Stochastic analysis based on deterministic Brownian motion (2nd version)

(Israel J. Math. 125 (2001), pp.317-346)

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October 21, 2010

Abstract: A deterministic version of the Itô calculus is presented. We consider a model $Y_t = H(\mathbf{N}_t, t)$ with a deterministic Brownian $\mathbf{N}_t$ and an unknown function H. We predict Y_c from the observation $\{Y_t;\ t \in [a, b]\}$, where $a < b < c$. We prove that there exists an estimator $\hat{Y}_t$ based on the observation such that $E[(\hat{Y}_t - Y_c)^2] = O((c-b)^2)$ as $c \downarrow b$.

1 Introdution

Deterministic Brownian motions are stochastic processes with non-correlated, stationary and strictly ergodic increments having 0-entropy and 0-expectation. The self-similarity of order 1/2 follows from these properties. Such processes have a lot of variety and have different properties. It is not the case of the Brownian motion where the process is characterized as a process with stationary and independent increments with 0-expectation and standard variance.

Among the deterministic Brownian motions, the simplest one is the N-process $(\mathbf{N}_t;\ t \in \mathbf{R})$ which is defined by the author in Example 8 of [1]. It comes from a piecewise linear function called N_1-function in Figure 1. It is time reversible.

The aim of this paper is to develop stochastic analysis based on N-process. We consider a process $Y_t = H(\mathbf{N}_t, t)$, where the function

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$H(x, s)$ is twice continuously differentible in x and once continuously differentible in s and $H_x(x, s) > 0$. The function H is consisered completely unknown except for these properties. We want to predict the value Y_c from the observation $Y_J := \{Y_t;\ t \in J\}$, where $J = [a, b]$ and $a < b < c$. We prove in Theorem 9 that there exists a estimator $\hat{Y}_c$ such that

$$\mathrm{E}[(\hat{Y}_c - Y_c)^2] = o((c-b)^2) + O(\frac{(c-b)^2}{b-a}) \tag{1}$$

as $c \downarrow b$ with the following $C(b)$ as the constant in $O(\ )$:

$$C(b) := 50 \sup_{|x| \leq |b|^{1/2}} |H_x(x, b)|^2. \tag{2}$$

One of the motivations of our paper is given by Benoit B. Mandelbrot [2], who mentioned that the simulation of stock market by the Brownian motion contains too much randomness. Actual market has a strong negative correlation between the fluctuations of price on a day and the next day. He is suggesting to use the N-shaped function as the base of the simulation.

Our model has a lot of similarities to the Itô process. For example, we have a Itô formula (Theorem 4). Nevertheless, there is a big difference between them. Our process has 0-entropy while Itô process has ∞-entropy. Therefore, we have much better possibility of predicting the future. Theoritically, if we have complete informations of the function H, and have comlete data of Y_t in the past, we should be able to predict the future without error. But the actual setting is with the unknown function H and the limited observation Y_t for a bounded interval J. The best we can do is the order $O((c-b)^2)$ in the above estimate (1), while $O(c-b)$ in the case of Itô process.

A sample path from N-process repeats N_1-function in various scales. The main idea for the prediction called **synchronization** is to find out the positions and the scales of appearances of N_1-function in the sample path. An appearance of N_1-function in a sample path is a part of bigger N_1-functions while containing smaller ones. Along the 3 line segments in an appearance of N_1-function, the sample path either increases at the first part, then decreases and increases, or decreases at the first part, then increases and decreases. Thus, it has a strong correlation along the synchronized intervals, while the process itself has noncorrelated increments.

Another motivation is to create a sample path of Brownian motion in a deterministic way without using random mechanism. Our N-process is strictly ergodic so that any chosen path realizes probablistic properties of the process. We don't need a randomization procedure but just take one, for example, N_∞ function itself. Of course, it is not exactly like a path of Brownian motion, but shares the quadratic structure with Brownian motion. If we take a derivative in some sense of the sample path, we get a white noise. thus, our N-process provides a method of generating a random number.

2 N-process

We consider the **N-process** $(\mathbf{N}_t;\ t \in \mathbf{R})$ which is the stochastic process defined in Example 8 of [1] for $\alpha = 1/2$. We repeat the definition in a little different way as follows.

Define a continuous piecewise linear function N_1 (see Figure 1) on the interval $[0,1]$ by

$$N_1(x) = \begin{cases} \frac{3}{2}x & 0 \le x \le 4/9 \\ -3x+2 & 4/9 \le x \le 5/9 \\ \frac{3}{2}x - \frac{1}{2} & 5/9 \le x \le 1. \end{cases}$$

Let N_2 be the continuous piecewise linear function on $[0,1]$ obtained by replacing 3 line segments in N_1 by self-affine images of N_1 or $-N_1$ keeping the 2 end points fixed, that is

$$N_2(x) = \begin{cases} \frac{2}{3}N_1(\frac{9}{4}x) & 0 \le x \le 4/9 \\ \frac{2}{3} - \frac{1}{3}N_1(9x-4) & 4/9 \le x \le 5/9 \\ \frac{1}{3} + \frac{2}{3}N_1(\frac{9}{4}x - \frac{5}{4}) & 5/9 \le x \le 1. \end{cases}$$

Let N_3 be the the continuous piecewise linear function on $[0,1]$ obtained by replacing 9 line segments in N_2 by self-affine images of N_1 or $-N_1$ as before. In the same way, we obtain N_n from N_{n-1} for $n = 4, 5, \cdots$. For covenience, we define N_0 by $N_0(t) = t$ for any $t \in [0,1]$.

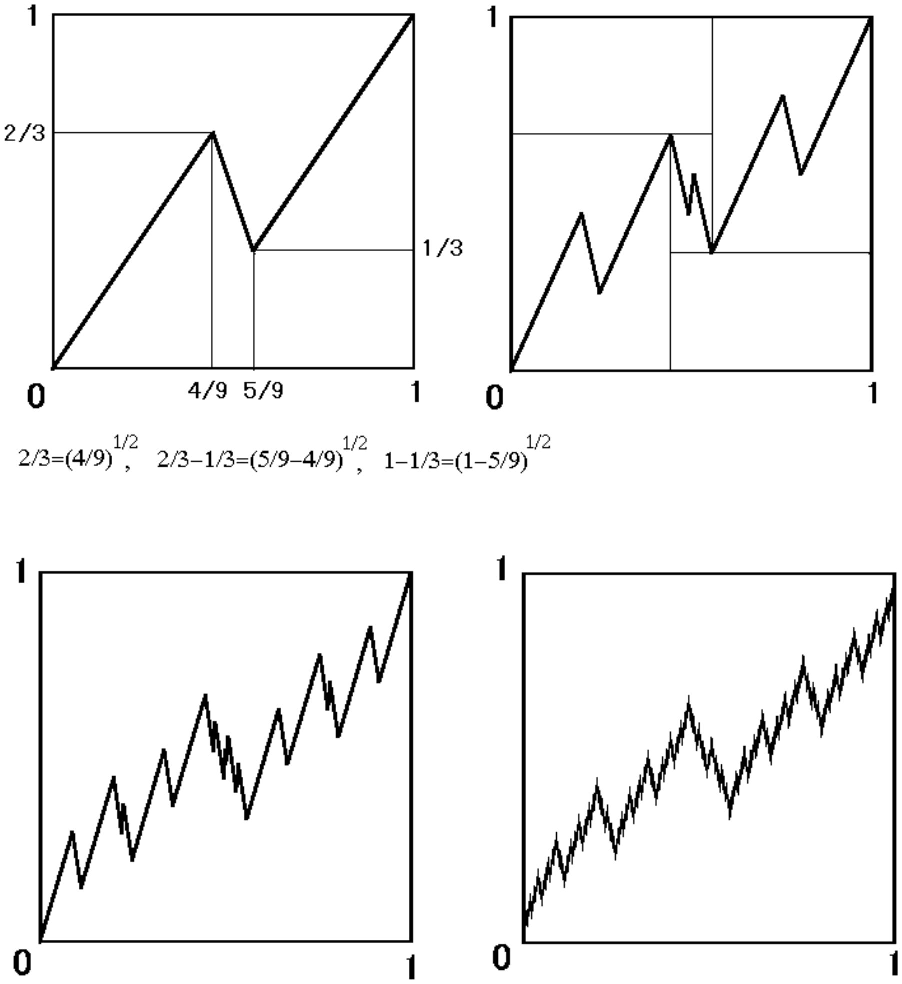

Figure 1: N_1, N_2, N_3 and N_∞

We prove that the function N_n converges pointwise as n tends to infinity to a continuous function, say N_∞ on $[0,1]$. Let $a, b \in [0,1]$ with $a < b$. The interval $[a,b]$ is called a **synchronized interval of level** n if $\overline{(a, N_n(a))(b, N_n(b))}$ is one of the 3^n line segments consisting of the graph of the function N_n for $n = 0, 1, 2, \cdots$. In this case, it holds for any $m \geq n$ that

1. $N_m(a) = N_n(a)$ and $N_m(b) = N_n(b)$,
2. $N_n(a) < N_m(t) < N_n(b)$ or $N_n(a) > N_m(t) > N_n(b)$ for any $t \in (a,b)$,
3. $|N_n(b) - N_n(a)| = |b-a|^{1/2}$, and
4. $b - a = \left(\frac{4}{9}\right)^i \left(\frac{1}{9}\right)^{n-i}$ for some $i = 0, 1, \cdots, n$.

Take any $t \in [0,1]$. For any $\varepsilon > 0$, there exists n and a synchronized interval of level n, say $[a,b]$ with $t \in [a,b]$ and $|b-a| < \varepsilon^2$. Then for any $m, m' \geq n$,

$$|N_m(t) - N_{m'}(t)| \leq |N_n(b) - N_n(a)| = |b-a|^{1/2} < \varepsilon.$$

Thus, $N_m(t)$ converges as $m \to \infty$. The limit will be denoted by $N_\infty(t)$.

Let us prove the continuity of the function N_∞. Take any $s, t \in [0,1]$ with $0 < t - s \leq (1/9)^n$ for some $n = 1, 2, \cdots$. Then there exists 2 neighboring synchronized intervals of level n, say $[a,b]$ and $[b,c]$ such that $[s,t] \subset [a,c]$. Then we have

$$\begin{aligned} & |N_\infty(t) - N_\infty(s)| \\ \leq\ & |N_n(b) - N_n(a)| + |N_n(c) - N_n(b)| \\ =\ & |b-a|^{1/2} + |c-b|^{1/2} \leq 2\left(\frac{4}{9}\right)^{n/2} \end{aligned}$$

Thus, the function N_∞ is continuous.

