Automata, algebraicity and distribution of sequences of powers

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Jean-Paul Allouche*
Jean-Marc Deshouillers‡
Teturo Kamae§
Tadahiro Koyanagi¶

Abstract

Let K be a finite field of characteristic p. Let $K((x))$ be the field of formal Laurent series $f(x)$ in x with coefficients in K. That is,

$$f(x) = \sum_{n=n_0}^{\infty} f_n x^n$$

with $n_0 \in \mathbf{Z}$ and $f_n \in K$ $(n = n_0, n_0 + 1, \cdots)$. We discuss the distribution of $(\{f^m\})_{m=0,1,2,\cdots}$ for $f \in K((x))$, where

$$\{f\} := \sum_{n=0}^{\infty} f_n x^n \in K[[x]]$$

denotes the nonnegative part of $f \in K((x))$. This is a little different from the real number case where the fractional part that excludes constant term (digit of order 0) is considered.

We give an alternative proof of a result by De Mathan obtaining the generic distribution for f with $f_n \neq 0$ for some

*CNRS, LRI, Bâtiment 490, F-91405 Orsay, France (allouche@lri.fr)
‡U. Bordeaux I, Math., F-33405 Talence, France (dezou@math.u-bordeaux.fr)
§Osaka City Univ., 558-8585 Osaka, Japan (kamae@sci.osaka-cu.ac.jp)
¶Osaka City Univ., 558-8585 Osaka, Japan

$n < 0$. This distribution is not the uniform measure on $K[[x]]$, but is equivalent to it. We have a different situation for $f \in K[[x]]$, where if $f_0 \neq 0$ and $f \neq f_0$, then the distribution for f is continuous but has a small support. We prove in this case, that the distribution for f^{-1} is identical with the distribution for $f_0^{-2} f$.

Christol, Kamae, Mendès France and Rauzy proved that the algebraicity of $f(x) \in K((x))$ over $K(x)$ is equivalent to the p-automaticity of the sequence (f_n). This result was generalized to the multidimensional case by Salon. Hence, if the Laurent series $f(x) \in K((x))$ is algebraic over $K(x)$, then

$$F(x, y) := \sum_{m=0}^{\infty} f(x)^m y^m$$

is 2-dimensionally p-automatic, since it is algebraic over the field $K(x, y)$. We construct a finite automaton recognizing the sequence of coefficients of this double series $F(x, y)$ to discuss the distribution of $(\{f^m\})_{m \geq 0}$. Thus, we generalize results by Houndonougbo and Deshouillers, and strengthen results by Allouche and Deshouillers.

1 Introduction

Let K be a finite field of characteristic p. Let $K((x))$ be the field of formal Laurent series in x. We call $f \in K((x))$ **algebraic** if it is algebraic over the rational function field $K(x)$. We say that

$$f(x) = \sum_{n=-\infty}^{\infty} f_n x^n \in K((x)),$$

where $f_n = 0$ if n is sufficiently small, is p-**automatic** (see for example [4] and the references therein), if there exists a finite automaton $M = (\Sigma, \phi, \sigma_0, \tau)$ over the alphabet $[p] := \{0, 1, \cdots, p-1\}$ such that

$$f_n = \tau(\phi(\cdots \phi(\phi(\sigma_0, n_0), n_1) \cdots, n_L)) \tag{1}$$

for any nonnegative integers n and L with

$$n = \sum_{i=0}^{\infty} n_i p^i = \sum_{i=0}^{L} n_i p^i, \quad n_i \in [p], \tag{2}$$

where Σ is a finite set, $\sigma_0 \in \Sigma$, $\phi : \Sigma \times [p] \to \Sigma$ and $\tau : \Sigma \to K$. In this case, we say that M **recognizes** f. The elements in Σ are called the **states** and σ_0 is called the **initial state** of M. We call τ the **output function** of M.

Remark 1. The usual definition for that M recognizes f is different from ours, but is that (1) holds for any nonnegative integers n and L with (2) together with $n_L \neq 0$. If we have a finite automaton M like this, then we can modify it to have a finite automaton $M' = (\Sigma \times \Sigma, \phi', (\sigma_0, \sigma_0), \tau')$ which recognizes f in our sense. In fact, we define

$$\phi'((\sigma, \sigma'), k) = \begin{cases} (\phi(\sigma, k), \sigma') & (k = 0) \\ (\phi(\sigma, k), \phi(\sigma, k)) & (k \neq 0) \end{cases}$$

and

$$\tau'(\sigma, \sigma') = \tau(\sigma').$$

Thus, f is recognized by some automaton in our sense if and only if f is recognized by some automaton in the usual sense.

The notion of "(p-)automaticity" does not change if automata read the highest digit first, i.e., if we replace (1) by

$$f_n = \tau(\phi(\cdots \phi(\phi(\sigma_0, n_L), n_{L-1}) \cdots, n_0)).$$

In this case, we say that M **dually recognizes** f. If M recognizes f, then the **dual automaton** M^* dually recognizes f (Section 6).

It holds that

Theorem 1 ([3, 4])**.** *The series $f \in K((x))$ is algebraic if and only if it is p-automatic.*

This theorem was generalized to the multi-dimensional case by Salon:

Theorem 2 ([12, 13])**.** *The formal power series $F(x,y) \in K[[x,y]]$ is algebraic if and only if it is p-automatic.*

F. von Haeseler and A. Petersen [8] and F. von Haeseler [9] also discussed the multi-dimensional generalization. In fact, they proved the equivalence between finite kernel property and automaticity in a general setting which essentially implies our Theorem 6, which generalize the "only if" part of Theorem 2 for

$$F(x,y) = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} F_{n,m} x^n y^m \in K((x))[[y]].$$

Here, $F_{n,m} \in K$ for any $n, m \in \mathbf{Z}$ with $m \geq 0$ and it holds that for any $m \geq 0$, there exists $n_0(m)$ such that $F_{n,m} = 0$ for any $n < n_0(m)$. The meaning of "p-automatic" for such an $F(x,y)$ is that there exists a finite automaton $M = (\Sigma, \phi, \sigma_0, \tau)$ over $[p] \times [p]$ such that

$$F_{n,m} = \tau(\phi(\cdots \phi(\phi(\sigma_0, n_0, m_0), n_1, m_1) \cdots, n_L, m_L)) \qquad (3)$$

for any nonnegative integers n, m and L with the following (4):

$$n = \sum_{i=0}^{\infty} n_i p^i = \sum_{i=0}^{L} n_i p^i, \quad m = \sum_{i=0}^{\infty} m_i p^i = \sum_{i=0}^{L} m_i p^i, \qquad (4)$$
$$\text{where} \quad n_i \in [p] \quad \text{and} \quad m_i \in [p].$$

The reader may compare the definition with [1] and [9]. Our definition of p-automaticity does not involve the part of $F_{n,m}$ with $n < 0$.

