Sequence entropy and the maximal pattern complexity of infinite words

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Abstract: For an infinite word $\alpha = \alpha_0\alpha_1\alpha_2\cdots$, over a finite alphabet A we define the maximal pattern complexity by

$$p^*_\alpha(k) = \sup_\tau \sharp\{\alpha_{n+\tau(0)}\alpha_{n+\tau(1)}\cdots\alpha_{n+\tau(k-1)};\ n = 0,1,2,\cdots\}$$

where the "sup" is taken over all subsequences $0 = \tau(0) < \tau(1) < \cdots < \tau(k-1)$ of integers of length k. We prove that α is eventually periodic if and only if $p^*_\alpha(k) \le 2k-1$ for some k. Infinite words α with $p^*_\alpha(k) = 2k$ for any k are called pattern Sturmian words and are studied.

1 Introduction

An increasing sequence of integers

$$\tau : 0 = \tau(0) < \tau(1) < \cdots < \tau(k-1)$$

with $k = 1,2,\cdots$ is called a *window* of size k. For $k = 1,2,\cdots$, we denote by $\langle k\rangle$ the window of size k such that

$$\langle k\rangle(i) = i \quad (i = 0,1,\cdots,k-1).$$

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Let $\alpha = \alpha_0\alpha_1\alpha_2\cdots$ be an infinite word over a finite alphabet A with $\sharp A \geq 2$, where $\sharp A$ denotes the number of elements in A. Let τ be a window of size k. We denote by $\alpha[n+\tau]$ the word $\alpha_{n+\tau(0)}\alpha_{n+\tau(1)}\cdots\alpha_{n+\tau(k-1)}$ over A of length k. A finite word $\eta_0\eta_1\cdots\eta_{k-1}$ is called a *τ-factor* of α if $\eta_0\eta_1\cdots\eta_{k-1} = \alpha[n+\tau]$ for some $n = 0, 1, 2, \cdots$. The set of τ-factors of α is denoted by $F_\alpha(\tau)$. We also denote $F_\alpha(k) := F_\alpha(\langle k\rangle)$.

We define the *complexity* $p_\alpha(k)$ in the usual sense and the *maximal pattern complexity* $p^*_\alpha(k)$ as a function on $k \in \{1, 2, 3, \cdots\}$ by

$$\begin{aligned} p_\alpha(k) &= \sharp F_\alpha(k) \\ p^*_\alpha(k) &= \sup_\tau \sharp F_\alpha(\tau) \end{aligned}$$

where the "sup" is taken over all windows τ of size k.

For windows τ and τ' of size k and $k+1$, respectively, such that $\tau(i) = \tau'(i)$ for $i = 0, 1, \cdots, k-1$, we call τ' an *immediate extension* of τ.

An infinite word α is called *recurrent* if for any $L \geq 1$, there exists $M \geq 1$ such that

$$\alpha_i = \alpha_{i+M} \quad (i = 0, 1, \cdots, L-1). \tag{1}$$

It is easy to see that if α is recurent, then there exist infinitely many M's as above.

It is known that α is eventually periodic if and only if $p_\alpha(k) \leq k$ for some k. In this paper, we prove the following theorem.

Theorem 1. *An infinite word α over a finite set A is eventually periodic if and only if $p^*_\alpha(k) \leq 2k-1$ for some $k = 1, 2, \cdots$.*

Proof. The "only if" part holds since if α is eventually periodic, then $p^*_\alpha(k)$ is bounded in $k = 1, 2, \cdots$.