We define a function $\tilde{N}_\infty : \mathbf{R} \to \mathbf{R}$ which is an extension of N_∞ by

$$\tilde{N}_\infty(t) = \begin{cases} 0 & t < 0 \\ N_\infty(t) & 0 \leq t \leq 1 \\ 1 & t > 1. \end{cases}$$

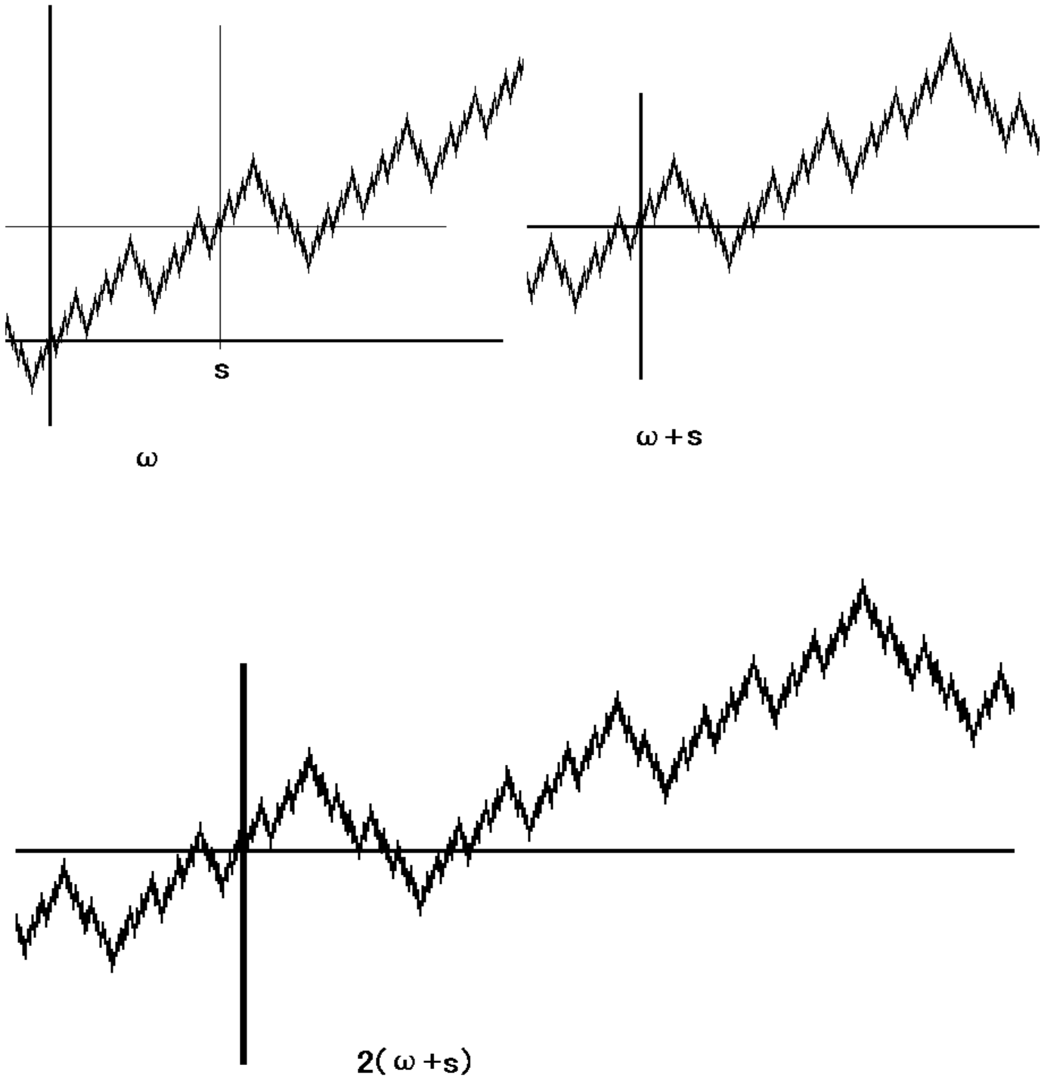

Figure 2: ω, $\omega + s$ and $2(\omega + s)$

Now we randomize $\tilde{N}_\infty$ to get the N-process $(\mathbf{N}_t;\ t \in \mathbf{R})$.

Let Θ be the set of continuos functions $\omega : \mathbf{R} \to \mathbf{R}$ with $\omega(0) = 0$. We consider Θ as a topological space with the compact open topology, that is, $\omega_n \in \Theta$ converges to $\omega \in \Theta$ as n tends to infinity if and only if $\omega_n(t)$ converges to $\omega(t)$ uniformly on each bounded set of t. For $\omega \in \Theta$ and $s \in \mathbf{R}$, we define the **addition** $\omega + s \in \Theta$ (see Figure 2) by

$$(\omega + s)(t) = \omega(s + t) - \omega(s).$$

For $\omega \in \Theta$ and $\lambda \in \mathbf{R}_+$, we define the **multiplication** $\lambda\omega \in \Theta$ by

$$(\lambda\omega)(t) = \lambda^{1/2}\omega(\lambda^{-1}t).$$

Choose $s \in [0, 1]$ randomly according to the Lebesgue measure on $[0, 1]$ and define $\tilde{N}_\infty + s$. Now take $L > 0$ and chose $\lambda \in [0, L]$ randomly according to the normilized Lebesgue measure on $[0, L]$ independently of s and define $e^\lambda(\tilde{N}_\infty + s)$. Now let L tend to infinity. We prove in Theorem 1 that the distribution of the random variable $e^\lambda(\tilde{N}_\infty + s)$ on Θ converges weakly (i.e. in the weak* sense) as L tends to infinity. Let P be the limiting distribution on Θ. Then the stochastic process $(\mathbf{N}_t;\ t \in \mathbf{R})$ on the probability space (Θ, P) is defined by $\mathbf{N}_t(\omega) = \omega(t)$ for any $\omega \in \Theta$ and $t \in \mathbf{R}$, which is called the **N-process**. Let Θ_0 be the topological support of the measure P.

Let $[a, b]$ be a synchronized interval of level i. We call it **increasing** if $N_\infty(a) < N_\infty(b)$ and **decreasing** if $N_\infty(a) > N_\infty(b)$. We call it **left, middle** or **right** if there exists a synchronized interval $[u, v]$ such that $[a, b]$ is equal to $[u, u']$, $[u', v']$ or $[v', v]$, respectively, where we put $u' = (5u+4v)/9$ and $v' = (4u+5v)/9$. For example, $[0, 1]$ is the only synchronized interval of level 0, which is increasing. There are 3 synchronized intervals of level 1, namely $[0, 4/9]$, $[4/9, 5/9]$, $[5/9, 1]$, which are increasing, decreasing and increasing, respectively and left, middle and right, respectively.

Let $X = e^\lambda(\tilde{N}_\infty + s)$ for some $s \in [0, 1]$ and $\lambda \in [0, \infty)$. Note that

$$\begin{aligned} e^\lambda &= (X(\infty) - X(-\infty))^2 \\ 1 - s &= e^{-\lambda} \min\{t;\ X(t) = X(\infty)\}, \end{aligned}$$

so that λ and s are determined by X. Let $[a,b]$ be a synchronized interval. Then we say that $[(a-s)e^{\lambda},(b-s)e^{\lambda}]$ is a **synchronized interval of** X. We also say that it is increasing, decreasing, left, middle or right synchronized interval of X if $[a,b]$ is so.

Lemma 1. (1) *$\tilde{N}_\infty(t)+\tilde{N}_\infty(1-t)=1$ for any $t\in\mathbf{R}$.*

(2) *Let $[a,b]$ be a synchronized interval. Then we have*

$$N_\infty(t)-N_\infty(a)=\xi(b-a)^{1/2}N_\infty\left(\frac{t-a}{b-a}\right)$$

for any $t\in[a,b]$, where ξ is 1 or -1 according as the interval $[a,b]$ is increasing or decreasing, respectively.

(3) *There exists a constant C such that*

$$|\tilde{N}_\infty(t)-\tilde{N}_\infty(s)|\le C|t-s|^{1/2}$$

for any $s,t\in\mathbf{R}$.

(4) *The set $K:=\{e^{\lambda}(\tilde{N}_\infty+s);\ s\in[0,1],\ \lambda>0\}$ is relatively compact in Θ.*

Remark 1. In Theorem 2, we prove that C in (3) of Lemma 1 can be taken as 1.

Proof. (1) Clear from the definitions of N_∞ and $\tilde{N}_\infty$.

(2) The graph of N_∞ restricted to the interval $[a,b]$ is the image of the graph of N_∞ by the affine transformation sending the point $(0,0)$ to $(a,N_\infty(a))$, $(0,1)$ to $(a,N_\infty(b))$, $(1,0)$ to $(b,N_\infty(a))$, and $(1,1)$ to $(b,N_\infty(b))$. Moreover, we already remarked that $N_\infty(b)-N_\infty(a)=\xi(b-a)^{1/2}$. Our conclusion follows from these properties.

(3) Assume without loss of generality that $0\le s<t\le 1$ and $t-s<1/2$, since otherwise, either the required inequality holds with $C=2$ or it follows from our case by the symmetry or with $s\vee 0$ for s and $t\wedge 1$ for t. Take the maximum n such that either there exist 2 neighboring synchronized intervals $[a,b]$ and $[b,c]$ of level n with $[s,t]\subset[a,c]$. Then we have $t-s>(1/9)((b-a)\wedge(c-b))$ since

otherwise, we can take a larger n as this. It follows that

$$\begin{aligned}
& |\tilde{N}_\infty(t) - \tilde{N}_\infty(s)| = |N_\infty(t) - N_\infty(s)| \\
\leq \;& |N_\infty(b) - N_\infty(a)| + |N_\infty(c) - N_\infty(b)| \\
= \;& |b-a|^{1/2} + |c-b|^{1/2} \\
= \;& 3((b-a) \wedge (c-b))^{1/2} \\
< \;& 9|t-s|^{1/2},
\end{aligned}$$

where we used the fact that either $c-b = 4(b-a)$ or $c-b = (1/4)(b-a)$ holds, since $[a,b]$ and $[c,d]$ are neighboring synchronized intervals of the same level (see (2) of Lamma 2).

(4) By (3), any function f in K satisfies that $|f(t) - f(s)| \leq C|t-s|^{1/2}$ for any $s, t \in \mathbf{R}$ together with $f(0) = 0$. This implies that K is relatively compact in Θ. ☐

Theorem 1. *The* N*-process introduced above is well defined and has the same distribution as the cocycle F for $\alpha = 1/2$ in* Example 8 *in* [1].

Proof. In Example 6 of [1], the weighted substition (φ, η) on $\{0,1\}$ was defined as

$$\begin{aligned}
0 &\to (0, \tfrac{4}{9})(1, \tfrac{1}{9})(0, \tfrac{4}{9}) \\
1 &\to (1, \tfrac{4}{9})(0, \tfrac{1}{9})(1, \tfrac{4}{9}).
\end{aligned}$$

Then, we defined $\Omega := \Omega(\varphi, \eta)$, the set of colored tilings associated to (φ, η) which is strictly ergodic with respect to the addition ($\mathbf{R}$-action). Let μ be the unique invariant measure on Ω with respect to the addition, which is also invarint under the multiplication ($\mathbf{R}_+$-action). Finally, we defined the 1/2-homogeneous cocycle F on Ω in Example 8 of [1]. Then it holds that

$$F(\omega, t) - F(\omega, c) = (-1)^\sigma (d-c)^{1/2} N_\infty \left(\frac{t-c}{d-c} \right) \tag{3}$$

for any $\omega \in \Omega$ and $t \in [c,d]$ if there exists a tile S of ω with color σ such that $S = (a,b] \times [c,d)$ for some a, b. For $\omega \in \Omega$, let $F(\omega)$ denote the function $\mathbf{R} \to \mathbf{R}$ such that $F(\omega)(t) = F(\omega, t)$. Then, $F(\omega) \in \Theta$.

Let μ_F be the distribution of the random variable $F(\omega)$ with values in Θ defined on the probability space (Ω, μ).

We want to prove that the process $(\mathbf{N}_t;\ t \in \mathbf{R})$ is well defined and has the distribution μ_F. For this purpose, we prove that the distribution of the random variable $X_L := e^{\lambda}(\tilde{N}_\infty + s)$ converges in the weak sense to μ_F as $L \to \infty$, where (s, λ) is a uiformly distributed random variable on $[0,1] \times [0, L]$. It is sufficient to prove that for any sequence $\{L_n;\ n = 1, 2, \cdots\}$ with $\lim_{n\to\infty} L_n = \infty$, there exists a subsequence $\{L_n^{'}\}$ of $\{L_n\}$ with $\lim_{n\to\infty} L_n^{'} = \infty$ such that the distribution of $X_{L_n^{'}}$ converges to μ_F weakly as n tends to infinity.

Take any sequence $\{L_n;\ n = 1, 2, \cdots\}$ with $\lim_{n\to\infty} L_n = \infty$. There exists a subsequence $\{L_n^{'}\}$ of $\{L_n\}$ with $\lim_{n\to\infty} L_n^{'} = \infty$ such that the distribution of $X_{L_n^{'}}$ converges weakly to, say P', as n tends to infinity by (4) of Lemma 1. We want to prove that $P' = \mu_F$.

Since Ω is strictly ergodic with respect to the addition ([1]) and the transformation $F : \Omega \to \Theta$ is continuous satisfying that $F(\omega + t) = F(\omega) + t$ $(\forall \omega \in \Omega,\ \forall t \in \mathbf{R})$, $F(\Omega)$ is strictly ergodic with respect to the addition. Hence it is sufficient to prove that

(i) P' is invariant under the addition, and

(ii) $P'(F(\Omega)) = 1$.