We apply this theorem to discuss the distribution of the sequence $(\{f^m\})_{m \geq 0}$ for $f \in K((x))$, where $\{f\}$ is the **nonnegative part** of f, i.e.,

$$\{f\} = \sum_{n=0}^{\infty} f_n x^n \in K[[x]].$$

The following result was proved by Allouche and Deshouillers [2] (see Deshouillers [5, 6, 7] for more precise results if f is rational).

Theorem 3 ([2])**.** *For any algebraic $f \in K((x))$, the logarithmic distribution of $(\{f^m\})_{m \geq 0}$ exists and its support has Hausdorff dimension zero.*

In the above, a Borel probability measure μ on $K[[x]]$ is called the **logarithmic distribution** of a sequence $(f^{(m)})_{m\geq 0}$ in $K[[x]]$ if for any finite sequence $(c_i)_{0\leq i<b}$ $(b>0)$ of elements in K, it holds that

$$\lim_{M\to\infty}\frac{1}{\log M}\sum_{\substack{m=0\\ f_i^{(m)}=c_i,\ \forall i\in[0,b)}}^{M-1}\frac{1}{m+1}=\mu\{\omega\in K((x));\ \omega_i=c_i,\ \forall i\in[0,b)\}.$$

Here, we call μ simply the **distribution** of a sequence $(f^{(m)})_{m\geq 0}$ in $K[[x]]$ if for any finite sequence $(c_i)_{0\leq i<b}$ of elements in K, it holds that

$$\lim_{M\to\infty}\frac{1}{M}\sum_{\substack{m=0\\ f_i^{(m)}=c_i,\ \forall i\in[0,b)}}^{M-1}1=\mu\{\omega\in K((x));\ \omega_i=c_i,\ \forall i\in[0,b)\}.$$

It is clear that, if a sequence has a distribution, then it has a logarithmic distribution and both distributions coincide.

In Section 2, we obtain the generic distribution of $(\{f^m\})_{m\geq 0}$ for random $f\in K((x))$ such that $\min\{n;\ f_n\neq 0\}<0$. The generic distribution is not the Haar measure on $K[[x]]$ but is equivalent to it, which is proved in Theorem 4 in Section 2.

In Section 3, we consider $(\{f^m\})_{m\geq 0}=(f^m)_{m\geq 0}$ when $f\in K[[x]]$. In this case, there always exists a continuous distribution if $f_0\neq 0$ and $f\neq f_0$. Moreover, the distributions of f^{-1} and $f_0^{-2}f$ coincide. In particular, if $f_0=1$, then f and f^{-1} have the same distribution.

In the further sections, we consider

$$F(x,y):=\sum_{m=0}^{\infty}f(x)^m y^m=\frac{1}{1-f(x)y}\in K((x))[[y]] \tag{5}$$

for an algebraic $f(x)\in K((x))$. We give an alternative proof of Theorem 3 using the fact that $F(x,y)$ is algebraic, and hence, p-automatic. In fact, we prove that the support of the logarithmic distribution is not only of Hausdorff dimension zero, but also of sublinear (block-)complexity. We construct a finite automaton which recognizes $F(x,y)$ for a rational $f(x) \in K((x))$ to discuss the distribution of the sequence $(\{f^m\})_{m\geq 0}$. Using it, we obtain a sufficient condition for the distribution to be the Dirac measure at 0 in the case where either the denominator or the numerator is a monomial. This generalizes results by Houndonougbo [10] and by Deshouillers [6] as well as simplifies the proofs.

2 Generic distribution

For any $n \in \mathbf{Z}$, denote $\mathbf{K}_n = \{f \in K((x));\ f_i = 0 \text{ for any } i < n\}$, which is identified with the product space $K^{\{n,n+1,n+2,\cdots\}}$. Let λ_n be the uniform distribution on $\mathbf{K}_n$. That is, λ_n is the product measure $(\lambda_K)^{\{n,n+1,n+2,\cdots\}}$, where λ_K is the uniform probability measure on K. The following Theorem is essentially due to De Mathan [11] (see Théorème 3 bis, p. 40).

Theorem 4. *For any $n < 0$ and for almost all $f \in \mathbf{K}_n \setminus \mathbf{K}_{n+1}$ with respect to λ_n, the distribution of $(\{f^m\})_{m\geq 0}$ exists and is equal to*

$$\mu = (p-1)\sum_{k=1}^{\infty} p^{-k}\lambda_0 \circ T^{1-k},$$

where $T : K[[x]] \to K[[x]]$ is defined by $T(f) = \sum_{i=0}^{\infty} f_i x^{pi}$. Hence, μ is equivalent to λ_0 and the support is the whole space $K[[x]]$.

Corollary 1. *The logarithmic distribution of $(\{f^m\})_{m\geq 0}$ for $f \in K[[x]]$ or algebraic $f \in K((x))$, for which we know that the support has Hausdorff dimension 0, is singular with respect to this generic distribution μ.*

Remark 2. The uniform distribution λ_0 cannot be a logarithmic distribution of the sequence $(\{f^m\})_{m\geq 0}$ for any $f \in K((x))$, since the relative frequency of m such that $(f^m)_1 = (f^m)_{p+1} = 0$ is at least $1/p$ as $(f^{jp})_1 = (f^{jp})_{p+1} = 0$ $(j = 1, 2, \cdots)$. On the other hand, the λ_0-measure of the set of $g \in K[[x]]$ such that $g_1 = g_{p+1} = 0$ is at most $1/p^2$.