If $p^*_\alpha(1) = 1$, then α is clearly periodic. Assume that $p^*_\alpha(1) \geq 2$ and $p^*_\alpha(k) \leq 2k-1$ for some $k = 2, 3, \cdots$. Then, there exists $k = 1, 2, \cdots$ such that $p^*_\alpha(k+1) \leq p^*_\alpha(k) + 1$. Let τ be a window of size k such that $\sharp F_\alpha(\tau) = p^*_\alpha(k)$. Then we have

$$\sharp F_\alpha(\tau') \leq p^*_\alpha(k+1) \leq p^*_\alpha(k) + 1 = \sharp F_\alpha(\tau) + 1$$

for any immediate extension τ' of τ. Thus, our theorem follows from the following Theorem 2. □

Theorem 2. *For an arbitrary infinite word α over a finite set A, if there exists $k = 1, 2, 3, \cdots$ and a window τ of size k such that*

$$\sharp F_\alpha(\tau') \leq \sharp F_\alpha(\tau) + 1$$

holds for any immediate extension τ' of τ, then α is eventually periodic.

The proof differs according to whether α is recurrent (Section 3) or not (Section 4).

An infinite word α with $p_\alpha(k) = k + 1$ for any k is known as a *Sturmian word* and has been studied extensively (Valerie Berthé [1] for a nice recent survey). We prove that for a Sturmian word α, $p^*_\alpha(k) = 2k$ holds for any k. Moreover, any infinite word of the labels given by a partition into 2 nonempty intervals of the circle along an orbit of an irrational rotation has this property (Example 2). An infinite word α with $p^*_\alpha(k) = 2k$ for any k is called a *pattern Sturmian word*. Another family of pattern Sturmian words is given in Example 5.

It follows from Kushnirenko [2] that an infinite word α which induces a dynamical system with a partially continuous spectrum satisfies that $\limsup_{k\to\infty}(1/k) \log p^*_\alpha(k) > 0$ (Corollary 1). In fact, Thue-Morse sequence α satisfies that $p^*_\alpha(k) = 2^k$ (Example 1), while $p_\alpha(k)$ is known to be of linear order in k.

2 Sequence entropy

Let A be a finite set with $\sharp A \geq 2$. Let $\mathbb{N} := \{0, 1, 2, \cdots\}$ and $A^{\mathbb{N}}$ be the product space. Let X_n ($n \in \mathbb{N}$) be the projection $A^{\mathbb{N}} \to A$ defined by $X_n(\alpha) = \alpha(n)$ for any $\alpha \in A^{\mathbb{N}}$. Let T be the shift on the space $A^{\mathbb{N}}$ and μ be a T-invariant probability Borel measure on $A^{\mathbb{N}}$. Let $\tau : 0 = \tau(0) < \tau(1) < \tau(2) < \cdots$ be an infinite sequence of integers. We define a *sequence entropy* $h(\mu, \tau)$ of μ with respect to τ by

$$h(\mu, \tau) := \limsup_{k \to \infty} \frac{1}{k} H(X_{\tau(0)}, X_{\tau(1)}, \cdots, X_{\tau(k-1)})$$

where $X_0, X_1, X_2, \cdots$ are considered as random variables on the probability space $(A^{\mathbb{N}}, \mu)$ and $H(\cdots)$ is the Shannon's entropy of random variables.

Theorem 3 (Kushnirenko[2]). *The sequence entropy $h(\mu, \tau)$ is 0 for any τ if and only if the dynamical system $(A^{\mathbb{N}}, \mu, T)$ has a discrete spectrum, that is, the eigen-functions with respect to the isometry on $L^2(A^{\mathbb{N}}, \mu)$ induced by the shift T span the whole space $L^2(A^{\mathbb{N}}, \mu)$.*

Corollary 1. *For $\alpha \in A^{\mathbb{N}}$, assume that*

$$\mu_\alpha := w\text{-}\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i \alpha}$$

exists, where δ_x is the unit measure at $x \in A^{\mathbb{N}}$ and the "w-lim" implies the weak limit on the space of measures. Assume further that the dynamical system $(A^{\mathbb{N}}, \mu_\alpha, T)$ has a partially continuous spectrum. That is, it has not a discrete spectrum. Then, we have