Let $\mathbf{L}$ be any bounded continuous funtional on Θ. Take any $t \in \mathbf{R}$ and $\eta \in \mathbf{R}_+$. Then we have

$$\begin{aligned}
&\int \mathbf{L}(\omega + t) dP'(\omega) \\
= \;& \lim_{n\to\infty} \frac{1}{L_n^{'}} \int_0^{L_n^{'}} \int_0^1 \mathbf{L}(e^{\lambda}(\tilde{N}_\infty + s) + t)) ds d\lambda \\
= \;& \lim_{n\to\infty} \frac{1}{L_n^{'}} \int_0^{L_n^{'}} \int_{te^{-\lambda}}^{1+te^{-\lambda}} \mathbf{L}(e^{\lambda}(\tilde{N}_\infty + s)) ds d\lambda \\
= \;& \int \mathbf{L}(\omega) dP'(\omega),
\end{aligned}$$

which proves (i).

Since $F(\Omega)$ is compact ([1]), to prove (ii), it is sufficient to prove that $P'(F(\Omega)_M) = 1$ for any $M > 0$, where $F(\Omega)_M$ is the set of $f \in \Theta$

such that there exists $\omega \in \Omega$ satisfying that the restrictions of f and $F(\omega)$ to $[-M, M]$ coincide.

Let $[a_L, b_L]$ be the minimal synchronized interval of X_L, if it exists, containing $[-M, M]$ and let $c_L = 0$ or 1 corresponding to whether $[a_L, b_L]$ is increasing or decreasing. Such an interval $[a_L, b_L]$ exists if and only if

$$[-M, M] \subset [-se^{\lambda}, (1-s)e^{\lambda}], \tag{4}$$

since $[-se^{\lambda}, (1-s)e^{\lambda}]$ is the unique synchronized interval of X of level 0. In this case, take $\omega \in \Omega$ such that there exists a tile S of ω with color c_L and $S = (a, b] \times [a_L, b_L)$ for some a, b. Then by Lemma 1 and (3), we have

$$F(\omega, t) - F(\omega, a_L) = X_L(t) - X_L(a_L)$$

for any $t \in [-M, M] \subset [a_L, b_L]$. Since $F(\omega, 0) = X_L(0) = 0$, we have $F(\omega, a_L) = X_L(a_L)$ by putting $t = 0$ in the above equality. Hence, we have $F(\omega, t) = X_L(t)$ for any $t \in [-M, M]$. Thus,

$$X_L \in F(\Omega)_M \tag{5}$$

if (4) holds.

Let us estimate the probability that (4) holds.

$$\begin{aligned}
& \Pr([-M, M] \subset [-se^{\lambda}, (1-s)e^{\lambda}]) \\
= \ & \Pr((s \wedge (1-s))e^{\lambda} \geq M) \\
= \ & \frac{1}{L}\int_0^L \int_0^1 1_{(s\wedge(1-s))e^{\lambda} \geq M} ds d\lambda \\
\geq \ & \frac{1}{L}\int_0^L (1 - 2Me^{-\lambda}) d\lambda \\
\geq \ & 1 - \frac{2M}{L}, \qquad (6)
\end{aligned}$$

which tends to 1 as L tends to infinity.

Since $F(\Omega)_M$ is a closed set, it holds by (5) and (6),

$$P'(F(\Omega)_M)) \geq \lim_{n\to\infty} \Pr(X_{L'_n} \in F(\Omega)_M) = 1,$$

which proves (ii). $\square$

Corollary 1. *The following statements hold.*

(1) $\Theta_0 = F(\Omega)$, *where* Θ_0 *is the topological support of the measure* P.

(2) *For any* $\theta \in \Theta_0$ *and* $a, b \in \mathbf{R}$ *with* $a < b$, *there exist* $s \in [0,1]$ *and* $\lambda \in [0, \infty)$ *such that the restriction of* θ *to the interval* $[a,b]$ *coincides with the resriction of* $e^{\lambda}(\tilde{N}_\infty + s)$ *to* $[a,b]$. *Moreover, in this case,* $[a,b] \subset [-se^{\lambda}, (1-s)e^{\lambda}]$ *holds.*

Corollary 2 ([1]). *The space* Θ_0 *is compact and invariant under the addition and the multiplication. The addition on* Θ_0 *is strictly ergodic with the unique invariant probability Borel measure* P. *Moreover,* P *is invariant under the multiplication. The entropy of the addition is* 0. *The stochastic process* $(\mathbf{N}_t;\ t \in \mathbf{R})$ *is self-similar with order* $1/2$ *and has stationary, strictly ergodic and noncorrelated increments with* 0 *entropy. Moreover,* $\mathrm{E}[\mathbf{N}_t] = 0$ *and* $\mathrm{V}[\mathbf{N}_t] = C|t|$ *for any* $t \in \mathbf{R}$, *where* $C > 0$ *is a constant. Furthermore, the process* $(\mathbf{N}_t;\ t \in \mathbf{R})$ *is time reversible.*

Remark 2. We do not know the exact value of C in Corollary 2. A numerical computation tells us that $C = 0.1243 \cdots$.

3 Synchronization

Lemma 2. (1) *For any synchronized intervals* I *and* J, *either* $I \subset J$, $I \supset J$ *or* $I^i \cap J^i = \emptyset$ *holds, where* I^i, J^i *are the sets of interior points of* I *and* J, *respectively.*

(2) *For any neighboring synchronized intervals* $[a,b]$ *and* $[b,c]$, *either* $(c-b)/(b-a) = (1/4)(4/9)^i$ *for some integer* i, *or* $(c-b)/(b-a) = 4(4/9)^i$ *for some integer* i *holds, where* i *is the level of* $[b,c]$ *relative to* $[a,b]$. *Moreover, one of them is increasing and the other is decreasing.*

Proof. (1) Clear from our construction of the function N_∞.

(2) Let $[u,v]$ be the minimal synchronized interval containing $[a,b] \cup [b,c]$ and let $[u,u']$, $[u',v']$, $[v',v]$ be the synchronized intervals of the next level, where $u' = (5u+4v)/9$, $v' = (4u+5v)/9$. Then, there are 2 cases:

Case 1 $[a,b] \subset [u,u']$ and $[b,c] \subset [u',v']$.

In this case, it holds that $b-a=(4/9)^h(4/9)(v-u)$ and $c-b=(4/9)^k(1/9)(v-u)$, so that $(c-b)/(b-a)=(1/4)(4/9)^i$ with $i:=k-h$, which is the level of $[b,c]$ relative to $[a,b]$.
Case 2 $[a,b]\subset[u',v']$ and $[b,c]\subset[v',v]$
In this case, it holds that $b-a=(4/9)^h(1/9)(v-u)$ and $c-b=(4/9)^k(4/9)(v-u)$, so that $(c-b)/(b-a)=4(4/9)^i$ with $i:=k-h$, which is the level of $[b,c]$ relative to $[a,b]$. $\square$

Lemma 3. *For any increasing (decreasing) synchronized interval* $[a,b]$, *we have* $N_\infty(a)<N_\infty(t)<N_\infty(b)$ $(N_\infty(a)>N_\infty(t)>N_\infty(b)$, *respectively) for any* $t\in(a,b)$. *In particular,* $0\le\tilde{N}_\infty(t)\le1$ *for any* $t\in\mathbf{R}$.

Proof. Let $[a,b]$ be an increasing synchronized interval of level n. Then, we remarked in Section 2 that $N_n(a)<N_m(t)<N_n(b)$ or $N_n(a)>N_m(t)>N_n(b)$ for any $t\in(a,b)$ and $m\ge n$. Since $N_n(a)=N_\infty(a)<N_\infty(b)=N_n(b)$, we have $N_\infty(a)<N_m(t)<N_\infty(b)$ for any $t\in(a,b)$ and $m\ge n$. Take any $t\in(a,b)$. There exists $m\ge n$ and a synchronized interval $[c,d]$ of level m such that $a<c\le t\le d<b$. Then, we have

$$N_\infty(a)<N_\infty(c)=N_m(c)\le N_M(t)\le N_m(d)=N_\infty(d)<N_\infty(a)$$

for any $M\ge m$. Letting $M\to\infty$, we have

$$N_\infty(a)<N_\infty(t)<N_\infty(a).$$

$\square$

Lemma 4. (1) *It holds for any* $0<t\le1$ *that* $N_\infty(t)\le t^{1/2}$. *The equality holds if and only if* $[0,t]$ *is a synchronized interval.*

(2) *It holds for any* $0\le t<1$ *that* $1-N_\infty(t)\le(1-t)^{1/2}$. *The equality holds if and only if* $[t,1]$ *is a synchronized interval.*

Proof. (1) If $t\in(4/9,5/9]$, then by Lemma 3,

$$N_\infty(t)/t^{1/2}<N_\infty(4/9)/(4/9)^{1/2}=1.$$

Let

$$a = \frac{5}{9} + \left(\frac{4}{9}\right)^3 = \frac{469}{729} \quad , \quad b = \frac{5}{9} + \left(\frac{4}{9}\right)^2 \frac{5}{9} = \frac{485}{729}$$

$$c = \frac{5}{9} + \left(\frac{4}{9}\right)^2 = \frac{61}{81} \quad , \quad d = 1 - \left(\frac{4}{9}\right)^2 = \frac{65}{81}$$

Then we have

$$\begin{aligned} N_\infty(a) &= \frac{1}{3} + \left(\frac{2}{3}\right)^3 = \frac{17}{27} = \max_{5/9 \le t \le b} N_\infty(t) \\ N_\infty(c) &= \frac{1}{3} + \left(\frac{2}{3}\right)^2 = \frac{7}{9} = \max_{b \le t \le d} N_\infty(t). \end{aligned}$$

Hence,

$$N_\infty(t)/t^{1/2} < N_\infty(a)/(5/9)^{1/2} = \frac{17/27}{(5/9)^{1/2}} < 1$$

for any $t \in (5/9, b]$, and

$$N_\infty(t)/t^{1/2} < N_\infty(c)/b^{1/2} = \frac{7/9}{(485/729)^{1/2}} < 1$$

for any $t \in (b, d]$. If $t \in (d, 1)$, then there exists $k = 2, 3, \cdots$ such that $1 - (4/9)^k < t \le 1 - (4/9)^{k+1}$, and it holds that

$$N_\infty(t)/t^{1/2} < N_\infty(1-(5/9)(4/9)^k)/(1-(4/9)^k)^{1/2} = \frac{1 - \frac{1}{3}(\frac{2}{3})^k}{(1-(4/9)^k)^{1/2}} < 1.$$

Therefore, $N_\infty(t)/t^{1/2} \ge 1$ holds only if $t = 1$ or $t \in (0, 4/9]$. For $t \in (0, 4/9]$, let $k = 1, 2, \cdots$ be such that $(4/9)^{k+1} < t \le (4/9)^k$. Then, since $[0, (4/9)^k]$ is a synchronized interval, it holds by Lemma 1 that

$$N_\infty(t)/t^{1/2} = N_\infty((9/4)^k t)/((9/4)^k t)^{1/2}.$$

Since $(9/4)^k t \in (4/9, 1]$, $N_\infty(t)/t^{1/2} \ge 1$ if and only if $(9/4)^k t = 1$. That is, $t = (4/9)^k$. This is equivalent to say that $[0, t]$ is a synchronized interval. Moreover, since the value of $N_\infty(t)/t^{1/2}$ at such t is 1, we complete the proof of (1).

(2) follows from (1) by (1) of Lemma 1. □

Lemma 5. *For any $a, b \in \mathbf{R}$ with $a < b$, $|\tilde{N}_\infty(b) - \tilde{N}_\infty(a)| \le (b-a)^{1/2}$. The equality holds if and only if $[a, b]$ is a synchronized interval.*

Proof. If $a < b < 0$ or $1 < a < b$, then $|\tilde{N}_\infty(b) - \tilde{N}_\infty(a)| = 0 < (b-a)^{1/2}$. If $a < 0 < 1 < b$, then $|\tilde{N}_\infty(b) - \tilde{N}_\infty(a)| = 1 < (b-a)^{1/2}$. If $a < 0 \le b \le 1$, then $|\tilde{N}_\infty(b) - \tilde{N}_\infty(a)| = \tilde{N}_\infty(b) \le b^{1/2} < (b-a)^{1/2}$ by Lemma 4. If $0 \le a \le 1 < b$, then $|\tilde{N}_\infty(b) - \tilde{N}_\infty(a)| = 1 - \tilde{N}_\infty(a) \le (1-a)^{1/2} < (b-a)^{1/2}$ by Lemma 4.