Proof of Theorem 4. Let $f = \sum_{i \geq n} Z_i x^i$ be a random variable on $\mathbf{K}_n \setminus \mathbf{K}_{n+1}$, where $Z_{n+1}, Z_{n+2}, Z_{n+3}, \cdots$ are independent random variables uniformly distributed on K and Z_n is a uniformly distributed random variable on $K \setminus \{0\}$ which is independent of $Z_{n+1}, Z_{n+2}, Z_{n+3}, \cdots$. Take any $m \geq 0$ which is not a multiple of p. Then, for any $i \geq 0$, we have

$$\begin{aligned} (f^m)_i &= A_{m,i} + mZ_n^{m-1} Z_{i-n(m-1)} \\ &= A_{m,i} + B_m Z_{i-n(m-1)}, \end{aligned}$$

where $A_{m,i} \in K$ and $B_m \in K \setminus \{0\}$ are random variables determined by $Z_n, Z_{n+1}, \cdots, Z_{i-n(m-1)-1}$. Therefore, for any k and $(c_0, c_1, \cdots, c_{k-1}) \in K^k$, we have

$$\begin{aligned} & E[\prod_{0 \leq i \leq k-1} 1_{(f^m)_i = c_i}] \\ &= E[E[\prod_{0 \leq i \leq k-1} 1_{(f^m)_i = c_i} | Z_n, Z_{n+1}, \cdots, Z_{k-n(m-1)-1}]] \\ &= E[\prod_{0 \leq i \leq k-2} 1_{(f^m)_i = c_i} \\ & \qquad P[A_{m,k-1} + B_m Z_{k-n(m-1)} = c_k | Z_n, Z_{n+1}, \cdots, Z_{k-n(m-1)-1}]] \\ &= E[\prod_{0 \leq i \leq k-2} 1_{(f^m)_i = c_i}](\sharp K)^{-1} = \quad \cdots \quad = (\sharp K)^{-k}, \end{aligned}$$

where $\sharp K$ denotes the number of elements in K. Now let us estimate the variance of $(1/M) \sum_{m \in a(M)} \prod_{0 \leq i \leq k-1} 1_{(f^m)_i = c_i}$, where we denote by $a(M)$ the set of the least M positive integers not divisible by p.

We denote $A = (\sharp K)^{-1}$ and $B = \frac{k}{-n}$. Then we have

$$
\begin{aligned}
&|E\left[\left(\sum_{m\in a(M)}\prod_{0\le i\le k-1} 1_{(f^m)_i=c_i} - MA^k\right)^2\right]| \\
&= |\sum_{m,h\in a(M)} E[(\prod_{0\le i\le k-1} 1_{(f^m)_i=c_i} - A^k)(\prod_{0\le i\le k-1}(1_{(f^h)_i=c_i} - A^k)]| \\
&\le (2B+1)M + 2|\sum_{m,h\in a(M);\ m-h>B} \\
&\quad E[(\prod_{0\le i\le k-1} 1_{(f^m)_i=c_i} - A^k)(\prod_{0\le i\le k-1}(1_{(f^h)_i=c_i} - A^k)]| \\
&= (2B+1)M.
\end{aligned}
$$

The last equality in the above holds since for any $m,\ h\in a(M)$ with $m-h>B$, the term

$$
(\prod_{0\le i\le k-1} 1_{(f^m)_i=c_i} - A^k)(\prod_{0\le i\le k-1}(1_{(f^h)_i=c_i} - A^k)
$$

can be written as the sum of terms:

$$
\begin{aligned}
&A^{2k-2-j-j'}(1_{(f^m)_j=c_j} - A)(1_{(f^h)_{j'}=c_{j'}} - A) \\
&\times \prod_{0\le i\le j-1} 1_{(f^m)_i=c_i} \prod_{0\le i\le j'-1} 1_{(f^h)_i=c_i},
\end{aligned}
$$

which has 0 expectation since all the terms but $(1_{(f^m)_j=c_j} - A)$ are determined by $Z_n, Z_{n+1}, \cdots, Z_{j-n(m-1)-1}$, while as above

$$
\begin{aligned}
&E[1_{(f^m)_j=c_j} - A|Z_n, Z_{n+1}, \cdots, Z_{j-n(m-1)-1}] \\
&= P[A_{m,j} + B_m Z_{j-n(m-1)} = c_j|Z_n, Z_{n+1}, \cdots, Z_{j-n(m-1)-1}] - A \\
&= 0.
\end{aligned}
$$

Thus the variance of $(1/M)\sum_{m\in a(M)}\prod_{0\le i\le k-1} 1_{(f^m)_i=c_i}$ is at most $(2B+1)/M$ and we have the law of large numbers. That is, with probability 1, $(1/M)\sum_{m\in a(M)}\prod_{0\le i\le k-1} 1_{(f^m)_i=c_i}$ converges to A^k. Since this holds for any finite sequence $(c_0, c_1, \cdots, c_{k-1}) \in K^k$, it holds with probability 1 that the distribution of $(\{f^m\})_{m\in a(\infty)}$ is λ_0, where $a(\infty)$ is the set of positive integers which are not multiples of p. Since

$$(f^{pm})_n = \begin{cases} (f^m)_{n/p} & (\text{if } p|n) \\ 0 & (\text{otherwise}), \end{cases}$$

the distribution of $(\{f^m\})_{m\in pa(\infty)}$ is $\lambda_0 \circ T^{-1}$ with probability 1. In the same way, the distribution of $(\{f^m\})_{m\in p^2a(\infty)}$ is $\lambda_0 \circ T^{-2}$ with probability 1. Hence, the distribution of $(\{f^m\})_{m\geq 0}$ is

$$\frac{p-1}{p}\lambda_0 + \frac{p-1}{p^2}\lambda_0 \circ T^{-1} + \frac{p-1}{p^3}\lambda_0 \circ T^{-2} + \cdots ,$$

which completes the poof. □

3 Case $K[[x]]$

In this section, we consider the case where $f \in K[[x]]$. That is,

$$f = \sum_{n=0}^{\infty} f_n x^n.$$

For a positive integer N, let $f|_N := \sum_{n=0}^{N-1} f_n x^n$. Let G be a transformation on the finite set $K_{[0,N)} := \{g|_N;\ g \in K[[x]]\}$ defined by $G(g) := (gf)|_N$. Then since we have $f^m|_N = G^m(1)$ for $m = 0, 1, 2, \cdots$), the sequence $f^m|_N \in K_{[0,N)}$ in $m = 0, 1, 2, \cdots$ is ultimately periodic. In fact, the period starts at least at p^N since $p^N \geq N$ and $f^{p^N}|_N = f^{2p^N}|_N = f_0$. Let c_N be the least period of the ultimately periodic sequence $f^m|_N$ in $m = 0, 1, 2, \cdots$. Then, c_N is the least positive integer m such that $f^{p^N+m}|_N = f_0$. Moreover, any element appearing in the smallest period appears just once. Hence, f has a distribution, say μ_f.

