k+2) + a_1a_2a_3a_6\ \gamma_3(4k+2)], \tag{29}$$
$$\gamma_3(8k+4) = (1/2)\ [a_1a_7\ \gamma_1(4k+2) + a_1a_7\ \gamma_3(4k+2)], \tag{30}$$

$$\gamma_0(8k+5) = (1/2)\ [a_2a_5\ \gamma_0(4k+2) + a_1a_2a_4\ \gamma_2(4k+2)], \tag{31}$$
$$\gamma_1(8k+5) = (1/2)\ [a_1a_3a_6\ \gamma_0(4k+3) + a_1a_5\ \gamma_2(4k+3)], \tag{32}$$
$$\gamma_2(8k+5) = (1/2)\ [a_1a_2a_3a_7\ \gamma_1(4k+2) + a_1a_6\ \gamma_3(4k+2)], \tag{33}$$
$$\gamma_3(8k+5) = (1/2)\ [a_1a_3\ \gamma_1(4k+3) + a_2a_3a_4a_7\ \gamma_3(4k+3)], \tag{34}$$

$$\gamma_0(8k+6) = (1/2)\ [a_3a_6\ \gamma_0(4k+3) + a_1a_4\ \gamma_2(4k+3)], \tag{35}$$
$$\gamma_1(8k+6) = (1/2)\ [a_1a_3a_7\ \gamma_0(4k+3) + a_1a_2a_3a_5\ \gamma_2(4k+3)], \tag{36}$$
$$\gamma_2(8k+6) = (1/2)\ [a_1a_2\ \gamma_1(4k+3) + a_2a_3a_4a_6\ \gamma_3(4k+3)], \tag{37}$$
$$\gamma_3(8k+6) = (1/2)\ [a_3\ \gamma_1(4k+3) + a_2a_3a_5a_7\ \gamma_3(4k+3)], \tag{38}$$

$$\gamma_0(8k+7) = (1/2)\ [a_3a_7\ \gamma_0(4k+3) + a_1a_2a_3a_4\ \gamma_2(4k+3)], \tag{39}$$
$$\gamma_1(8k+7) = (1/2)\ [a_1\ \gamma_0(4k+4) + a_4a_5\ \gamma_2(4k+4)], \tag{40}$$
$$\gamma_2(8k+7) = (1/2)\ [a_2\ \gamma_1(4k+3) + a_2a_3a_5a_6\ \gamma_3(4k+3)], \tag{41}$$
$$\gamma_3(8k+7) = (1/2)\ [a_2a_3\ \gamma_1(4k+4) + a_6a_7\ \gamma_3(4k+4)]. \tag{42}$$

Now, we are ready to prove the sufficiency part of Theorem 1.5. To apply the formulas (15) – (42), note that

$$\gamma_0(0) = \gamma_1(0) = \gamma_2(0) = \gamma_3(0) = 1.$$

We show that if a sequence satisfies one of the conditions (1)(2)(3) of Theorem 1.5, then the correlation function $\gamma(h)$ vanishes except for $h = 0$.

Assume the condition (1). Applying the formulas (15) − (18) with $k = 0$, we get

$$\gamma_0(1) = \gamma_1(1) = \gamma_2(1) = \gamma_3(1) = 0.$$

Since $\gamma(1) = (1/4)\ [\ \gamma_0(1) + \gamma_1(1) + \gamma_2(1) + \gamma_3(1)\ ]$, we have $\gamma(1) = 0$. Based on these results it follows from (19) − (22) that

$$\gamma_0(2) = \gamma_1(2) = \gamma_2(2) = \gamma_3(2) = 0,$$

and hence, $\gamma(2) = 0$. In the same way, using (15) − (42) repeatedly we have $\gamma(h) = 0$ for any $h = 1, 2, \cdots$.

If the binary pattern sequence of degree 3 satisfies the condition (2), then $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ takes one of the following 4 values:

$$(-1, -1, -1, -1, 1, -1, -1),$$
$$(-1, -1, -1, 1, -1, 1, 1),$$
$$(-1, 1, 1, -1, 1, 1, 1),$$
$$(-1, 1, 1, 1, -1, -1, -1).$$

Calculating their correlation functions by (15) − (26), we have

$$\gamma_0(1) = -1, \gamma_1(1) = 0, \gamma_2(1) = 1, \gamma_3(1) = 0,$$
$$\gamma_0(2) = \gamma_1(2) = \gamma_2(2) = \gamma_3(2) = 0,$$
$$\gamma_0(3) = \gamma_1(3) = \gamma_2(3) = \gamma_3(3) = 0.$$

In the same way, using (15) − (42) repeatedly we have $\gamma(h) = 0$ for any $h = 1, 2, \cdots$.