Finally, assume that $0 \le a < b \le 1$ and $\tilde{N}_\infty(a) = N_\infty(a)$, $\tilde{N}_\infty(b) = N_\infty(b)$. Let $[c, d]$ be the minimal synchronized interval containing $[a, b]$. We assume without loss of generality that the interval $[c, d]$ is increasing. Let $c' = (5c + 4d)/9$ and $d' = (4c + 5d)/9$. Then, the intervals $[c, c']$, $[c', d']$, $[d', d]$ are synchronized. By the assumption, $[a, b]$ is not contained in any of these intervals. Hence, there are 3 cases:

<u>Case 1</u> $a < c' < b \le d'$,
<u>Case 2</u> $c' \le a < d' < b$, and
<u>Case 3</u> $a < c' < d' < b$.

In Case 1, by Lemmas 1, 3 and 4, we have

$$\begin{aligned}
&|N_\infty(b) - N_\infty(a)| \le (N_\infty(c') - N_\infty(a)) \vee \ (N_\infty(c') - N_\infty(b)) \\
&= (c'-c)^{1/2} N_\infty\left(\frac{c'-a}{c'-c}\right) \vee \ (d'-c')^{1/2} N_\infty\left(\frac{b-c'}{d'-c'}\right) \\
&\le (c'-c)^{1/2}\left(\frac{c'-a}{c'-c}\right)^{1/2} \vee \ (d'-c')^{1/2}\left(\frac{b-c'}{d'-c'}\right)^{1/2} \\
&= (c'-a)^{1/2} \vee \ (b-c')^{1/2} \\
&< (b-a)^{1/2}.
\end{aligned}$$

In Case 2, by Lemmas 1, 3 and 4, we have

$$\begin{aligned}
&|N_\infty(b) - N_\infty(a)| \le (N_\infty(a) - N_\infty(d')) \vee \ (N_\infty(b) - N_\infty(d')) \\
&= (d'-c')^{1/2} N_\infty\left(\frac{d'-a}{d'-c'}\right) \vee \ (d-d')^{1/2} N_\infty\left(\frac{b-d'}{d-d'}\right) \\
&\le (d'-c')^{1/2}\left(\frac{d'-a}{d'-c'}\right)^{1/2} \vee \ (d-d')^{1/2}\left(\frac{b-d'}{d-d'}\right)^{1/2} \\
&= (d'-a)^{1/2} \vee \ (b-d')^{1/2} \\
&< (b-a)^{1/2}.
\end{aligned}$$

Let us considered Case 3. Let $A := N_\infty(c') - N_\infty(a)$ and $B := N_\infty(b) - N_\infty(d')$. Then we have $A > 0$ and $B > 0$ by Lemma 3. By Lemmas 1 and 4, it holds that $A^2 \leq c' - a$ and $B^2 \leq b - d'$. Moreover, $N_\infty(d') - N_\infty(c') = -(d' - c')^{1/2}$. Hence, we have

$$\begin{aligned}
&(N_\infty(b) - N_\infty(a))^2 = (A + B - (d' - c')^{1/2})^2 \\
&= A^2 + B^2 + (d' - c') + 2AB - 2(A + B)(d' - c')^{1/2} \\
&\leq b - a + 2AB - 2(A + B)(d' - c')^{1/2}. \qquad (7)
\end{aligned}$$

Since $A \leq (c' - c)^{1/2} = 2(d' - c')^{1/2}$ and $B \leq (d - d')^{1/2} = 2(d' - c')^{1/2}$, we have

$$\begin{aligned}
& 2AB - 2(A + B)(d' - c')^{1/2} \\
\leq\ & 2(d' - c')^{1/2} \cdot B + A \cdot 2(d' - c')^{1/2} - 2(A + B)(d' - c')^{1/2} = 0
\end{aligned}$$

with the equality only if $A = (c' - c)^{1/2}$ and $B = (d - d')^{1/2}$. Therefore by (7), we have $|N_\infty(b) - N_\infty(a)| \leq (b - a)^{1/2}$ with the equality only if $a = c$ and $b = d$ and that the interval $[a, b]$ is synchronized. □

Lemma 6. *Let* $s \in [0, 1]$ *and* $\lambda \in [0, \infty)$ *be arbitrary and let* $X := e^\lambda(\tilde{N}_\infty + s)$.

(1) *For any interval* $[a, b]$ $(a < b)$, *we have* $|X(b) - X(a)| \leq (b - a)^{1/2}$.

(2) *The following statements for an interval* $[a, b]$ $(a < b)$ *are equivalent to each other.*

(i) $[a, b]$ *is a synchronized interval of* X.

(ii) *It holds that* $X(a) \neq X(b)$ *and*

$$X(t) - X(a) = \xi(b - a)^{1/2}\tilde{N}_\infty\left(\frac{t - a}{b - a}\right)$$

for any $t \in [a, b]$, *where we put* $\xi := \operatorname{sgn}(X(b) - X(a))$.

(iii) $|X(b) - X(a)| = (b - a)^{1/2}$.

Proof. (1) follows from Lemma 5.

(2) It is clear that (ii) implies (iii). That (i) implies (ii) follows from Lemma 1. That (iii) implies (i) follows from Lemma 5. □

Let $\omega = (\mathbf{N}_t(\omega);\ t \in \mathbf{R})$ be an arbitrary sample path of the N-process belonging to Θ_0. Then by Corollary 1, its restriction to any bounded set is a restriction to the same set of some of X in Lemma 6. An interval $[a,b]$ $(a < b)$ is called a **synchronized interval of** ω if it is a synchronized interval of a function X as in Lemma 6 which coincides with ω on $[a-4(b-a),\ b+4(b-a)]$. This is well defined since it is independent of the choice of X by Lemma 6. It is called **increasing**, **decreasing**, **left**, **middle** or **right** if it is so in X as above. We cannot count the level of a synchronized interval of ω, but we can compare the levels between synchronized intervals. For two synchronized intervals I and J of ω, J is said to have **level** n $(n \in \mathbf{Z})$ **relative** to I if there exists X as in Lemma 6 which coincides with ω on an interval containing $I \cup J$ and $m \geq 0$ such that I and J are synchronized intervals of X with levels m and $m+n$, respectively. In special, they are said to have a **same level** if $n = 0$ in the above. If two synchronized intervals I and J of ω satisfy that $I \subset J$ and that I has level n relative to J, we say that J is the n-th **ancester** of I.

Theorem 2. *For any $\omega \in \Theta_0$ and an interval $[a,b]$ $(a < b)$, it holds that $|\omega(b) - \omega(a)| \leq (b-a)^{1/2}$ with the equality if and only if $[a,b]$ is a synchronized interval of ω. If $[a,b]$ is a synchronized interval of ω, then*

$$\omega(t) - \omega(a) = \xi (b-a)^{1/2} N_\infty\left(\frac{t-a}{b-a}\right)$$

for any $t \in [a,b]$, where we put $\xi := \operatorname{sgn}(\omega(b) - \omega(a))$.

Proof. Clear from Lemma 6. ☐

Lemma 7. *For any t with $0 < t \leq 1$, it holds that $N_\infty(t) \geq (1/3)t^{1/2}$.*

Proof. Take $k = 0, 1, 2, \cdots$ such that $(4/9)^{k+1} < t \leq (4/9)^k$. The minimum value of $N_\infty(s)$ for $(4/9)^{k+1} < s \leq (4/9)^k$ is $(1/3)(2/3)^k$ attained when $s = (5/9)(4/9)^k$. Therefore, we have

$$N_\infty(t) \geq (1/3)\left(\frac{2}{3}\right)^k = (1/3)\left(\frac{4}{9}\right)^{k/2} \geq (1/3)t^{1/2}.$$

$\square$

For any $\omega \in \Theta_0$ and $\varepsilon > 0$, a closed interval I is call an $(1-\varepsilon)$-**synchronized interval** of ω if there exists a synchronized interval J of ω with $|I \cap J| \ / \ |I \cup J| \geq 1 - \varepsilon$.

Theorem 3. *Let $\omega \in \Theta_0$. Then, the following statements holds.*

(1) *For any $\varepsilon > 0$, there exists $\delta > 0$ such that for any interval $[a,b]$ $(a < b)$, if $|\omega(b) - \omega(a)| > (1-\delta)(b-a)^{1/2}$, then $[a,b]$ is an $(1-\varepsilon)$-synchronized interval of ω. In fact, for $\varepsilon < 1/10$, we can take $\delta = \varepsilon/18$.*

(2) *For any $\delta > 0$, there exists $\varepsilon > 0$ such that for any interval $[a,b]$ $(a < b)$, if $[a,b]$ is an $(1-\varepsilon)$-synchronized interval of ω, then $|\omega(b) - \omega(a)| > (1-\delta)(b-a)^{1/2}$. In fact, for $\delta < 1$, we can take $\varepsilon = (\delta/4)^2$.*

(3) *If $I = [u,v]$ is an $(1-\varepsilon)$-synchronized interval of ω with $0 < \varepsilon < 1/10$. Then, there exists a unique solution in u' and v' of the equation:*

$$\begin{aligned} & u', \ v' \in [u - (1/7)(v-u), \ v + (1/7)(v-u)] \qquad (8) \\ & \omega(u') = \min\{\omega(t); \ t \in [u - (1/7)(v-u), \ v + (1/7)(v-u)]\} \\ & \omega(v') = \max\{\omega(t); \ t \in [u - (1/7)(v-u), \ v + (1/7)(v-u)]\}. \end{aligned}$$

Let this solution be u', v'. Then, the interval J defined as $J = [u', v']$ if $u' < v'$ and $J = [v', u']$ if $v' < u'$ is a synchronized interval of ω such that $|I \cap J|/|I \cup J| \geq 1 - \varepsilon$.

Proof. (1) Take any ε with $0 < \varepsilon < 1/20$. Assume that $[a,b]$ is not an $(1-2\varepsilon)$-synchronized interval of ω. Let $[c,d]$ be a minimal synchronized interval of ω containing $[a + \varepsilon(b-a), \ b - \varepsilon(b-a)]$. We assume without loss of generality that $[c,d]$ is increasing. Let $c' = (5c+4d)/9$ and $d' = (4c+5d)/9$. Then, by the minimality of $[c,d]$ and the assumption that $[a,b]$ is not $(1-2\varepsilon)$-synchronized, we have 6 cases.

<u>Case 1</u> $c - \varepsilon(b-a) \leq a \leq c + \varepsilon(b-a)$ and $c' + \varepsilon(b-a) < b \leq d'$.
<u>Case 2</u> $c - \varepsilon(b-a) \leq a \leq c + \varepsilon(b-a)$ and $d' < b < d - \varepsilon(b-a)$.
<u>Case 3</u> $c + \varepsilon(b-a) < a < c' - \varepsilon(b-a)$ and $c' + \varepsilon(b-a) < b \leq d'$.

Case 4 $c + \varepsilon(b-a) < a < c' - \varepsilon(b-a)$ and $d' < b \leq d + \varepsilon(b-a)$.
Case 5 $c' - \varepsilon(b-a) \leq a \leq c' + \varepsilon(b-a)$ and $d' + \varepsilon(b-a) < b \leq d + \varepsilon(b-a)$.
Case 6 $c' + \varepsilon(b-a) < a < d' - \varepsilon(b-a)$ and $d' + \varepsilon(b-a) < b \leq d + \varepsilon(b-a)$.

In Case 1, by Theorem 2 and Lemma 7, we have

$$\begin{aligned} & |\omega(b) - \omega(a)| = (\omega(c') - \omega(a)) - (\omega(c') - \omega(b)) \\ \leq \ & (c'-a)^{1/2} - (d'-c')^{1/2} N_\infty \left(\frac{b-c'}{d'-c'} \right) \\ \leq \ & (c'-a)^{1/2} - (d'-c')^{1/2} (1/3) \left(\frac{b-c'}{d'-c'} \right)^{1/2} \\ = \ & (c'-a)^{1/2} - (1/3)(b-c')^{1/2} \\ \leq \ & (b-a)^{1/2} - (1/3)(\varepsilon(b-a))^{1/2} \\ \leq \ & (b-a)^{1/2}(1 - (\varepsilon/9)^{1/2}). \end{aligned}$$

Hence, taking $\delta := (\varepsilon/9)^{1/2} > \varepsilon/9$ for 2ε, we have (1).