We prove that μ_f is continuous if $f_0 \neq 0$ and $f \neq f_0$. To prove this, it suffices to show that the least period c_N tends to infinity as N tends to infinity. Let n be the least positive integer such that $f_n \neq 0$. Let $c_N = p^L c'$ with c' which is not a multiple of p. Suppose that $p^L n_0 < N$. Then, we have a contradiction that $(f^{p^N+c_N})_{p^L n_0} = (f^{c_N})_{p^L n_0} \neq 0$. Therefore, we have $p^L n_0 \geq N$. Since c_N is the minimum positive integer such that $c_N = p^L c'$ with c' which is not a multiple of p and L satisfying $p^L n_0 \geq N$, we have $c_N = p^L$ with L which is the minimum integer such that $p^L n_0 \geq N$.

Thus we have $N/n_0 \leq c_N < pN/n_0$ and $c_N \to \infty$ as $N \to \infty$.

The **complexity** $C_N(\Omega)$ of a closed subset Ω of $K[[x]]$ is defined by

$$C_N(\Omega) = \sharp \left\{ \begin{array}{c} (H_0, H_1, \cdots, H_{N-1}) \in K^N;\ \text{there exists } \omega \in \Omega \\ \text{such that } \omega_i = H_i, \quad \forall i = 0, 1, \cdots, N-1 \end{array} \right\}. \tag{6}$$

Let $\Omega(f)$ be the topological support of the measure μ_f on $K[[x]]$. Then it is clear that $C_N(\Omega(f)) = c_N$ for any $N = 1, 2, \cdots$.

When we discuss the Hausdorff dimension of subsets in $K[[x]]$, it is with respect to the metric ρ defined by

$$\rho(\omega, \omega') := p^{-\min\{n \geq 0;\ \omega_n \neq \omega'_n\}}$$

for any $\omega \neq \omega' \in K[[x]]$. For the α-Hausdorff measure Λ_α of $\Omega(f)$, we have

$$\begin{aligned} \Lambda_\alpha(\Omega(f)) &\leq \lim_{n\to\infty} \sum_{\substack{(H_0,\cdots,H_{n-1}) \in K^n \\ \exists \omega \in \Omega(f),\ \omega_i = H_i,\ i=0,\cdots,n-1}} p^{-n\alpha} \\ &= \lim_{n\to\infty} C_n(\Omega(f)) p^{-n\alpha} \\ &\leq \lim_{n\to\infty} pn \cdot p^{-n\alpha} \\ &= 0 \end{aligned}$$

for any $\alpha > 0$. Thus, $\dim \Omega(f) = 0$.

Theorem 5. *For $f \in K[[x]]$, the sequence $(\{f^m\})_{m=0,1,\cdots}$ has a distribution μ_f. If $f_0 = 0$, then μ_f is the Dirac measure at $0 \in K((x))$. If $f_0 \neq 0$ and $f \neq f_0$, then μ_f is a continuous distribution supported by $\Omega(f)$ while $\Omega(f)$ has a sublinear complexity and hence 0-Hausdorff dimension. In fact, $C_N(\Omega(f)) < pN$ for any $N = 1, 2, \cdots$. Moreover, in this case, it holds that $\mu_{f^{-1}} = \mu_{f_0^{-2} f}$.*

Proof. We only have to prove that $\mu_{f^{-1}} = \mu_{f_0^{-2}f}$. It suffices to prove this in the case $f_0 = 1$. Since $f^{p^k}|_{p^k} = 1$, $f^{p^k-m}|_{p^k} = f^{-m}|_{p^k}$ for any $k = 1, 2, \cdots$ and m with $0 \leq m < p^k$. This implies that $\mu_{f^{-1}} = \mu_f$. □

4 Construction of automata

For $i \in [p]$, define the linear operators X_i and Y_i on $K((x))[[y]]$ by

$$X_i\left(\sum_{n,m=-\infty}^{\infty} H_{n,m}\ x^n y^m\right) := \sum_{n,m=-\infty}^{\infty} H_{np+i,m}\ x^n y^m$$

and

$$Y_i\left(\sum_{n,m=-\infty}^{\infty} H_{n,m}\ x^n y^m\right) := \sum_{n,m=-\infty}^{\infty} H_{n,mp+i}\ x^n y^m.$$

Lemma 1.

(i) $X_iY_j(x^ny^m) = \begin{cases} x^{(n-i)/p}y^{(m-j)/p} & \text{if } n \equiv i \text{ and } m \equiv j \bmod p, \\ 0 & \text{otherwise.} \end{cases}$

(ii) *For any $i, j \in [p]$, we have $X_iY_j = Y_jX_i$.*

(iii) *For any $i, j \in [p]$ and for any $H, G \in K((x))[[y]]$, we have $X_iY_j(HG^p) = X_iY_j(H)G$.*

Proof. Assertions (i) and (ii) are clear from the definition. For the proof of (iii), it is sufficient to remark that $G(x,y)^p = G(x^p, y^p)$ holds for any $G \in K((x))[[y]]$. □

We state now a theorem to be compared with [3, 4, 9, 12, 13]. The proof either follows from them or at least is essentially the same with them. But for the readers' convenience, we give the proof.