If the binary pattern sequence of degree 3 satisfies the condition (3), then $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ takes one of the following 4 values:

$$(1, -1, 1, 1, 1, 1, -1),$$
$$(1, -1, 1, -1, -1, -1, 1),$$
$$(1, 1, -1, 1, 1, -1, 1),$$
$$(1, 1, -1, -1, -1, 1, -1).$$

Calculating their correlation functions by (15) − (26), we have

$$\gamma_0(1) = 1, \gamma_1(1) = 0, \gamma_2(1) = -1, \gamma_3(1) = 0,$$
$$\gamma_0(2) = \gamma_1(2) = \gamma_2(2) = \gamma_3(2) = 0,$$
$$\gamma_0(3) = \gamma_1(3) = \gamma_2(3) = \gamma_3(3) = 0.$$

In the same way, using (15) − (42) repeatedly we have $\gamma(h) = 0$ for any $h = 1, 2, \cdots$. Hence the sufficiency is proved.

Next we will prove the necessity part of Theorem 1.5.

If any of the conditions (1)(2)(3) is failed to hold, then we have one of the following 3 cases:

Case 1. $a_1 = a_4a_5$ and $a_2a_3 = -a_6a_7$.

Case 2. $a_1 = -a_4a_5$ and $a_2a_3 = a_6a_7$.

Case 3. $a_1 = a_4a_5$, $a_2a_3 = a_6a_7$ and neither (2) nor (3) of Theorem 1.5 holds.

Case 1. $a_1 = a_4a_5$ and $a_2a_3 = -a_6a_7$. We divide it into 2 subcases.

Subcase 1.1. $a_1 = a_4a_5 = 1$ and $a_2a_3 = -a_6a_7$.

By (15) − (18), we get

$$\gamma_0(1) = 1,$$
$$\gamma_1(1) = (1/2)\ a_2,$$

$$\gamma_2(1) = 0,$$
$$\gamma_3(1) = (a_3a_4)/(4 - 2a_3a_7).$$

If $a_2 = 1$, then $\gamma(1) \neq 0$ as $(a_3a_4)/(2 - a_3a_7) \neq -3$.
If $a_2 = -1$ and $(a_3a_4)/(2 - a_3a_7) \neq -1$, then $\gamma(1) \neq 0$.
If $a_2 = -1$ and $(a_3a_4)/(2 - a_3a_7) = -1$, then $\gamma(1) = 0$ but $\gamma(2) \neq 0$. In fact, there are only two sequences

$$(1, -1, 1, -1, -1, 1, 1) \ \text{ and } \ (1, -1, -1, 1, 1, 1, -1)$$

satisfying this subcase. Putting these initial values into formulas (19) – (22) we calculate the values of $\gamma(2)$ are $-1/8$ and $-3/8$ respectively. That is $\gamma(2) \neq 0$.

Subcase 1.2. $a_1 = a_4a_5 = -1$ and $a_2a_3 = -a_6a_7$.
In the same way we have

$$\gamma_0(1) = -1,$$
$$\gamma_1(1) = (-1/2)\ a_2,$$
$$\gamma_2(1) = 0,$$
$$\gamma_3(1) = (a_3a_4)/(4 - 2a_3a_7).$$

If $a_2 = 1$, then $\gamma(1) \neq 0$ as $(a_3a_4)/(2 - a_3a_7) \neq 3$.
If $a_2 = -1$ and $(a_3a_4)/(2 - a_3a_7) \neq 1$, then $\gamma(1) \neq 0$.
If $a_2 = -1$ and $(a_3a_4)/(2 - a_3a_7) = 1$, then $\gamma(1) = 0$ but $\gamma(2) \neq 0$. Using the same argument as in subcase 1.1 we also find only two sequences

$$(-1, -1, 1, 1, -1, 1, 1) \ \text{ and } \ (-1, -1, -1, -1, 1, 1, -1).$$

Similarly it is easy to check that the values of $\gamma(2)$ are $-3/8$ and $-1/8$ respectively.

Case 2. $a_1 = -a_4a_5$ and $a_2a_3 = a_6a_7$. We divide it into 2 subcases.