In Case 2, by Theorem 2 and Lemma 7, we have

$$\begin{aligned} & (\omega(b) - \omega(a))^2 = (A + B - C)^2 \\ & \leq (c'-a) + (b-d') + (d'-c') + 2AB - 2AC - 2BC \\ & = b - a + 2AB - (A+B)(2/3)(d-c)^{1/2} \\ & \leq b - a + (2/3)(d-c)^{1/2} B + AB - (A+B)(2/3)(d-c)^{1/2} \\ & \leq b - a - A(\omega(d) - \omega(d') - B) \\ & = b - a - ((\omega(c') - \omega(c)) - (\omega(a) - \omega(c)))(\omega(d) - \omega(b)) \\ & \leq b - a - ((2/3)(d-c)^{1/2} - |a-c|^{1/2})(1/3)(d-b)^{1/2} \\ & \leq b - a - \left((2/3)((1-2\varepsilon)(b-a))^{1/2} - (\varepsilon(b-a))^{1/2}\right)(1/3)(\varepsilon(b-a))^{1/2} \\ & \leq b - a - (1/12)\varepsilon^{1/2}(b-a) \\ & \leq (1 - \varepsilon^{1/2}/12)(b-a) \end{aligned}$$

where we put $A := \omega(c') - \omega(a)$, $B := \omega(b) - \omega(d')$ and $C := \omega(c') - \omega(d')$. Hence, taking $\delta := \varepsilon^{1/2}/12 > \varepsilon/9$ for 2ε, we have (1).

For Case 3, by Theorem 2 and Lemma 7, we have

$$\begin{aligned} & (\omega(b)-\omega(a))^2 = (A-B)^2 \\ \leq\ & (c'-a)+(b-c')-2AB \\ \leq\ & b-a-2(1/3)(c'-a)^{1/2}(1/3)(b-c')^{1/2} \\ \leq\ & b-a-2((1/3)\varepsilon^{1/2}(b-a)^{1/2})^2 \\ \leq\ & (1-(2\varepsilon/9))(b-a), \end{aligned}$$

where we put $A := \omega(c')-\omega(a)$ and $B := \omega(c')-\omega(b)$. Hence, taking $\delta := 2\varepsilon/9$ for 2ε, we have (1).

In Case 4, if $d' < b \leq d' + \varepsilon(b-a)$, then there exists b' with $c'+\varepsilon(b-a) < b' < d'$ and $\omega(b') = \omega(b)$. Hence, (1) follows from Case 3 since

$$\begin{aligned} |\omega(b)-\omega(a)| &= |\omega(b')-\omega(a)| \\ &\leq (1-(2\varepsilon/9))(b'-a) \\ &\leq (1-(2\varepsilon/9))(b-a). \end{aligned}$$

Now assume that $d'+\varepsilon(b-a) < b \leq d+\varepsilon(b-a)$. By Theorem 2 and Lemma 7, we have

$$\begin{aligned} & (\omega(b)-\omega(a))^2 = (A+B-C)^2 \\ \leq\ & (c'-a)+(b-d')+(d'-c')+2AB-2AC-2BC \\ =\ & b-a+2AB-(A+B)(2/3)(d-c)^{1/2} \\ \leq\ & b-a+A(2/3)(d-c)^{1/2}+AB-(A+B)(2/3)(d-c)^{1/2} \\ \leq\ & b-a-(\omega(c')-\omega(c)-A)B \\ \leq\ & b-a-(1/3)(a-c)^{1/2}(1/3)(b-d')^{1/2} \\ \leq\ & b-a-(1/9)\varepsilon(b-a) \\ =\ & (1-(\varepsilon/9))(b-a), \end{aligned}$$

where we put $A := \omega(c')-\omega(a)$, $B := \omega(b)-\omega(d')$, and $C := \omega(c')-\omega(d')$. Hence, taking $\delta := \varepsilon/9$ for 2ε, we have (1).

Case 5 and Case 6 follow from the previous cases by symmetry.

(2) Let $0 < \varepsilon < 1/10$ and Let $[a,b]$ be a $(1-\varepsilon)$-synchronized interval. Then, there exists a synchronized interval $[c,d]$ with $|a-c| < 2\varepsilon(b-a)$ and $|b-d| < 2\varepsilon(b-a)$. Then by Theorem 2, we have

$$\begin{aligned} & |\omega(b)-\omega(a)| \\ \geq\ & |\omega(d)-\omega(c)| - |\omega(a)-\omega(c)| - |\omega(b)-\omega(d)| \\ \geq\ & (d-c)^{1/2} - |a-c|^{1/2} - |b-d|^{1/2} \\ \geq\ & (b-a-\varepsilon(b-a))^{1/2} - 2(2\varepsilon(b-a))^{1/2} \\ \geq\ & (1-4\varepsilon^{1/2})(b-a)^{1/2}. \end{aligned}$$

Thus, for any δ with $0 < \delta < 1$, we have (2) by taking $\varepsilon = (\delta/4)^2$.

(3) Assume without loss of generality that $\omega(a) < \omega(b)$. Then, there exists a synchronized interval $J = [u', v']$ such that $|I \cap J|/|I \cup J| \geq 1 - \varepsilon$. Moreover, u', v' is the unique solution of the equation (8). $\square$

4 Stochastic Integral

Let $L = L(\omega)$ be a measurable function of $\omega \in \Theta_0$ taking value in positve integers. Let $\{\zeta_0 < \zeta_1 < \cdots\}$ be a finite or infinite sequence of measurable functions of $\omega \in \Theta_0$ such that $[\zeta_i, \zeta_{i+1}]$ is a synchronized interval of $\omega \in \Theta_0$ for any $i = 0, 1, \cdots$ and ζ_L is defined for any $\omega \in \Theta_0$. We call a sequence $\zeta := \{\zeta_0 < \zeta_1 < \cdots < \zeta_L\}$ a **synchronized net**. If for an interval I, $I \subset [\zeta_0, \zeta_L]$ holds for any $\omega \in \Theta_0$, we say that ζ **covers** I. We denote $\| \zeta \| := \| \max_{0 \leq i \leq L-1}(\zeta_{i+1} - \zeta_i) \|_\infty$. Let $\mathcal{C}$ be a sub-σ-field of the probability space (Θ_0, P). If the above L and $\zeta_{i \wedge L}$ $(i = 0, 1, \cdots)$ are measurable with respect to $\mathcal{C}$, then we say that ζ is **measurable** with respect to $\mathcal{C}$ or ζ is $\mathcal{C}$-**measurable**. If $\{Y\}$ is a set of measurable functions on the probability space (Θ_0, P), then we say that ζ is $\{Y\}$-measurable if it is measurable with respect to the σ-field generated by the functions in $\{Y\}$. Let $\zeta = \{\zeta_0 < \zeta_1 < \cdots < \zeta_L\}$ and $\eta = \{\eta_0 < \eta_1 < \cdots < \eta_K\}$ be synchronized nets. If for any $\omega \in \Theta_0$, $\zeta \subset \eta$ holds between the sets of values of functions in ζ and η, and if η is measurable with respect to ζ, we say that η is a **refinement** of ζ.

Lemma 8. *Let J be a bounded closed interval with $J = [a, b]$ $(a < b)$. Then, for any bounded closed intervals I with $I \subset J^i$ and $\varepsilon > 0$, there exists a synchronized net ζ covering I with $\| \zeta \| < \varepsilon$ which is measurable with respect to $d\mathbf{N}|_J$, where $d\mathbf{N}|_J := \{\mathbf{N}_t - \mathbf{N}_s;\ s, t \in J\}$, where $d\mathbf{N}|_J := \{\mathbf{N}_t - \mathbf{N}_s;\ s, t \in J\}$*

Proof. We may assume that $\varepsilon > 0$ is small enough so that $I \subset [a + 2\varepsilon, b - 2\varepsilon]$.

1st step: Let $\{(u_n, v_n);\ n = 1, 2, \cdots\}$ be a countable dense subset of $\{(x, y); -\varepsilon/2 < x < 0 < y < \varepsilon/2,\ \varepsilon/18 \leq y - x < \varepsilon/2\}$. Since there exists an synchronized interval $[c, d]$ of ω containing $a + \varepsilon$ with $\varepsilon/18 \leq d - c < \varepsilon/2$, for δ with $0 < \delta < 1/200$, there exists $n = 1, 2, \cdots$ such that $|\omega(a + \varepsilon + v_n) - \omega(a + \varepsilon + u_n)| > (1 - \delta)(v_n - u_n)^{1/2}$. Take the minimum n as this and define $d\mathbf{N}|_J$-measurable functions $u := a + \varepsilon + u_n$ and $v := a + \varepsilon + v_n$. Then by Theorem 3, $[u, v]$ is $(1 - \delta')$-synchronized interval of ω for some $\delta' < 1/10$. Let u' and v' be the unique solution of the equation (8) in Theorem 3 for this $(1 - \delta')$-synchronized interval $[u, v]$. Then, the functions u' and v' of $\omega \in \Theta$ are measurable with respect to $d\mathbf{N}|_J$. We define $\zeta_0 = u',\ \zeta_1 = v'$ if $u' < v'$ and $\zeta_0 = v',\ \zeta_1 = u'$ if $v' < u'$.

2nd step: Assume that a sequence of $d\mathbf{N}|_J$-measurable functions $\zeta_0 < \zeta_1 < \cdots < \zeta_k$ is defined so that $\zeta_0 < a + 2\varepsilon$ and $[\zeta_{i-1}, \zeta_i]$ is a synchronized interval with $\zeta_i - \zeta_{i-1} < \varepsilon$ for any $i = 1, 2, \cdots, k$. This is done for $k = 1$ in the 1st step.

We add ζ_{k+1} to get a longer sequence with this properties. Take the minimum nonnegative integer i such that $4(4/9)^i(\zeta_k - \zeta_{k-1}) < \varepsilon$. Since $[\zeta_{k-1}, \zeta_k]$ is a synchronized interval, for exactly one of ξ in $\{1/4, 4\}$, $[\zeta_k, \zeta_k + \xi(4/9)^i(\zeta_k - \zeta_{k-1})]$ is a synchronized interval. Define $\zeta_{k+1} = \zeta_k + \xi(4/9)^i(\zeta_k - \zeta_{k-1})$ with this ξ. Since ξ can be chosen in a $d\mathbf{N}|_J$-measurable way by Theorem 2, ζ_{k+1} is measurable with respect to $d\mathbf{N}|_J$ such that $\zeta_{k+1} - \zeta_k < \varepsilon$.

final step: We prove that we can continue this process until we get $\zeta_{L+1} > b - 2\varepsilon$. Then, $\zeta := \{\zeta_0 < \zeta_1 < \cdots < \zeta_L\}$ satisfies the required properties.

The only possible obstruction against this is that ζ_k converges to some point, say $\eta \leq b - \varepsilon$ as $k \to \infty$. We prove that this is impossible.

To the contrary, suppose that this is the case. Then, there exists K such that for any $k \geq K$, the i in the description of the 2nd step is chosen as $i = 0$, so that all synchronized intervals $[\zeta_k, \zeta_{k+1}]$ for $k = K, K+1, \cdots$ have the same level. All consecutive $2 \cdot 3^n$ synchronized intervals of the same level contains a synchronized interval of level $-n$ relative to them for any $n = 1, 2, \cdots$. A synchronized interval of level $-n$ relative to the synchronized interval $[\zeta_K, \zeta_{K+1}]$ has length at least $(9/4)^n(\zeta_{K+1} - \zeta_K)$. Therefore, it holds that $\zeta_{K+2\cdot 3^n} - \zeta_K \geq (9/4)^n(\zeta_{K+1} - \zeta_K)$, which is a contradiction since letting $n \to \infty$ we have $\eta - \zeta_k$ in the left hand side while ∞ in the right hand side. $\square$

Let $A(\omega, s)$ be a function on $\Theta_0 \times \mathbf{R}$ which is measurable in ω and continuous in s for any fixed ω. Then for any $a, b \in \mathbf{R}$ with $a < b$, we define a **stochastic integral** $\int_a^b A d\mathbf{N}_t$ as follows:

$$\int_a^b A d\mathbf{N}_t := \lim_{\substack{\|\zeta\| \to 0 \\ \zeta_0 \to a \\ \zeta_L \to b}} \sum_{i=0}^{L-1} A(\omega, \zeta_i)(\mathbf{N}_{\zeta_{i+1}} - \mathbf{N}_{\zeta_i}) \tag{9}$$

if the limit in the right-hand side exists, where $\zeta = \{\zeta_0 < \zeta_1 < \cdots < \zeta_L\}$ is a synchronized net.