Theorem 6. *If $F \in K((x))[[y]]$ is algebraic, then it is p-automatic.*

Proof. Assume that a nonzero element $F \in K((x))[[y]]$ is algebraic over $K(x,y)$ with degree h_0. Then, the elements $F, F^p, F^{p^2}, \cdots, F^{p^{h_0}}$ are linearly dependent over $K(x,y)$. Let h be the least integer such that $F, F^p, F^{p^2}, \cdots, F^{p^h}$ are linearly dependent over $K(x,y)$. Then, there exist $A_0, A_1, A_2, \cdots, A_h \in K[x,y]$ with at least one of them nonzero such that

$$A_0 F + A_1 F^p + A_2 F^{p^2} + \cdots + A_h F^{p^h} = 0. \tag{7}$$

We may also assume that $A_0, A_1, A_2, \cdots, A_h$ have no nontrivial common factor.

We prove that $A_0 \neq 0$. Suppose that $A_0 = 0$. Then we have

$$A_1 F^p + A_2 F^{p^2} + \cdots + A_h F^{p^h} = 0.$$

Since at least one of $A_1, A_2, \cdots, A_h$ is nonzero, there exist $i, j \in [p]$ such that at least one of $X_iY_j(A_1), X_iY_j(A_2), \cdots, X_iY_j(A_h)$ is nonzero. Then, by Lemma 1,

$$\begin{aligned} 0 &= X_iY_j(A_1F^p + A_2F^{p^2} + \cdots + A_hF^{p^h}) \\ &= X_iY_j(A_1)F + X_iY_j(A_2)F^{p^2} + \cdots + X_iY_j(A_h)F^{p^{h-1}}, \end{aligned}$$

which contradicts the minimality of h.

Thus, we have (7) with $A_0 \neq 0$. Let $G := F/A_0 \in K((x))[[y]]$. Then, it holds that

$$\begin{aligned} G &= -A_0^{p-2}A_1G^p - A_0^{p^2-2}A_2G^{p^2} - \cdots - A_0^{p^h-2}A_hG^{p^h} \\ &=: B_1G^p + B_2G^{p^2} + \cdots + B_hG^{p^h} \end{aligned}$$

and $F = A_0G$ with $A_0, B_1, B_2, \cdots, B_h \in K[x,y]$.

Let $d := \max\{\deg A_0, \deg B_1, \deg B_2, \cdots, \deg B_h\}$ and

$$\begin{aligned} \overline{\mathbf{S}}(f) \;:=\; &\{a_0G + a_1G^p + \cdots + a_{h-1}G^{p^{h-1}} \in K((x))[[y]]; \\ &\quad a_i \in K[x,y] \text{ and } \deg a_i \leq d,\ i = 0, 1, \cdots, h-1\}. \end{aligned}$$

Note that $\overline{\mathbf{S}}(f)$ is a finite set containing F. For any $i, j \in [p]$ and $H \in \overline{\mathbf{S}}(f)$ with

$$H = a_0G + a_1G^p + a_2G^{p^2} + \cdots + a_{h-1}G^{p^{h-1}},$$

it holds by Lemma 1 that

$$\begin{aligned}
X_iY_j(H) &= X_iY_j(a_0G + a_1G^p + a_2G^{p^2} + \cdots + a_{h-1}G^{p^{h-1}}) \\
&= X_iY_j(a_0(B_1G^p + \cdots + B_hG^{p^h}) + \\
&\qquad a_1G^p + a_2G^{p^2} + \cdots + a_{h-1}G^{p^{h-1}}) \\
&= X_iY_j(a_0B_1 + a_1)G + X_iY_j(a_0B_2 + a_2)G^p + \cdots + \\
&\qquad X_iY_j(a_0B_h)G^{p^{h-1}} \\
&\in \overline{\mathbf{S}}(f),
\end{aligned}$$

since, for any $k = 0, 1, \cdots, h-1$,

$$\deg X_iY_j(a_0B_k) \leq \frac{1}{p}(\deg a_0 + \deg B_k) \leq \frac{2d}{p} \leq d.$$

Let

$$\overline{\mathbf{M}}(f) := (\overline{\mathbf{S}}(f), \phi, F, \eta) \tag{8}$$

be the finite automaton over $[p] \times [p]$ such that

$$\phi(H, i, j) := X_iY_j(H) \text{ and } \eta(H) = H_{0,0}$$

for any $H = \sum H_{n,m}x^ny^m \in \overline{\mathbf{S}}(f)$ and $i, j \in [p]$. Let $\mathbf{S}(f)$ be the set of states in $\overline{\mathbf{S}}(f)$ which are **attainable** from the initial state F in $\overline{\mathbf{M}}(f)$, i.e., the set of states $S \in \overline{\mathbf{S}}(f)$ such that there exists a finite sequence of inputs in $[p] \times [p]$ which sends the state F to S. Let $\mathbf{M}(f) := (\mathbf{S}(f), \phi, F, \eta)$ be the automaton obtained from $\overline{\mathbf{M}}(f)$ by restricting the set of states to be $\mathbf{S}(f)$.

We prove that $\mathbf{M}(f)$ recognizes F. Take any nonnegative integers n, m and L with (4). It holds that

$$\begin{aligned}
F_{n,m} &= (X_{n_L}Y_{m_L} \cdots X_{n_1}Y_{m_1}X_{n_0}Y_{m_0}(F))_{0,0} \\
&= \eta(\phi(\cdots\phi(\phi(F, n_0, m_0), n_1, m_1) \cdots, n_L, m_L)),
\end{aligned}$$

which completes the proof. $\square$

5 Rational functions

Let

$$f(x) = \frac{P(x)}{Q(x)}, \text{ where } P, Q \in K[x], \text{ are coprime} \tag{9}$$

be a rational function in $K((x))$. Then, $F(x, y)$ defined in (5) satisfies

$$F(x, y) = \frac{1}{1 - f(x)y} = \frac{Q(x)}{Q(x) - P(x)y}.$$

Let

$$G(x, y) = \frac{1}{Q(x) - P(x)y}.$$

Let $\overline{S}(f)$ be the set of all $H \in K[x]$ with $\deg H \leq \max\{\deg P, \deg Q\}$. Define $\phi : \Sigma \times [p] \times [p] \to \Sigma$ by

$$\phi(H, i, j) = X_i(HQ^{p-1-j}P^j). \tag{10}$$

Let $\tau : \Sigma \to K$ be $\tau(H) = (\frac{H}{Q})_0$, i.e., the coefficient of $\frac{H}{Q} \in K((x))$ of degree 0. Thus, we define a finite automaton $(\overline{S}(f), \phi, Q, \tau)$ over $[p] \times [p]$. Let $S(f)$ be the set of states in $\overline{S}(f)$ which are attainable from the initial state Q in this automaton. Let $M(f) := (S(f), \phi, Q, \tau)$ be the automaton obtained from $(\overline{S}(f), \phi, Q, \tau)$ by restricting the set of states to be $S(f)$.