Subcase 2.1. $a_1 = -a_4a_5$ and $a_2a_3 = a_6a_7 = 1$.
From (15) – (18), we get

$$\gamma_0(1) = 0,$$
$$\gamma_1(1) = (1/2)\ a_5a_6,$$
$$\gamma_2(1) = 1,$$
$$\gamma_3(1) = -a_6/(4 - 2a_3a_7).$$

If $a_5a_6 = 1$, then $\gamma(1) \neq 0$ as $-a_6/(2 - a_3a_7) \neq -3$.
If $a_5a_6 = -1$ and $-a_6/(2 - a_3a_7) \neq -1$, then $\gamma(1) \neq 0$.
If $a_5a_6 = -1$ and $-a_6/(2 - a_3a_7) = -1$, then $\gamma(1) = 0$ but $\gamma(2) \neq 0$. In fact, there are only two sequences

$$(1, 1, 1, 1, -1, 1, 1) \ \text{ and } \ (-1, 1, 1, -1, -1, 1, 1)$$

satisfying this subcase. Thus by (19) – (22) the values of $\gamma(2)$ are $-1/8$ and $-3/8$ respectively.

Subcase 2.2. $a_1 = -a_4a_5$ and $a_2a_3 = a_6a_7 = -1$.
By (15) – (18) we obtain

$$\gamma_0(1) = 0,$$

$$\gamma_1(1) = (1/2)\ a_5a_6,$$

$$\gamma_2(1) = -1,$$

$$\gamma_3(1) = a_6/(4 - 2a_3a_7).$$

If $a_5a_6 = -1$, then $\gamma(1) \neq 0$ as $a_6/(2 - a_3a_7) \neq 3$.

If $a_5a_6 = 1$ and $a_6/(2 - a_3a_7) \neq 1$, then $\gamma(1) \neq 0$.

If $a_5a_6 = 1$ and $a_6/(2 - a_3a_7) = 1$, then $\gamma(1) = 0$ but $\gamma(2) \neq 0$. Using the same argument in subcase 2.1 we get two sequences$(1, 1, -1, -1, 1, 1, -1)$ and $(-1, 1, -1, 1, 1, 1, -1)$. It can easily be shown that the values of $\gamma(2)$ are $-3/8$ and $-1/8$ respectively.

Case 3. $a_1 = a_4a_5,\ a_2a_3 = a_6a_7$ and neither (2) nor (3) of Theorem 1.5 holds. We will show that at least one of $\gamma(1)$, $\gamma(2)$ and $\gamma(4)$ is nonzero. We divide it into 4 subcases.

Subcase 3.1. $a_1 = a_2a_3 = a_4a_5 = a_6a_7 = 1$.

By (15) – (18), we have

$$\gamma_0(1) = 1,$$

$$\gamma_1(1) = (1/2)\ (a_2 + a_5a_6),$$

$$\gamma_2(1) = 1,$$

$$\gamma_3(1) = (a_3a_4 + a_6)/(4 - 2a_3a_7).$$

If $\gamma_1(1) + \gamma_3(1) \neq -2$, then $\gamma(1) \neq 0$.

If $\gamma_1(1) + \gamma_3(1) = -2$, then $\gamma(1) = 0$. We'll show that $\gamma(2) \neq 0$ in this case. Since $\gamma_1(1) + \gamma_3(1) = -2$ and $|\gamma_1(1)| \leq 1,\ |\gamma_3(1)| \leq 1$, we have $\gamma_1(1) = \gamma_3(1) = -1$. Hence, $a_2 = a_5a_6 = -1$ and $a_3a_4 = a_6 = -1,\ a_3a_7 = 1$. Hence, together with $a_1 = a_2a_3 = a_4a_5 = a_6a_7 = 1$, the sequence $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ is determined as

$$(1, -1, -1, 1, 1, -1, -1).$$

Then by (15) – (22), we have $\gamma(1) = 0$ and $\gamma(2) = -1$.

Subcase 3.2. $a_1 = a_2a_3 = a_4a_5 = a_6a_7 = -1$.

Using (15) – (18) we have

$$\gamma_0(1) = -1,$$

$$\gamma_1(1) = (1/2)\ (a_5a_6 - a_2),$$

$$\gamma_2(1) = -1,$$

$$\gamma_3(1) = (-a_6 - a_2a_4)/(4 - 2a_3a_7).$$

If $\gamma_1(1) + \gamma_3(1) \neq 2$, then $\gamma(1) \neq 0$.

If $\gamma_1(1) + \gamma_3(1) = 2$, then $\gamma(1) = 0$. We'll show that $\gamma(2) \neq 0$ in this case. Since $\gamma_1(1) + \gamma_3(1) = 2$ and $|\gamma_1(1)| \leq 1,\ |\gamma_3(1)| \leq 1$, we have $\gamma_1(1) = \gamma_3(1) = 1$. Hence, $-a_2 = a_5a_6 = 1$ and $a_2a_4 = a_6 = -1,\ a_3a_7 = 1$. Hence, together with $a_1 = a_2a_3 = a_4a_5 = a_6a_7 = -1$, the sequence $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ is determined as

$$(-1, -1, 1, 1, -1, -1, 1).$$

Then by $(15)-(22)$, we have $\gamma(1)=0$ and $\gamma(2)=-1$.