$$\limsup_{k \to \infty} (1/k) \log p_\alpha^*(k) > 0.$$

Proof. Assume that $(A^{\mathbb{N}}, \mu_\alpha, T)$ has a partially continuous spectrum. We consider X_n's as random variables on the probability space $(A^{\mathbb{N}}, \mu_\alpha)$. Let τ be a window of size k. Since the random variable $X_{\tau(0)} X_{\tau(1)} \cdots X_{\tau(k-1)}$ on the space of words over A of length k has a distribution which is supported by $F_\alpha(\tau)$, we have

$$\begin{aligned} h(\mu, \tau) &= \limsup_{k \to \infty} \frac{1}{k} H(X_{\tau(0)}, X_{\tau(1)}, \cdots, X_{\tau(k-1)}) \\ &\leq \limsup_{k \to \infty} \frac{1}{k} \log \sharp F_\alpha(\tau) \\ &\leq \limsup_{k \to \infty} \frac{1}{k} \log p_\alpha^*(k). \qquad (2) \end{aligned}$$

By Theorem 3, there exists an infinite sequence $\tau : 0 = \tau(0) < \tau(1) < \tau(2) < \cdots$ such that $h(\mu, \tau) > 0$. Therefore, by (2)

$$\limsup_{k\to\infty}(1/k)\log p^*_\alpha(k) > 0.$$

□

Remark 1. The converse of Corollay 1 is not true. In fact, in [4], it is proved that for any irrational number θ, there exists a closed set S in $\mathbb{R}/\mathbb{Z}$ such that the symbolic representation α of the rotation by angle θ with respect to the patition $\{S, S^c\}$ has the maximal pattern complexity $p^*_\alpha(k) = 2^k$ $(k = 1, 2, \cdots)$ for almost all starting point of the rotation.

Example 1. Let $\alpha = 0110100110010110\cdots$ be the Thue-Morse sequence over $A := \{0, 1\}$. Then, we have $p^*_\alpha(k) = 2^k$ for $k = 1, 2, \cdots$.

Proof. Note that $\alpha_n \equiv \xi(n) \pmod 2$, where $\xi(n)$ is the number of digits 1 in the 2-adic representation of n. Let $\tau(i) = 2^{2i} - 1$ $(i = 0, 1, 2, \cdots)$. For any $k = 1, 2, \cdots$, let U_k be the set of $u := \sum_{i=0}^k u_i 2^{2i} + 1$ with $(u_0, u_1, \cdots, u_k) \in \{0, 1\}^{k+1}$ satisfying that $\sum_{i=0}^k u_i \equiv 1 \pmod 2$. Then for any $u \in U_k$ and $i < k$, we have

$$\xi(u + \tau(i)) \equiv u_i \pmod 2.$$

Hence we have

$$\alpha_{u+\tau(0)}\alpha_{u+\tau(1)}\cdots\alpha_{u+\tau(k-1)} = u_0u_1\cdots u_{k-1}.$$

Since $\{u_0u_1\cdots u_{k-1};\ u \in U_k\} = \{0, 1\}^k$, we have

$$p^*_\alpha(k) = \sharp\{u_0u_1\cdots u_{k-1};\ u \in U_k\} = 2^k.$$

□

3 Recurrent case

We prove Theorem 2 in the case that α is recurrent. Let α be a recurrent infinite word over a finite set A. Assume that there exist $k = 1, 2, \cdots$ and a window τ of size k such that

$$\sharp F_\alpha(\tau') \le \sharp F_\alpha(\tau) + 1$$

holds for any immediate extension τ' of τ.

If $\sharp F_\alpha(1) = 1$, then α is clearly periodic. Therefore, assume that $\sharp F_\alpha(1) \geq 2$.