Theorem 4. *Let $H(x, s)$ be a real valued function of $x, s \in \mathbf{R}$ which is twice continuously differentiable in x and once continuously differentiable in s. Then for any $a < b$, the stochastic integral $\int_a^b H_x(\mathbf{N}_t, t) d\mathbf{N}_t$ exists and is $(H_x)_J \vee d\mathbf{N}|_J$-measurable with $J = [a, b]$, where $(H_x)_J := \{H_x(\mathbf{N}_t, t);\ t \in J\}$. Moreover, the following formula holds:*

$$\begin{aligned} & H(\mathbf{N}_b, b) - H(\mathbf{N}_a, a) = \\ & \int_a^b H_x(\mathbf{N}_t, t) d\mathbf{N}_t + \int_a^b (\frac{1}{2} H_{xx}(\mathbf{N}_t, t) + H_s(\mathbf{N}_t, t)) dt \end{aligned} \tag{10}$$

Proof. (1) The $(H_x)_J \vee d\mathbf{N}|_J$-measurability of the stochastic integral follows from Lemma 8 if it exists, by taking the limit $\zeta_0 \downarrow a$ and $\zeta_L \uparrow b$.

(2) The stochastic integral $\int_a^b H_x(\mathbf{N}_t, t) d\mathbf{N}_t$ is Y_J-measurable by Lemma 8 and Theorems 5 and 6 which are proved later if it exists.

Therefore, it suffices to prove the existence of the the stochacstic integral and the formula (10). For a net $\zeta = \{\zeta_0 < \zeta_1 < \cdots < \zeta_L\}$, denote

$$B(\zeta) := \sum_{i=0}^{L-1} H_x(\mathbf{N}_{\zeta_i}, \zeta_i)(\mathbf{N}_{\zeta_{i+1}} - \mathbf{N}_{\zeta_i}).$$

Then, by the Taylor expansion of H and the continuity of H, H_{xx} and H_s in (x, s) as well as the sample path $\mathbf{N}_t$ in t, it holds as $\| \zeta \| \to 0$, $\zeta_0 \to a$ and $\zeta_L \to b$ that

$$\begin{aligned}
& H(\mathbf{N}_b, b) - H(\mathbf{N}_a, a) \\
= & \sum_{i=0}^{L-1} \big(H(\mathbf{N}_{\zeta_{i+1}}, \zeta_{i+1}) - H(\mathbf{N}_{\zeta_i}, \zeta_i)\big) + o(1) \\
= & \sum_{i=0}^{L-1} \big(H_x(\mathbf{N}_{\zeta_i}, \zeta_i)(\mathbf{N}_{\zeta_{i+1}} - \mathbf{N}_{\zeta_i}) + \frac{1}{2} H_{xx}(\mathbf{N}_{\zeta_i}, \zeta_i)(\mathbf{N}_{\zeta_{i+1}} - \mathbf{N}_{\zeta_i})^2 \\
& + H_t(\mathbf{N}_{\zeta_i}, \zeta_i)(\zeta_{i+1} - \zeta_i) + o(\zeta_{i+1} - \zeta_i)\big) + o(1) \\
= & B(\zeta) + \sum_{i=0}^{L-1} \big(\frac{1}{2} H_{xx}(\mathbf{N}_{\zeta_i}, \zeta_i) + H_t(\mathbf{N}_{\zeta_i}, \zeta_i)\big)(\zeta_{i+1} - \zeta_i) + o(1) \\
= & B(\zeta) + \int_a^b \big(\frac{1}{2} H_{xx}(\mathbf{N}_t, t) + H_t(\mathbf{N}_t, t)\big) dt + o(1),
\end{aligned}$$

where we used the fact that $(\mathbf{N}_{\zeta_{i+1}} - \mathbf{N}_{\zeta_i})^2 = \zeta_{i+1} - \zeta_i$. Hence, $B(\zeta)$ converges. Thus, the stochastic integral exists and we have (10). $\square$

5 Prediction

Let $H(x, s)$ be a real valued function of $x, s \in \mathbf{R}$ such that
(H1) H is twice continuously differentiable in x and once continuously differentiable in s, and
(H2) $H_x(x, s) > 0$ for any $x, s \in \mathbf{R}$.

We consider the stochastic process $Y_t = H(\mathbf{N}_t, t)$ $(t \in \mathbf{R})$. Our problem is to predict Y_t for $t \notin J$ from the observation $Y_J := \{Y_t;\ t \in J\}$, where J is a bouded closed interval with nonempty interior. The function H is considered to be unknown except for the property (H1) and (H2). All the measurable functions of the observation Y_J we construct in the following do not need any further knowledge on the unknown function H.

Theorem 5. *For any* $\omega \in \Theta_0$ *and* $t \in \mathbf{R}$*, it holds that*

$$H_x(\mathbf{N}_t, t) = \limsup_{\substack{u,v\to t\\ u<v}} \frac{|Y_v - Y_u|}{(v-u)^{1/2}} \ .$$

Let t_1*,* t_2 *with* $t_1 < t_2$ *tend to* t *attaining the* lim sup *in the right hand side of the above equality. Let* ${t_1}' = (5t_1 + 4t_2)/9$ *and* ${t_2}' = (4t_1 + 5t_2)/9$*. Then, it holds that*

$$\begin{aligned} H_{xx}(\mathbf{N}_t, t) &= \frac{9}{4} \lim \frac{-Y_{t_1} + Y_{{t_1}'} + Y_{{t_2}'} - Y_{t_2}}{(t_2 - t_1)^{1/2}} \\ H_s(\mathbf{N}_t, t) &= \frac{3}{8} \lim \frac{Y_{t_1} - 9Y_{{t_1}'} + 3Y_{{t_2}'} + 5Y_{t_2}}{(t_2 - t_1)^{1/2}} \ . \end{aligned}$$

Therefore, if $t \in J$*, then those quantities as* $H_x(\mathbf{N}_t, t)$*,* $H_{xx}(\mathbf{N}_t, t)$ *and* $H_s(\mathbf{N}_t, t)$ *are measurable functions of the observation* Y_J*.*

Proof. Since by the Taylor expansion of H, we have

$$\begin{aligned} Y_v - Y_u &= H(\mathbf{N}_v, v) - H(\mathbf{N}_u, u) \\ &= H_x(\mathbf{N}_t, t)(\mathbf{N}_v - \mathbf{N}_u) + \frac{1}{2} H_{xx}(\mathbf{N}_t, t)(\mathbf{N}_v - \mathbf{N}_u)^2 \\ &\quad + H_s(\mathbf{N}_t, t)(v-u) + o(v-u) \end{aligned}$$

as $u, v \to t$, by Theorem 2 and (H2), we have

$$\begin{aligned} &\limsup_{\substack{u,v\to t\\ u<v}} \frac{|Y_v - Y_u|}{(v-u)^{1/2}} \\ = \ & H_x(\mathbf{N}_t, t) \limsup_{\substack{u,v\to t\\ u<v}} \frac{|\mathbf{N}_v - \mathbf{N}_u|}{(v-u)^{1/2}} \\ = \ & H_x(\mathbf{N}_t, t). \end{aligned}$$

By Theorem 3, the $\limsup$ is attained if and only if $u, v \to t$ so that $[u, v]$ is an $(1-\varepsilon)$-synchronized interval of ω with $\varepsilon \to 0$. Therefore, the interval $[t_1, t_2]$ as in the statement of our theorem satisfies this condition. Furthermore, since we can approximate the $(1-\varepsilon)$-synchronized interval $[t_1, t_2]$ by a synchronized interval close to it and approximate the following quantities for the former by those for the latter with small errors, we may assume that $[t_1, t_2]$ itself is synchronized. Consider the Taylor expansions for

$$\begin{array}{rcl} H(\mathbf{N}_{t_2{}'}, t_2{}') & - & H(\mathbf{N}_{t_1{}'}, t_1{}') \\ H(\mathbf{N}_{t_2}, t_2) & - & H(\mathbf{N}_{t_1{}'}, t_1{}') \\ H(\mathbf{N}_{t_2}, t_2) & - & H(\mathbf{N}_{t_1}, t_1) \end{array}$$

and using the relations that

$$\begin{array}{rcl} t_2{}' - t_1{}' & = & (1/9)(t_2 - t_1) \\ t_2 - t_1{}' & = & (5/9)(t_2 - t_1) \\ \mathbf{N}_{t_2{}'} - \mathbf{N}_{t_1{}'} & = & -(1/3)\xi(t_2 - t_1)^{1/2} \\ \mathbf{N}_{t_2} - \mathbf{N}_{t_1{}'} & = & (1/3)\xi(t_2 - t_1)^{1/2} \\ \mathbf{N}_{t_2} - \mathbf{N}_{t_1} & = & \xi(t_2 - t_1)^{1/2}, \end{array}$$

where $\xi = \operatorname{sgn}(\mathbf{N}_{t_2} - \mathbf{N}_{t_1})$, we have

$$\begin{array}{rcl} Y_{t_2{}'} - Y_{t_1{}'} & = & -(1/3)\xi H_x(\mathbf{N}_t, t)(t_2 - t_1)^{1/2} + (1/18)H_{xx}(\mathbf{N}_t, t)(t_2 - t_1) \\ & & +(1/9)H_s(\mathbf{N}_t, t)(t_2 - t_1) + o(t_2 - t_1), \\ Y_{t_2} - Y_{t_1{}'} & = & (1/3)\xi H_x(\mathbf{N}_t, t)(t_2 - t_1)^{1/2} + (1/18)H_{xx}(\mathbf{N}_t, t)(t_2 - t_1) \\ & & +(5/9)H_s(\mathbf{N}_t, t)(t_2 - t_1) + o(t_2 - t_1), \end{array}$$

and

$$\begin{array}{rcl} Y_{t_2} - Y_{t_1} & = & \xi H_x(\mathbf{N}_t, t)(t_2 - t_1)^{1/2} + \frac{1}{2}H_{xx}(\mathbf{N}_t, t)(t_2 - t_1) \\ & & +H_s(\mathbf{N}_t, t)(t_2 - t_1) + o(t_2 - t_1). \end{array}$$

By solving the above linear equation on $H_x(\mathbf{N}_t, t)$, $H_{xx}(\mathbf{N}_t, t)$, $H_s(\mathbf{N}_t, t)$ and letting $t_2 - t_1 \to 0$, we get the required formulas for $H_{xx}(\mathbf{N}_t, t)$ and $H_t(\mathbf{N}_t, t)$.

It is clear from the above formulas that if t belongs to the interior of J, then the quantities as $H_{xx}(\mathbf{N}_t, t)$ and $H_t(\mathbf{N}_t, t)$ are measurable with respect to the observation Y_J. It follows from the continuity that the same result holds for any $t \in J$. $\square$

Theorem 6. *Let I, J be closed intervals with $J = [a, b]$ $(a < b)$ and $\emptyset \neq I^i \subset I \subset (a, b)$.*

(1) *For any $\delta > 0$, there exists $\varepsilon > 0$ such that for any $t \in J$ and $u, v \in (t - \varepsilon, t + \varepsilon)$, it holds that*

$$Y_v - Y_u = H_x(\mathbf{N}_t, t)(\mathbf{N}_v - \mathbf{N}_u) + \Xi$$

with

$$|\Xi| \leq \delta(|\mathbf{N}_v - \mathbf{N}_u| + |v - u|^{1/2})$$

(2) *For any $\varepsilon > 0$, there exists a Y_J-measurable synchronized net covering I with $\| \zeta \| < \varepsilon$.*

(3) *$d\mathbf{N}|_J$ is measurable with respect to the observation Y_J. Hence, both terms in the right-hand side of* (10) *are Y_J-measurable.*

Proof. (1) For any given $\delta > 0$, take ε with $0 < \varepsilon < 1$ satisfying that

(i) $|H_x(x', s') - H_x(x, s)| < \delta$ for any (x, s) and (x', s') with

$$s, s' \in J', \ |s - s'| < \varepsilon, \ |x|, |x'| \leq (|a'|^{\vee}|b'|)^{1/2} \text{ and } |x - x'| < \varepsilon^{1/2},$$