Theorem 7. *The finite automaton $M(f)$ recognizes $F(x, y)$.*

Proof. For $H \in S(f)$ and $i, j \in [p]$, it holds by Lemma 1 that

$$\begin{aligned} X_iY_j(HG) &= X_iY_j(H(Q - Py)^{p-1}G^p) \\ &= X_iY_j(H(Q - Py)^{p-1})G \\ &= X_i(H \binom{p-1}{j} Q^{p-1-j}(-P)^j)G \\ &= X_i(HQ^{p-1-j}P^j)G = \phi(H, i, j)G. \end{aligned}$$

Take any nonnegative integers n, m and L with (4). Then it holds that

$$\begin{aligned}
F_{n,m} &= X_{n_L}Y_{m_L}\cdots X_{n_1}Y_{m_1}X_{n_0}Y_{m_0}(F)_{0,0} \\
&= X_{n_L}Y_{m_L}\cdots X_{n_1}Y_{m_1}X_{n_0}Y_{m_0}(QG)_{0,0} \\
&= X_{n_L}Y_{m_L}\cdots X_{n_1}Y_{m_1}(\phi(Q,n_0,m_0)G)_{0,0} \\
&= \cdots \\
&= (\phi(\cdots\phi(\phi(Q,n_0,m_0),n_1,m_1)\cdots,n_L,m_L)G)_{0,0} \\
&= \tau(\phi(\cdots\phi(\phi(Q,n_0,m_0),n_1,m_1)\cdots,n_L,m_L)),
\end{aligned}$$

which completes the proof. □

Let f be as in (9), F be as in (5) for this f, and the finite automaton $M := M(f)$ be as above. For each $i \in [p]$, let $M_i := (S(f), \phi_i, Q, \tau)$ be the finite automaton over $[p]$ such that $\phi_i(H,j) = \phi(H,i,j)$ for any $j \in [p]$ and $H \in S(f)$. Then, the sequence $(F_{n,m})_{m\geq 0}$ in K for a fixed nonnegative integer n with (2) is "recognizable" by the sequence of automata related to n:

$$M_{n_0}, M_{n_1}, \cdots, M_{n_L}, M_0, M_0, \cdots$$

in the sense that

$$F_{n,m} = \tau(\phi_{n_N}(\cdots\phi_{n_1}(\phi_{n_0}(Q,m_0),m_1)\cdots,m_N))$$

for any nonnegative integers m and $N \geq L$ with

$$m = \sum_{i=0}^{\infty} m_i p^i = \sum_{i=0}^{N} m_i p^i, \quad m_i \in [p].$$

Theorem 8.

(i) *The distribution of the sequence* $(\{f^m\})_{m\geq 0}$ *is equal to* δ_0, *the Dirac measure at* $0 \in K((x))$ *if in the finite automaton* M_0 *as above,* 0 *is attainable from any state in* $S(f)$.

(ii) *If* $P = 1$ *and* $Q \neq 0$ *satisfies* $Q(0) = 0$, *then the distribution of the sequence* $(\{f^m\})_{m\geq 0}$ *is equal to* δ_0.

(iii) *If* $Q = x^u$ *with* $u \geq 1$, $P(0) \neq 0$ *and for some* $k = 1, 2, \cdots$, P^k *lacks the term* x^{ku}, *i.e.,* $(P^k)_{ku} = 0$, *then the distribution of the sequence* $(\{f^m\})_{m\geq 0}$ *is equal to* δ_0.

Proof. (i) Assume that 0 is attainable from any state in $S(f)$ in M_0. By the above consideration, 0 is the only "sink" of the sequence of automata related to any $n \geq 0$. Since $\tau(0) = 0$, this implies that for any $n \geq 0$ the frequency of 0 in the sequence $(F_{n,m})_{m\geq 0}$ is equal to 1. Thus, the distribution of the sequence $(\{f^m\})_{m\geq 0}$ is equal to δ_0.

(ii) Since $\phi_0(x^c, p-1)$ is $x^{c/p}$ if $p \mid c$ and 0 otherwise,

$$\underbrace{\phi_0(\cdots\phi_0(\phi_0}_{k\ \text{times}}(x^c, p-1), p-1)\cdots, p-1) \neq 0$$

only if $p^k \mid c$. Therefore, for any $H \in S(f)$ and for any sufficiently large integer k, it holds that

$$\underbrace{\phi_0(\cdots\phi_0(\phi_0}_{k\ \text{times}}(H, p-1), p-1)\cdots, p-1) = H(0).$$

Assume that $H = C$ (constant). Then, since

$$\phi_0(H, p-2) = CX_0(Q) =: J,$$

the relation $J(0) = 0$ follows from the assumption $Q(0) = 0$.

Thus, 0 is attainable from any element H in $S(f)$ in M_0 by reading $(p-1)$ sufficiently many times followed by reading $(p-2)$ once and again $(p-1)$ sufficiently many times.

(iii) Assume that $(P^k)_{ku} = 0$ for some $k = 1, 2, \cdots$. Since $\phi_0(x^c, 0)$ is $x^{u+(c-u)/p}$ if $p \mid c-u$ and 0 otherwise,

$$\underbrace{\phi_0(\cdots\phi_0(\phi_0}_{j\ \text{times}}(x^c, 0), 0)\cdots, 0) \neq 0$$

only if $p^j \mid c-u$. Therefore, for any $H \in S(f)$ and for any sufficiently large integer j, it holds that

$$\underbrace{\phi_0(\cdots\phi_0(\phi_0}_{j\ \text{times}}(H, 0), 0)\cdots, 0) = H_u x^u.$$

Therefore, for any state in $S(f)$, there exists $C \in K$ such that Cx^u is attainable from it. Hence, it suffices to prove that 0 is attainable from x^u.