Suppose that α is not eventually periodic. Take L such that

$$\{\alpha[n+\tau];\ n = 0, 1, \cdots, L-1\} = F_\alpha(\tau). \tag{3}$$

Since α is recurrent, there exists M with $M > \tau(k-1)$ such that (1) holds for this L. Define a window τ' of size $k+1$ which is an immediate extension of τ by

$$\tau'(i) = \begin{cases} \tau(i) & (i = 0, 1, \cdots, k-1) \\ M & (i = k). \end{cases}$$

Then, for any n with $0 \leq n \leq L-1$ we have

$$\alpha_{n+\tau'(k)} = \alpha_{n+M} = \alpha_n = \alpha_{n+\tau'(0)}. \tag{4}$$

For each $b \in F_\alpha(1)$, let

$$G_b = \{\xi_1\xi_2\cdots\xi_{k-1};\ b\xi_1\xi_2\cdots\xi_{k-1} \in F_\alpha(\tau)\}$$

Then by (3) and (4), we have

$$\bigcup_{b \in F_\alpha(1)} bG_b b \subset F_\alpha(\tau').$$

Supose that

$$F_\alpha(\tau') = \bigcup_{b \in F_\alpha(1)} bG_b b.$$

Then, the first and last letter of each word in $F_\alpha(\tau')$ are equal. Therefore, for any $n \in \mathbb{N}$, we have $\alpha_n = \alpha_{n+M}$ since $\tau'(0) = 0$ and $\tau'(k) = M$. Thus, α is periodic, contradicting with our supposition.

Suppose that

$$F_\alpha(\tau') = \bigcup_{b \in F_\alpha(1)} bG_b b \cup \{\xi\}$$

with some word ξ of length $k+1$ beginning by letter c and ending by letter d. If $c = d$, we have the same contradiction as above. Assume that $c \neq d$. Then, the possible combinations of $\alpha_n \alpha_{n+M}$ for each $n \in \mathbb{N}$ are limited to bb for each $b \in F_\alpha(1)$ or cd. For $n \in \mathbb{N}$ with $0 \leq n < M$, consider the infinite word $\alpha_n \alpha_{n+M} \alpha_{n+2M} \cdots$. Then, we have only the following possibilities for it:

$$\begin{array}{ll} bbbb\cdots & (b \in F_\alpha(1)) \\ \underbrace{c\cdots c}_{m} ddd\cdots & (m \in \mathbb{N}). \end{array}$$

Therefore, for each $n \in \mathbb{N}$ with $0 \leq n < M$, there exists $m(n)$ such that $\alpha_{i+M} = \alpha_i$ for any i with $i \equiv n \pmod{M}$ and $i \geq m(n)$. Let $m_0 = \max_{0 \leq n < M} m(n)$. Then, $\alpha_{i+M} = \alpha_i$ holds for any $i \geq m_0$, which implies that α is eventually periodic, contradicting with our supposition.

Thus, $F_\alpha(\tau')$ must contain at least 2 more elements which are not contained in $\bigcup_{b \in F_\alpha(1)} bG_b b$. Since $\bigcup_{b \in F_\alpha(1)} bG_b = F_\alpha(\tau)$, we have $\sharp F_\alpha(\tau') \geq \sharp F_\alpha(\tau) + 2$, contradicting with our assumption which completes the proof of Theorem 2 in the case that α is recurrent.

4 Nonrecurrent case

Now assume that α is not recurrent. We may further assume that the shift of α, $T^n\alpha = \alpha_n \alpha_{n+1} \alpha_{n+2} \cdots$ is not recurrent for any $n \in \mathbb{N}$.

This is because if $T^n\alpha$ is recurrent for some $n \in \mathbb{N}$ and α is not eventually periodic, then it is already prove that for any window τ of size k, there exists τ' which is an immediate extension of τ such that

$$\sharp F_{T^n\alpha}(\tau') \geq \sharp F_{T^n\alpha}(\tau) + 2,$$

which implies that

$$\sharp F_\alpha(\tau') \geq \sharp F_\alpha(\tau) + 2$$

since in the following commutative diagram, the projections π, φ are surjective while the inclusions i and j are injective.

$$\begin{array}{ccc} F_\alpha(\tau') & \xrightarrow{\pi} & F_\alpha(\tau) \\ i \uparrow & & \uparrow j \\ F_{T^n\alpha}(\tau') & \xrightarrow{\varphi} & F_{T^n\alpha}(\tau) \end{array}$$

Assume that
(i) $T^n\alpha$ *is not recurrent for any* $n \in \mathbb{N}$, and that
(ii) *there exists* $k = 1, 2, 3, \cdots$ *and a window* τ *of size* k *such that*

$$\sharp F_\alpha(\tau') \leq \sharp F_\alpha(\tau) + 1$$

holds for any immediate extension τ' *of* τ.
We claim that α is eventually periodic from these assumptions. Suppose to the contrary that α is not eventually periodic; we shall derive a contradiction.