(ii)

$$\sup_{s \in J', \ |x| \leq (|a'|^{\vee}|b'|)^{1/2}} |H_s(x, s)| \cdot (2\varepsilon)^{1/2} < \delta,$$

where $a' = a - 1, \ b' = b + 1, \ J' := [a', b']$. Then for any $t \in J$ and $u, v \in (t - \varepsilon, t + \varepsilon)$, it holds that

$$\begin{aligned}
& Y_v - Y_u \\
= \ & H(\mathbf{N}_v, v) - H(\mathbf{N}_u, u) \\
= \ & (H(\mathbf{N}_v, v) - H(\mathbf{N}_v, u)) + (H(\mathbf{N}_v, u) - H(\mathbf{N}_u, u)) \\
= \ & H_s(\mathbf{N}_v, t')(v - u) + H_x(x', u)(\mathbf{N}_v - \mathbf{N}_u) \\
= \ & H_x(\mathbf{N}_t, t)(\mathbf{N}_v - \mathbf{N}_u) + \Xi
\end{aligned}$$

with

$$\Xi := H_s(\mathbf{N}_v, t')(v - u) + (H_x(x', u) - H_x(\mathbf{N}_t, t))(\mathbf{N}_v - \mathbf{N}_u),$$

where t' and x' satisfies that $|t'-t|<\varepsilon$ and $|x'-\mathbf{N}_t|<\varepsilon^{1/2}$. Then using (i) and (ii), we have

$$\begin{aligned} & |\Xi| \\ \leq\ & |H_s(\mathbf{N}_v,t')||v-u| + |H_x(x',u)-H_x(\mathbf{N}_t,t)||\mathbf{N}_v-\mathbf{N}_u| \\ \leq\ & |H_s(\mathbf{N}_v,t')|(2\varepsilon)^{1/2}|v-u|^{1/2} + \delta|\mathbf{N}_v-\mathbf{N}_u| \\ \leq\ & \delta(|\mathbf{N}_v-\mathbf{N}_u| + |v-u|^{1/2}). \end{aligned}$$

(2) Take sufficiently small $\delta>0$ determined finally in the following 2nd step. At this moment, we assume that

$$0<\delta<\inf_{t\in J,\ |x|\leq(|a|\vee|b|)^{1/2}} H_x(x,t)/1200. \tag{11}$$

We may assume that $\varepsilon>0$ is small enough so that the statement (1) holds with this δ and that $I\subset[a+2\varepsilon,b-2\varepsilon]$. We use a similar construction as in the proof of Lemma 8.

1st step: Let $\{(u_n,v_n);\ n=1,2,\cdots\}$ be a countable dense subset of $\{(x,y); -\varepsilon/2<x<0<y<\varepsilon/2,\ \varepsilon/18\leq y-x<\varepsilon/2\}$. There exists an synchronized interval $[c,d]$ of ω containing $t:=a+\varepsilon$ with $\varepsilon/18\leq d-c<\varepsilon/2$. Then, we have by (1)

$$\begin{aligned} & |Y_d-Y_c| \\ \geq\ & (H_x(\mathbf{N}_t,t)-\delta)|\mathbf{N}_d-\mathbf{N}_c| - \delta(d-c)^{1/2} \\ =\ & (H_x(\mathbf{N}_t,t)-\delta)(d-c)^{1/2} - \delta(d-c)^{1/2} \\ =\ & (H_x(\mathbf{N}_t,t)-2\delta)(d-c)^{1/2}. \end{aligned}$$

Hence, there exists $n=1,2,\cdots$ such that $|Y_{t+v_n}-Y_{t+u_n}|>(H_x(\mathbf{N}_t,t)-3\delta)(v_n-u_n)^{1/2}$. Take the minimum n as this and define functions $u:=t+u_n$ and $v:=t+v_n$, which are Y_J-measurable by Theorem 5.

Since as above we have

$$\begin{aligned} & (H_x(\mathbf{N}_t,t)-3\delta)(v-u)^{1/2} \\ <\ & |Y_v-Y_u| \\ \leq\ & (H_x(\mathbf{N}_t,t)+\delta)|\mathbf{N}_v-\mathbf{N}_u| + \delta(v-u)^{1/2}, \end{aligned}$$

it holds by (11) that

$$|\mathbf{N}_v-\mathbf{N}_u|>(1-1/200)(v-u)^{1/2}.$$

Then by Theorem 3, $[u,v]$ is a $(1-1/11)$-synchronized interval of ω. Let u' and v' be the unique solution of the equation (8) in Theorem 3 for this $(1-1/11)$-synchronized interval $[u,v]$.

We prove that u', v' is also the unique solution of the equation:

$$\begin{aligned}
&u',\ v' \in [u-(1/7)(v-u),\ v+(1/7)(v-u)] \qquad (12)\\
&Y_{u'} = \min\{Y_s;\ s\in[u-(1/7)(v-u),\ v+(1/7)(v-u)]\}\\
&Y_{v'} = \max\{Y_s;\ s\in[u-(1/7)(v-u),\ v+(1/7)(v-u)]\}.
\end{aligned}$$

Take any $s\in[u-(1/7)(v-u),\ v+(1/7)(v-u)]$ with $s\neq u'$. Then by Lemma 7, $\mathbf{N}_s-\mathbf{N}_{u'}\geq(1/3)|s-u'|^{1/2}$. Therefore as above, we have

$$\begin{aligned}
& Y_s - Y_{u'}\\
\geq\ & (H_x(t,\mathbf{N}_t)-\delta)(\mathbf{N}_s-\mathbf{N}_{u'})-\delta|s-u'|^{1/2}\\
\geq\ & (H_x(\mathbf{N}_t,t)-\delta)(1/3)|s-u'|^{1/2}-\delta|s-u'|^{1/2}\\
=\ & (H_x(\mathbf{N}_t,t)-4\delta)(1/3)|s-u'|^{1/2}\\
\geq\ & (1200-4)\delta(1/3)|s-u'|^{1/2},
\end{aligned}$$

so that u' is the unique solution of the equation (12). Similarly, v' is the unique solution of the equation (12). Thus, u' and v' are Y_J-measurable functions on $\omega\in\Theta$.

We define $\zeta_0=u'$, $\zeta_1=v'$ if $u'<v'$ and $\zeta_0=v'$, $\zeta_1=u'$ if $v'<u'$.

2nd step: Assume that a sequence of Y_J-measurable functions $\zeta_0<\zeta_1<\cdots<\zeta_k$ is defined so that $\zeta_0<a+2\varepsilon$ and $[\zeta_{i-1},\zeta_i]$ is a synchronized interval with $\zeta_{i-1}-\zeta_i<\varepsilon$ for any $i=1,2,\cdots,k$. This is done for $k=1$ in the 1st step.

We add ζ_{k+1} to get a longer sequence with these properties. Take the minimum nonnegative integer i such that $4(4/9)^i(\zeta_k-\zeta_{k-1})<\varepsilon$. Since $[\zeta_{k-1},\zeta_k]$ is a synchronized interval, for exactly one of ξ in $\{1/4,4\}$, $[\zeta_k,\zeta_k+\xi(4/9)^i(\zeta_k-\zeta_{k-1})]$ is a synchronized interval. Define $\zeta_{k+1}=\zeta_k+\xi(4/9)^i(\zeta_k-\zeta_{k-1})$ with this ξ.

What we have to prove is that ξ is chosen in a Y_J-measurable way. Let $\xi\in\{1/4,4\}$ be such that $[t,\zeta]$ is a synchronized interval and let $\xi'\in\{1/4,4\}$ be $\xi'\neq\xi$, so that $[t,\zeta']$ is not a synchronized interval, where we put $t:=\zeta_k$, $\zeta:=t+\xi(4/9)^i(t-\zeta_{k-1})$ and $\zeta'=$

$t+\xi'(4/9)^i(t-\zeta_{k-1})$. Let $[t,\zeta'']$ be the minimal synchronized interval containing $[t,\zeta']$. Then, we can prove that there exists $p>0$ such that $(4/9)+p<(\zeta'-t)/(\zeta''-t)<1-p$. Therefore, by Theorem 2, there exists q with $1/2<q<1$ such that

$$|\mathbf{N}_{\zeta'}-\mathbf{N}_t|<q|\zeta'-t|^{1/2}$$

while

$$|\mathbf{N}_{\zeta}-\mathbf{N}_t|=|\zeta-t|^{1/2}.$$

Then, as we proved in the 1st step, we have

$$\begin{aligned}
&|Y_{\zeta'}-Y_t|\\
\leq\ &(H_x(\mathbf{N}_t,t)+\delta)|\mathbf{N}_{\zeta'}-\mathbf{N}_t|+\delta(\zeta'-t)^{1/2}\\
\leq\ &(H_x(\mathbf{N}_t,t)+3\delta)q(\zeta'-t)^{1/2},
\end{aligned}$$

while

$$\begin{aligned}
&|Y_{\zeta}-Y_t|\\
\geq\ &(H_x(\mathbf{N}_t,t)-\delta)|\mathbf{N}_{\zeta}-\mathbf{N}_t|-\delta(\zeta-t)^{1/2}\\
=\ &(H_x(\mathbf{N}_t,t)-2\delta)(\zeta-t)^{1/2}.
\end{aligned}$$

Therefore, by choosing small $\delta>0$, we have

$$\begin{aligned}
|Y_{\zeta'}-Y_t|/(\zeta'-t)^{1/2} &\leq H_x(\mathbf{N}_t,t)(1+2q)/3\\
|Y_{\zeta}-Y_t|/(\zeta-t)^{1/2} &\geq H_x(\mathbf{N}_t,t)(2+q)/3,
\end{aligned}$$

so that we can ditinguish these 2 cases by the observation Y_J. Hence, ξ is Y_J-measurable.

Thus, the function ζ_{k+1} on $\omega\in\Theta$ is Y_J-measurable such that $[\zeta_k,\zeta_{k+1}]$ is a synchronized interval of with $\zeta_{k+1}-\zeta_k<\varepsilon$.

final step: We continue this process until we get $\zeta_{L+1}>b-\varepsilon$. Then, $\zeta:=\{\zeta_0<\zeta_1<\cdots<\zeta_L\}$ satisfies the required properties. This can be done by the same reason as in the final step of the proof of Lemma 8.

(3) Let $\zeta=\{\zeta_0<\zeta_1<\cdots<\zeta_L\}$ be a Y_I-measurable synchronized net covering J. If necessary, we repeat the division of a

synhronized interval $[\zeta_i, \zeta_{i+1}]$ by $[\zeta_i, \zeta'_i]$, $[\zeta'_i, \zeta'_{i+1}]$, $[\zeta'_{i+1}, \zeta_{i+1}]$ with $\zeta'_i = (5\zeta_i + 4\zeta_{i+1})/9$ and $\zeta'_{i+1} = (4\zeta_i + 5\zeta_{i+1})/9$, we may assume that there exists $[\zeta_i, \zeta_{i+1}] \subset I^i$ such that $\zeta_{i+1} - \zeta_i$ is sufficiently small so that $Y_{\zeta_{i+1}} - Y_{\zeta_i}$ has the same sign with $\mathbf{N}_{\zeta_{i+1}} - \mathbf{N}_{\zeta_i}$. Then, we know from the observation Y_I whether the synchronized interval $[\zeta_i, \zeta_{i+1}]$ is increasing or decreasing. Since the synchronized intervals $[\zeta_j, \zeta_{j+1}]$'s are increasing and decreasing alternatively, we know $\xi = \mathrm{sgn}(\mathbf{N}_{j+1} - \mathbf{N}_j)$ for all $j = 0, 1, \cdots, L-1$. Since

$$\mathbf{N}_t - \mathbf{N}_{\zeta_j} = \xi (t - \zeta_j)^{1/2} N_\infty \left(\frac{t - \zeta_j}{\zeta_{j+1} - \zeta_j} \right)$$

for any $t \in [\zeta_j, \zeta_{j+1}]$ by Theorem 2, we get $d\mathbf{N}|_J$ from the observation Y_I, hence by Y_J considering the limit. ☐

Lemma 9. (1) *Let* $\{\zeta_0 < \zeta_0 < \zeta_1 < \cdots < \zeta_L\}$ *be a synchronized net. Let* $(\zeta_{i+1} - \zeta_i)/(\zeta_i - \zeta_{i-1}) = \xi(4/9)^j$ *with* $\xi \in \{1/4, 4\}$ *and* $j \in \mathbf{Z}$ *for some* $i = 1, 2, \cdots, L-1$ *and* $\omega \in \Theta$. *If* $j > 0$, *then for* $\eta := \zeta_i + \xi(\zeta_i - \zeta_{i-1})$, $[\zeta_i,\ \eta]$ *is a synchronized interval of* $\omega \in \Theta_0$ *and if* $\eta \le \zeta_L$, *then there exists* n *with* $i + 1 < n \le L$ *such that* $\eta = \zeta_n$. *If* $j < 0$, *then for* $\eta := \zeta_i - \xi(\zeta_{i+1} - \zeta_i)$, $[\eta,\ \zeta_i]$ *is a synchronized interval of* $\omega \in \Theta_0$ *and if* $\eta \ge \zeta_0$, *then there exists* n *with* $0 \le n < i - 1$ *such that* $\eta = \zeta_n$.