Let $k = \sum_{i=0}^{j-1} k_i p^i$ with $k_i \in [p]$. Then we have

$$
\begin{aligned}
H &:= \underbrace{\phi_0(\cdots \phi_0(\phi_0}_{j \text{ times}}(x^u, k_0), k_1) \cdots, k_{j-1}) \\
&= X_0(\cdots X_0(X_0(x^{(p-k_0)u} P^{k_0}) x^{(p-k_1-1)u} P^{k_1}) \cdots x^{(p-k_{j-1}-1)u} P^{k_{j-1}}) \\
&= X_0(\cdots X_0(X_0(x^{(p-k_0)u} P^{k_0} (x^{(p-k_1-1)u} P^{k_1})^p) \cdots x^{(p-k_{j-1}-1)u} P^{k_{j-1}}) \\
&= X_0(\cdots X_0(X_0(x^{(p^2-k_0-k_1 p)u} P^{k_0+k_1 p})) \cdots x^{(p-k_{j-1}-1)u} P^{k_{j-1}}) \\
&= X_0(\cdots X_0(X_0(x^{(p^j-k_0-k_1 p-\cdots-k_{j-1}p^{j-1})u} P^{k_0+k_1 p+\cdots k_{j-1}p^{j-1}})) \cdots) \\
&= x^u X_0^j (x^{-ku} P^k).
\end{aligned}
$$

Therefore, $H_u = 0$ follows from the assumption $(P^k)_{ku} = 0$. Thus, 0 is attainable by applying the preceding procedure again, which completes the proof. □

Remark 3. To cover the case where one of P or Q is a monomial, we have to consider the following subcases in addition to (ii) and (iii) in Theorem 8:

(iv) $P = 1$ and $Q(0) \neq 0$,

(v) $P = x^u$ with $u \geq 1$ and $Q(0) \neq 0$,

(vi) $Q = 1$ and $P(0) \neq 0$,

(vii) $Q = 1$ and $P(0) = 0$, and

(viii) $Q = x^u$ with $u \geq 1$ and $(P^k)_{ku} \neq 0$ for any $k = 1, 2, \cdots$.

The distribution is δ_0 in the case (v) and (vii), since $(f^m)_n = 0$ if $m > n$. In the cases (iv) and (vi), the distributions are continuous by Theorem 5 if f is nonconstant. In the case (viii), the distribution is always continuous by [6]

The case (iii) in Theorem 8 is due to Deshouillers [6]. Here we gave an alternative and simpler proof.

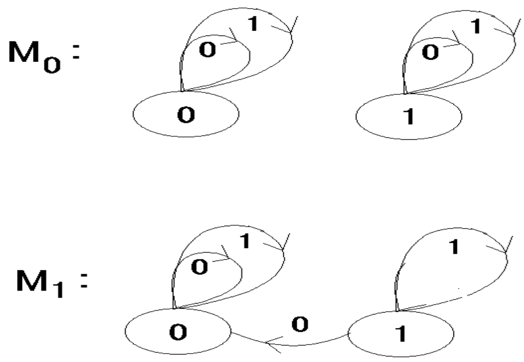

Figure 1: Automaton in Example 1

Example 1. *(Pascal triangle)*
Let $p = 2$, $K = \{0, 1\}$ and $f = 1 + x$, $(P = 1 + x,\ Q = 1)$. Then, the table $(F_{n,m})_{n,m\geq 0}$ is the Pascal triangle modulo 2. In the automaton $M = M(f)$, the initial state is 1, $S(f) = \{0, 1\}$, and it holds that

$$\phi(0, i, j) = 0,\ \ \phi(1, i, j) = \begin{cases} 1 & \text{if } i \leq j \\ 0 & \text{otherwise} \end{cases}$$

for any $i, j \in [2]$. Therefore, M_0 has two sinks 0 and 1. Furthermore we have $\tau(0) = 0$ and $\tau(1) = 1$.

The distribution μ for this f is determined using the automaton. In fact, we have

$$F_{n,m} = \begin{cases} 1 & \text{if } n_i \leq m_i \quad \forall i \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Define a partial order $\preceq$ on the nonnegative integers by

$$n \preceq m \text{ if and only if } n_i \leq m_i \text{ for all } i,$$

where we use the notation in (4). Then, for any fixed $m \geq 0$, the function F_m defined by $F_m(n) = F_{n,m}$ is monotone decreasing with respect to the partial order $\preceq$ on the set of nonnegative integers. It is not difficult to see that $\Omega(f)$ consists of all $\sum_{n\geq 0} g_n x^n$ such that the function $n \mapsto g_n$ is monotone decreasing in this sense. The distribution μ is the uniform distribution on $\Omega(f)$ in some sense.

By the arguments in Section 3, the function $m \mapsto F_m|_{2^k}$ is purely periodic with least period at most 2^k for $k = 1, 2, \cdots$. In our case, it is exactly 2^k since otherwise, there exists m with $0 < m < 2^k$ such that $F_m|_{2^k} = F_0|_{2^k} = \delta_0$. But this is impossible since $F_m(m) = F_{m,m} = 1$ by (11). The μ-measure of the cylinder determined by $F_m|_{2^k}$ is 2^{-k} for $m = 0, 1, \cdots, 2^k - 1$ using the periodicity.

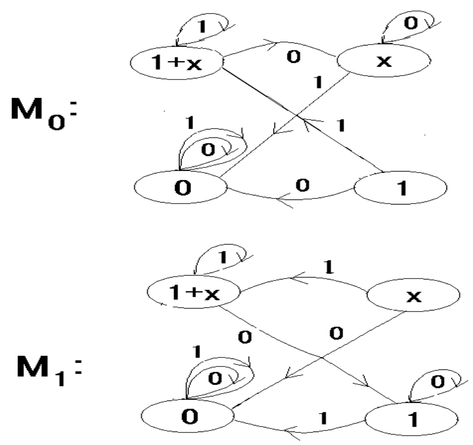

Figure 2: Automaton in Example 2

Example 2. Let $p = 2$, $K = \{0, 1\}$ and $f = (1+x^2)/x$, ($P = 1 + x^2$,

$Q = x$). Then, we have

$$\begin{aligned}
&\phi(x,i,0) = X_i(x^2) = \begin{cases} x & \text{if } i=0, \\ 0 & \text{if } i=1, \end{cases} \\
&\phi(x,i,1) = X_i(x+x^3) = \begin{cases} 0 & \text{if } i=0, \\ 1+x & \text{if } i=1, \end{cases} \\
&\phi(0,i,j) = 0 \qquad \forall i,j \in [2] \\
&\phi(1+x,i,0) = X_i(x+x^2) = \begin{cases} x & \text{if } i=0, \\ 1 & \text{if } i=1, \end{cases} \\
&\phi(1+x,i,1) = X_i(1+x+x^2+x^3) = 1+x \qquad \forall i \in [2] \\
&\phi(1,i,0) = X_i(x) = \begin{cases} 0 & \text{if } i=0, \\ 1 & \text{if } i=1, \end{cases} \\
&\phi(1,i,1) = X_i(1+x^2) = \begin{cases} 1+x & \text{if } i=0, \\ 0 & \text{if } i=1, \end{cases}
\end{aligned}$$