Take any K with $K > 2\tau(k-1)$.

Since α is not eventually periodic, there exists at least one element $\xi \in F_\alpha(K)$ having more than one extension in $F_\alpha(K+1)$.

Suppose that there exists $\xi = \xi_0\xi_1 \cdots \xi_{K-1} \in F_\alpha(K)$ such that ξ has more than 2 extensions in $F_\alpha(K+1)$. Define a window τ' which is an immediate extension of τ with $\tau'(k) = K$. Then, $\xi_{\tau(0)}\xi_{\tau(1)} \cdots \xi_{\tau(k-1)} \in F_\alpha(\tau)$ has more than 2 extensions in $F_\alpha(\tau')$ which implies that $\sharp F_\alpha(\tau') \geq \sharp F_\alpha(\tau) + 2$, contradicting with our assumption (ii).

Suppose that there exists $\xi, \eta \in F_\alpha(K)$ with $\xi \neq \eta$ such that both of ξ and η have 2 extensions in $F_\alpha(K+1)$. Then, there exists $n \in \mathbb{N}$ with $0 \leq n < K - \tau(k-1)$ such that $\xi[n+\tau] \neq \eta[n+\tau]$. Define a window τ' which is an immediate extension of τ by $\tau'(k) = K - n$. Since both of $\xi[n+\tau]$ and $\eta[n+\tau]$ have 2 extensions in $F_\alpha(\tau')$, we have $\sharp F_\alpha(\tau') \geq \sharp F_\alpha(\tau) + 2$, contradicting with our assumption (ii).

Thus, we have

Lemma 1. *For any* $K > 2\tau(k-1)$, *there exists a unique element in* $F_\alpha(K)$ *which has exactly 2 extensions in* $F_\alpha(K+1)$ *while any other element in* $F_\alpha(K)$ *has a unique extension in* $F_\alpha(K+1)$.

Fix $K > 2\tau(k-1)$. There exists a $\xi \in F_\alpha(K)$ which occurs at least twice in α. Hence, there exist $m > n \geq 0$ such that both $T^m\alpha$ and $T^n\alpha$ begins in ξ. Since $T^n\alpha$ is not recurrent, there exists a shortest prefix u' of $T^n\alpha$ which occurs only once in $T^n\alpha$. Since ξ is a prefix of $T^n\alpha$ which occurs twice in $T^n\alpha$, it follows that the length of u' is larger than K, the length of ξ. Let $L+1$ be the length of u'. Then, $L \geq K > 2\tau(k-1)$ and by the minimality of the length of u', the prefix u of u' of length L occurs more than once in $T^n\alpha$. But since u' occurs only once, it follows that there exists letters $a \neq b$ such that ua and ub are in $F_\alpha(L+1)$. Thus by Lemma 1, $u \in F_\alpha(L)$ has exactly two extenions in $F_\alpha(L+1)$ (namely, ua and ub one of which is u'), and all other elements in $F_\alpha(L)$ have a unique extension in $F_\alpha(L+1)$. Since u' does not occur in $T^{n+1}\alpha$, all elements in $F_{T^{n+1}\alpha}(L)$ have a unique extension in $F_{T^{n+1}\alpha}(L+1)$, which implies that $T^{n+1}\alpha$ is eventually periodic. Thus, we have a contradiction that α is eventually periodic, which completes the proof.