Subcase 3.3. $a_1=a_4a_5=-1$, $a_2a_3=a_6a_7=1$ and $a_2=-a_5a_6$.

Using the formulas $(15)-(18)$ we have

$$\begin{aligned}\gamma_0(1)&=-1,\\ \gamma_1(1)&=-a_2,\\ \gamma_2(1)&=1,\\ \gamma_3(1)&=a_2a_4/(2-a_3a_7).\end{aligned}$$

Then $\gamma(1)=0$ only if $\gamma_1(1)+\gamma_3(1)=0$, which implies that $a_3a_7=1$ and $a_4=1$. Hence together with $a_1=a_4a_5=-1$, $a_2a_3=a_6a_7=1$ and $a_2=-a_5a_6$, we have 2 cases:

$$(-1,1,1,1,-1,1,1) \ \text{ and } \ (-1,-1,-1,1,-1,-1,-1)$$

By the formulas $(19)-(30)$, we have $\gamma(2)=\gamma(3)=0$ and $\gamma(4)=1$.

Subcase 3.4. $a_1=a_4a_5=1$ and $a_2a_3=a_6a_7=-1$ and $a_2=a_5a_6$.

Using the formulas $(15)-(18)$ we have

$$\begin{aligned}\gamma_0(1)&=1,\\ \gamma_1(1)&=a_2,\\ \gamma_2(1)&=-1,\\ \gamma_3(1)&=-a_2a_4/(2-a_3a_7).\end{aligned}$$

Then $\gamma(1)=0$ only if $\gamma_1(1)+\gamma_3(1)=0$, which implies that $a_3a_7=1$ and $a_4=1$. Hence together with $a_1=a_4a_5=1$, $a_2a_3=a_6a_7=-1$ and $a_2=a_5a_6$, we have 2 cases:

$$(1,1,-1,1,1,1,-1) \ \text{ and } \ (1,-1,1,1,1,-1,1)$$

By the formulas $(15)-(30)$, we have $\gamma(1)=\gamma(2)=\gamma(3)=0,\ \gamma(4)=1$ for the both cases.

Thus, we complete the proof of the necessity.

REFERENCES

[1] J.-P. Allouche and P. Liardet, Generalized Rudin-Shapiro sequences, *Acta Arith.*, **60** (1991), 1–27.

[2] D. W. Boyd, J. H. Cook and P. Morton, On sequences of ±1's defined by binary patterns, *Dissertationes Math.*, **283** (1989), 64pp.

[3] J. Coquet, T. Kamae and M. Mendès France, Sur la mesure spectrale de certaines suites arithmétiques, *Bull. Soc. Math. France*, **105** (1977), 369–384.

[4] N. P. Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, Springer-Verlag, Berlin, 2002.

[5] C. Godrèche and J. M. Luck, Multifractal analysis in reciprocal space and the nature of the Fourier transform of self-similar structures, *J. Phys. A*, **23** (1990), 3769–3797.

[6] P. Morton, Connections between binary patterns and paperfolding, *Sém. Théor. Nombres Bordeaux (2)*, **2** (1990), 1–12.

[7] P. Morton and W. J. Mourant, Paper folding, digit patterns and groups of arithmetic fractals, *Proc. London Math. Soc. (3)*, **59** (1989), 253–293.

[8] M. Niu and Z. X. Wen, Correlation dimension of the spectral measure for m-multiplicative sequences,(Chinese) *Acta Math. Sci. Ser. A (Chin. Ed.)*, **27** (2007), 862–870.

[9] L. Peng and T. Kamae, Spectral measure of the Thue-Morse sequence and the dynamical system and random walk related to it, *Ergodic Theory Dynam. Systems*, **36** (2016), 1247–1259.

[10] K. Petersen, *Ergodic Theory*, Cambridge University Press, Cambridge, 1983.

[11] M. Queffélec, *Substitution Dynamical Systems-Spectral Analysis*, Springer-Verlag, Berlin, 1987.

[12] M. A. Zaks, A. S. Pikovsky, J. Kurths, On the correlation dimension of the spectral measure for the Thue-Morse sequence, *J. Statist. Phys.*, **88** (1997), 1387–1392.

Received December 2017; 1st revision April 2018; 2nd revision May 2018.

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