In this case, f has a distribution equal to δ_0.

6 Dual automata and complexity

Let $\mathbf{M}(f) := (\mathbf{S}(f), \phi, F, \eta)$ be the automaton constructed in Section 4 which recognizes F in (5) for an algebraic $f \in K((x))$. We construct the dual automaton $\mathbf{M}(f)^* := (\mathbf{S}(f)^*, \phi^*, \eta, F^*)$ over $[p] \times [p]$ which dually recognizes F.

Let

$$\begin{aligned}
&\overline{\mathbf{S}(f)}^* := \{\xi : \mathbf{S}(f) \to K\} \\
&\phi^*(\xi, i, j) := \xi \circ \phi_{i,j} \ (\forall i,j \in [p],\ \xi \in \overline{\mathbf{S}(f)}^*) \\
&F^*(\xi) := \xi(F) \ (\forall \xi \in \overline{\mathbf{S}(f)}^*),
\end{aligned}$$

where $\phi_{i,j} : \mathbf{S}(f) \to \mathbf{S}(f)$ is defined by $\phi_{i,j}(H) = \phi(H,i,j)$, $\forall H \in \mathbf{S}(f)$. Let $\mathbf{S}(f)^*$ be the set of all states which are attainable from the initial state η in the automaton $(\overline{\mathbf{S}(f)}^*, \phi^*, \eta, F^*)$ over $[p] \times [p]$. Let $\mathbf{M}(f)^* := (\mathbf{S}(f)^*, \phi^*, \eta, F^*)$ be the restriction of this automaton.

Then for any nonnegative integers n, m and L with (4), we have

$$\begin{aligned}
& F^*(\phi^*(\cdots\phi^*(\phi*(\eta,n_L,m_L),n_{L-1},m_{L-1})\cdots,n_0,m_0)) \\
= \; & F^*(\phi^*(\cdots\phi^*(\eta\circ\phi_{n_L,m_L},n_{L-1},m_{L-1})\cdots,n_0,m_0)) \\
= \; & \cdots \\
= \; & F^*(\eta\circ\phi_{n_L,m_L}\circ\phi_{n_{L-1},m_{L-1}}\circ\cdots\circ\phi_{n_0,m_0}) \\
= \; & \eta\circ\phi_{n_L,m_L}\circ\phi_{n_{L-1},m_{L-1}}\circ\cdots\circ\phi_{n_0,m_0}(F) \\
= \; & \eta(\phi(\cdots\phi(\phi(F,n_0,m_0),n_1,m_1)\cdots,n_L,m_L)) \\
= \; & F_{n,m}.
\end{aligned}$$

Thus, $\mathbf{M}(f)^*$ dually recognizes F.

Theorem 9. *If f is algebraic, then it holds that*

$$C_n(\Omega(f)) \leq pn\sharp\mathbf{S}(f)^*$$

for any $n = 1, 2, 3, \cdots$, where the notation is as in (6)*. In particular, the logarithmic distribution of the sequence $(\{f^m\})_{m\geq 0}$ is supported by $\Omega(f)$ which has Hausdorff dimension zero.*

Proof. Since the table $(F_{u,v})_{0\leq u<p^k,\ mp^k\leq v<(m+1)p^k}$ for $m \geq 0$ with

$$m = \sum_{i=0}^{\infty} m_i p^i = \sum_{i=0}^{L} m_i p^i \quad m_i \in [p],\ m_L \neq 0$$

is determined by

$$\phi^*(\cdots\phi^*(\phi^*(\eta,0,m_L),0,m_{L-1})\cdots,0,m_0) \in \mathbf{S}(f)^*,$$

there exist at most $\sharp\mathbf{S}(f)^*$ different tables as above. Hence, there exist at most $p^k\sharp\mathbf{S}(f)^*$ different sequences among $(F_{u,v})_{0\leq u<p^k}$ ($v = 0, 1, 2, \cdots$). Take any positive integer n. Then, there are at most $pn\sharp\mathbf{S}(f)^*$ different sequences among $(F_{u,v})_{0\leq u\leq n-1}$ ($v = 0, 1, 2, \cdots$), since there exists a positive integer k such that $p^{k-1} \leq n < p^k \leq pn$. Thus, we have

$$C_n(\Omega(f)) \leq pn\sharp\mathbf{S}(f)^*$$

for any $n = 1, 2, 3, \cdots$. For the α-Hausdorff measure Λ_α of $\Omega(f)$, we have

$$\begin{aligned}
\Lambda_\alpha(\Omega(f)) &\leq \lim_{n\to\infty} \sum_{\substack{(H_0,\cdots,H_{n-1})\in K^n \\ \exists\omega\in\Omega(f),\ \omega_i=H_i,\ i=0,\cdots,n-1}} p^{-n\alpha} \\
&= \lim_{n\to\infty} C_n(\Omega(f))p^{-n\alpha} \\
&\leq \lim_{n\to\infty} pn\sharp\mathbf{S}(f)^* p^{-n\alpha} \\
&= 0
\end{aligned}$$

for any $\alpha > 0$. Thus, $\dim \Omega(f) = 0$. □

Problem: it seems to be true that if $f \in K((x))$ is algebraic, then the sequence $(\{f^m\})_{m\geq 0}$ has a distribution which is either δ_0 or continuous. We do not have a proof of this assertion.

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