5 Examples

Example 2. Let θ be an irrational number and $[a,b)$ be an interval with $0 < b-a < 1$. Define an infinite word $\alpha = \alpha_0\alpha_1\alpha_2\cdots$ over $\{0,1\}$ by

$$\alpha_n = \begin{cases} 1 & n\theta \in [a,b) \pmod 1 \\ 0 & n\theta \notin [a,b) \pmod 1. \end{cases}$$

Then, we have $p^*_\alpha(k) = 2k$ for any $k = 1,2,3,\cdots$. Note that α is recurrent.

Proof. We consider $I_1 := [a,b)$ as a subset in the torus $\mathbb{R}/\mathbb{Z}$ and I_0 be its complement in the torus. Thus, $\mathcal{I} := \{I_0, I_1\}$ is a partition of the torus. Take an arbtrary window $\tau :\ 0 = \tau(0) < \tau(1) < \cdots < \tau(k-1)$ of size $k \geq 1$ and define the points on the torus :

$$\begin{array}{l} a-\tau(0),\ a-\tau(1)\theta,\ a-\tau(2)\theta,\ \cdots,\ a-\tau(k-1)\theta \\ b-\tau(0),\ b-\tau(1)\theta,\ b-\tau(2)\theta,\ \cdots,\ b-\tau(k-1)\theta \end{array}$$

considered in modulo 1. Let K be the number of different points in this list. Then, K is the number of nonempty elements in the partition

$$(\mathcal{I}-\tau(0)\theta) \vee (\mathcal{I}-\tau(1)\theta) \vee \cdots \vee (\mathcal{I}-\tau(k-1)\theta),$$

where "$\vee$" implies the common refinement of the partitions, if each element in the partition is connected, which we assume since otherwise, the number of non-empty elements in the partition is less than K. If $n\theta$ is in a element of this partition, say

$$n\theta \in (I_{c_0} - \tau(0)\theta) \cap (I_{c_1} - \tau(1)\theta) \cap \cdots \cap (I_{c_{k-1}} - \tau(k-1)\theta) \pmod 1,$$

then we have $\alpha_{n+\tau(i)} = c_i$ $(i = 0, 1, \cdots, k-1)$. Since the set of $\{n\theta;\ n = 0, 1, 2, \cdots\}$ is dense in the torus, this implies that

$$K = \sharp\{\alpha_{n+\tau(0)}\alpha_{n+\tau(1)} \cdots \alpha_{n+\tau(k-1)};\ n \in \mathbb{N}\}.$$

Since $K \leq 2k$ for any τ and $K = 2k$ for some τ, we have $p_\alpha^*(k) = 2k$ for any $k = 1, 2, 3, \cdots$. It is clear that α is recurrent. □

Example 3. Let α be a Sturmian word over $\{0, 1\}$. Then, it is known [1] that it is represented as Example 2 with $b - a = \theta \pmod 1$. Thus, α is recurrent and $p_\alpha^*(k) = 2k$ for any $k = 1, 2, 3, \cdots$.

Example 4. (Yu-Mei Xue [3]) Let θ and η be rationally independent irrational numbers. Consider the rotation $R : (\mathbb{R}/\mathbb{Z})^2 \to (\mathbb{R}/\mathbb{Z})^2$ by $(x, y) \mapsto (x + \theta, y + \eta)$. Let S be a closed convex subset of $(0, 1)^2$ with diameter less than $1/2$. Let α be the infinite word over $\{0, 1\}$ defined by

$$\alpha_n = \begin{cases} 1 & (n\theta, n\eta) \in S \\ 0 & (n\theta, n\eta) \notin S. \end{cases}$$

Then, $p_\alpha^*(k) \leq k^2 - k + 2$ $(k = 1, 2, 3, \cdots)$. Moreover, if S is a closed ball with diameter less than $1/2$ and larger than 0, then $p_\alpha^*(k) = k^2 - k + 2$ $(k = 1, 2, 3, \cdots)$.

Example 5. Let $0 < c_1 < c_2 < c_3 < \cdots$ be a sequence of integers such that $c_{k+1} > 2c_k$ $(k = 1, 2, 3, \cdots)$. Define an infinite word α over $\{0, 1\}$ by

$$\alpha_n = \begin{cases} 1 & n = c_k \text{ for some } k = 1, 2, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have $p_\alpha^*(k) = 2k$ $(k = 1, 2, 3, \cdots)$. Note that α is not recurrent.

Proof. For any $k = 1, 2, 3, \cdots$ and any window $\tau : \ 0 = \tau(0) < \tau(1) < \tau(2) < \cdots < \tau(k-1)$ of size k, it is clear that any word over $\{0, 1\}$ with length k which contains 1 at most once belongs to $F_\alpha(\tau)$. There are $k + 1$ such words.

Let Λ_k be the set of words over $\{0, 1\}$ of length k containing 1 at least twice belonging to $F_\alpha(\tau)$. For $\eta := \eta_0\eta_1 \cdots \eta_{k-1} \in \Lambda_k$, let $\phi(\eta)$ be the maximum j such that $\eta_j = 1$. Let $\eta \in \Lambda_k$ satisfy that $\phi(\eta) = j$. Then, $\eta_j = 1$ and there exists i with $0 \leq i < j$ such that $\eta_i = 1$. Therefore, there exists $n \in \mathbb{N}$ such that $n+\tau(i) = c_a,\ n+\tau(j) = c_b$ for some $a,\ b = 1, 2, 3, \cdots$. Let d be the minimum d such that $\tau(j) \leq c_d$.

We prove that $b = d$. Suppose that this is not the case. Since $\tau(j) \leq c_b$ and $d \neq b$, $b \geq d + 1$ holds. Since $c_a - \tau(i) = c_b - \tau(j)$ and $\tau(j) > \tau(i)$, we have $c_a < c_b$. On the other hand, since

$$c_a = c_b - \tau(j) + \tau(i) \geq c_b - c_d \geq c_b - c_{b-1} > c_{b-1}.$$

we have $c_a \geq c_b$, which contradicts with $c_a < c_b$.

Thus, we have $b = d$. This implies that $n = c_d - \tau(j)$ and n is determined by j. Thus, $\eta = \alpha_{n+\tau(0)}\alpha_{n+\tau(1)} \cdots \alpha_{n+\tau(k-1)}$ holds and η is determined by $\phi(\eta)$. This implies that Λ_k has at most $k - 1$ elements, since the image of ϕ is contained $\{1, 2, \cdots, k-1\}$. Thus, the number of words with length k which appears in α through τ is at most $k + 1 + k - 1 = 2k$.

For any $k = 1, 2, 3, \cdots$, define a window $\tau : \ 0 = \tau(0) < \tau(1) < \tau(2) < \cdots < \tau(k-1)$ of size k inductively by

$$\begin{aligned} \tau(0) &= 0, \quad \tau(1) = c_1 \\ \tau(i+1) &= c_{2i+1} - c_{2i} + \tau(i) \quad (i = 1, 2, \cdots, k-2) \end{aligned}$$

Then, Λ_k defined above for this τ consists of $k - 1$ elements :

$$\begin{array}{r} 1\ 1\ 0\ 0\ 0\ \ \cdots\ \ 0\ 0\ 0 \\ 1\ 1\ 0\ 0\ \ \cdots\ \ 0\ 0\ 0 \\ 1\ 1\ 0\ \ \cdots\ \ 0\ 0\ 0 \\ 1\ 1\ \ \cdots\ \ 0\ 0\ 0 \\ \cdots \\ * \qquad\quad 1\ 1\ 0 \\ 1\ 1 \end{array}$$

Thus, we have $\sharp F_\alpha(k) = k+1+k-1 = 2k$. This completes the proof that $p^*_\alpha(k) = 2k$ $(k = 1, 2, 3, \cdots)$. □

There are unsolved problems:

Problem 1: What is the general structure of recurrent pattern Sturmian words?

Problem 2: Is it true that an exponential growth in maximal pattern complexity of an infinite word over 2 letters forces the full maximal pattern complexity 2^k?

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