(2) *For any neighboring synchronized intervals* $[a, b]$, $[b, c]$ *and* $[c, d]$ *of* $\omega \in \Theta_0$, *if* $(c - b)/(b - a) = 1/4$ *and* $(d - c)/(c - b) = 4$, *then* $[a, d]$ *is a synchronized interval of* ω.

(3) *For any neighboring synchronized intervals* $[a, b]$, $[b, c]$ *and* $[c, d]$ *of* $\omega \in \Theta_0$, *if* $(c - b)/(b - a) = 1/4$ *and* $(d - c)/(c - b) = 1/4$, *then* $[a - (9/4)(b - a), b]$ *and* $[b, b + (9/4)(c - b)]$ *are synchronized intervals of* ω.

(4) *For any neighboring synchronized intervals* $[a, b]$, $[b, c]$ *and* $[c, d]$ *of* $\omega \in \Theta_0$, *if* $(c - b)/(b - a) = 4$ *and* $(d - c)/(c - b) = 4$, *then* $[b - (9/4)(c - b), c]$ *and* $[c, c + (9/4)(d - c)]$ *are synchronized intervals of* ω.

Proof.
(1) Assume that $j > 0$. Let K be the nearest common ancester of

$[\zeta_{i-1},\ \zeta_i]$ and $[\zeta_i,\ \zeta_{i+1}]$. Let $[\zeta_{i-1},\ \zeta_i]$ have level k relative to K. Then by (2) of Lemma 2, $[\zeta_i,\ \zeta_{i+1}]$ has level $k+j$ relative to K. Since $k > 0$, the j-th ancester of $[\zeta_i,\ \zeta_{i+1}]$, is neighboring to $[\zeta_{i-1},\ \zeta_i]$. Let it be $[\zeta_i,\ \eta]$. Then, $\eta - \zeta_i = \xi(\zeta_i - \zeta_{i-1})$. If $\eta \leq \zeta_L$, then by (1) of Lemma 2, there exists n with $i+1 < n \leq L$ such that $\eta = \zeta_n$. The proof for the case $j < 0$ is similar.

(2) Let K be the nearest common ancester of $[a,b]$, $[b,c]$ and $[c,d]$. It is sufficient to prove that $K = [a,d]$. Suppose to the contrary that $K \neq [a,d]$. Then, $[b,c]$ has level $j > 1$ relative to K and is not middle. Assume that it is left. Then, $[c,d]$ is middle since $[b,c]$ and $[c,d]$ have the same level. Thus, $(d-c)/(c-b) = 1/4$ contradicting with the assumption. If $[b,c]$ is right, we have $(c-b)/(b-a) = 4$ contradicting with the assumption.

(3) Since neither of $[a,b]$ and $[b,c]$ is middle by the assumption, it holds that $[a,b]$ is right and $[b,c]$ is left. Then, the first ancestor of $[a,b]$ is $[b-(9/4)(b-a),\ b]$ and the first ancestor of $[b,c]$ is $[b,\ b+(9/4)(c-b)]$.

(4) Let K be the nearest common ancester of $[a,b]$ and $[b,c]$. If K is not the first ancestor of $[a,b]$ and $[b,c]$, then $[b,c]$ is left, which contraicts with $(d-c)/(c-b) = 4$. Hence, K is the first ancestor of $[a,b]$ and $[b,c]$. This implies that $K = [c-(9/4)(c-b),\ c]$ and that K is not an ancestor of $[c,d]$, since $(c-b)/(b-a) = 4$. Therefore, the nearest common ancestor of $[b,c]$ and $[c,d]$ is not the first ancestor of them. Thus, $[c,d]$ is left and the first ancestor of $[c,d]$ is $[c,\ c+(9/4)(d-c)]$ □

Let $\zeta = \{\zeta_0 < \zeta_1 < \cdots < \zeta_L\}$ and $\eta = \{\eta_0 < \eta_1 < \cdots < \eta_M\}$ be synchronized nets such that η is measurable with respect to ζ. We say that η is a **reduction** of ζ if $\eta_0 \leq \zeta_0 < \zeta_L \leq \eta_M$ and $\{\eta_1 < \eta_2 < \cdots < \eta_{M-1}\} \subset \{\zeta_1 < \zeta_2 < \cdots < \zeta_{L-1}\}$ holds.

Theorem 7. *For any Y_J-measurable synchronized net $\zeta = \{\zeta_0 < \zeta_1 < \cdots < \zeta_L\}$, there exists a reduction of it consisting at most of* 3 *synchronized intervals with the same level.*

Proof. Let $\eta = \{\eta_0 < \eta_1 < \cdots < \eta_M\}$ be a reduction of ζ with the smallest number of intervals M. If the levels of the synchronized intervals contained in it are not the same, then there exists $i = 0, 1, \cdots, M-1$ such that $(\eta_{i+1} - \eta_i)/(\eta_i - \eta_{i-1}) = \xi(4/9)^j$ with $\xi \in \{1/4, 4\}$ and $j \neq 0$.

If $j > 0$, then by Lemma 9, $[\eta_i,\ \eta_i + \xi(\eta_i - \eta_{i-1})]$ is a synchronized interval and either there exists n with $i+1 < n \leq M$ such that $\eta_n = \eta_i + \xi(\eta_i - \eta_{i-1})$ or $\eta_i + \xi(\eta_i - \eta_{i-1}) > \eta_L$. In the former case, we have a further reduction of ζ, $\{\eta_0 < \eta_1 < \cdots < \eta_i < \eta_n < \cdots < \eta_M\}$ with a number of intervals less than M contradicting with the assumption on M. In the latter case, we have a futher reduction of ζ, $\eta' := \{\eta_0 < \eta_1 < \cdots < \eta_i < \eta_i + \xi(\eta_i - \eta_{i-1})\}$, which has a number of intervals at most M. By the assumption on M, it is exactly M and $i = M-1$.

If $j < 0$, then by Lemma 9, $[\eta_i - \xi(\eta_{i+1} - \eta_i),\ \eta_i]$ is a synchronized interval and either there exists n with $0 \leq n < i-1$ such that $\eta_n = \eta_i - \xi(\eta_{i+1} - \eta_i)$ or $\eta_i - \xi(\eta_{i+1} - \eta_i) < \eta_0$. In the former case, we have a further reduction of ζ, $\{\eta_0 < \eta_1 < \cdots < \eta_n < \eta_i < \cdots < \eta_M\}$ with a number of intervals less than M contradicting with the assumption on M. In the latter case, we have a futher reduction of ζ, $\eta' := \{\eta_i - \xi(\eta_{i+1} - \eta_i) < \eta_i < \cdots < \eta_M\}$, which has a number of intervals at most M. By the assumption on M, it is exactly M and $i = 1$.

If the levels of the synchronized intervals contained in η' are not the same, we repeat the above procedure to get finally a futher reduction of ζ such that it has M number of synchronized intervals with the same level. Hence, we may assume that $\eta = \{\eta_0 < \eta_1 < \cdots < \eta_M\}$ is a reduction of ζ wich has the smallest number of intervals M with the same level.

Suppose that $M \geq 4$. Then, in the sequence of $(\eta_{i+1} - \eta_i)/(\eta_i - \eta_{i-1})$ $(i = 1, 2, \cdots, M-1)$, there exists $i = 1, 2, \cdots, M-2$ such that the combination $((\eta_{i+1} - \eta_i)/(\eta_i - \eta_{i-1}),\ (\eta_{i+2} - \eta_{i+1})/(\eta_{i+1} - \eta_i))$ is either $(1/4,\ 4)$, $(1/4,\ 1/4)$ or $(4,\ 4)$. Then by Lemma 9, we find a further reduction of ζ with a smaller number of intervals, contradicting with the assumption on M. Hence $M \leq 3$. $\square$

Theorem 8. *For any bounded closed interval $J = [a, b]$ with $a < b$, there exists measurable functionals $\tau : C(J) \to [0, \infty)$ and G :*

$C(J) \to \Theta$ *such that*
(1) $\Pr[\ G(Y_J)(t) = \mathbf{N}_{b+t} - \mathbf{N}_b \mid t \le \tau(Y_J)\ ] = 1$ *for any* $t > 0$, *and*
(2) $\Pr[\ \tau(Y_J) < t\ ] \le 9t/(4B)$ *for any* $t > 0$,
where $C(J)$ *is the space of continuous functions on* J *and we put* $B := (b-a)/21$.

Proof. By Theorem 6, there exists a Y_J-measurable synchronized net covering $[a,b]$. Taking its reduction obtained in Theorem 7, we get a Y_J-measurable synchronized net $\eta := \{\eta_0 < \eta_1 < \cdots < \eta_M\}$ satisfying that
(i) $M \le 3$,
(ii) the synchronized intervals in η have the same level, and
(iii) $\eta_0 \le a < b \le \eta_M$.

Define $\tau = \tau(Y_J) := \eta_M - b$ and

$$G(Y_J)(t) := \begin{cases} 0 & t < 0 \\ \mathbf{N}_{b+t} - \mathbf{N}_b & 0 \le t \le \tau \\ \mathbf{N}_{b+\tau} - \mathbf{N}_b & t > \tau. \end{cases}$$

Then, (1) is clear from the definitions of τ and G together with (3) of Theorem 6.

Let $b \in [\eta_i,\ \eta_{i+1}]$. Then, it holds that

$$\eta_{i+1} - \eta_i \ge (\eta_M - \eta_0)/(1 + 4 + 4^2) \ge (b-a)/21 = B.$$

Let $[u,v]$ be the minimal synchronized interval containing b with $v - u \ge B$. Since $[u,v] \subset [\eta_i,\ \eta_{i+1}]$, we have

$$\tau' := v - b \le \eta_{i+1} - b \le \eta_M - b = \tau.$$

Take $t > 0$ with $t \le (4/9)B$ and let $n = [B/t]$. If $\tau'(\omega) \in [0,t)$, then $\tau'(\omega - jt) \in [jt, (j+1)t)$ for any $j = 0, 1, \cdots, n-1$. Hence, for any $j = 0, 1, \cdots, n-1$, we have

$$\begin{aligned} & Pr(\tau'(\omega) \in [0,t)) \\ \le\ & Pr(\tau'(\omega - jt) \in [jt, (j+1)t)) \\ =\ & Pr(\tau'(\omega) \in [jt, (j+1)t)), \end{aligned}$$

where we used the fact that the probability measure P is invariant under the addition. Therefore, we have $Pr(\tau' < t) \leq 1/n$, since

$$\begin{aligned} & nPr(\tau' \in [0,t)) \\ \leq & \sum_{i=0}^{n-1} P(\tau' \in [jt,(j+1)t)) \\ \leq & Pr(\tau' \in [0,B)) \leq 1. \end{aligned}$$

Thus we have (2), since

$$Pr(\tau < t) \leq Pr(\tau' < t) \leq 1/n \leq 9t/(4B)$$

for any $t < 4B/9$. For $t \geq 4B/9$, (2) holds trivially since $9t/(4B) \geq 1$. $\square$

We construct a predictor for Y_c with $c > b$ based on the observation Y_J, where $J = [a,b]$. We use $G(Y_J)(c)$ for to estimate $\mathbf{N}_c - \mathbf{N}_b$. By Theorem 8, if $c-b \leq \tau(Y_J)$. then the estimation is exact. To estimate $Y_c = H(\mathbf{N}_c, c)$, we use the Taylor expansion at $(\mathbf{N}_b,\ b)$ with $G(Y_J)(c)$ for $\mathbf{N}_c - \mathbf{N}_b$:

$$\hat{Y}_c := Y_b + H_x(\mathbf{N}_b,b)G(Y_J)(c) + \frac{1}{2}H_{xx}(\mathbf{N}_b,b)G(Y_J)(c)^2 + H_s(\mathbf{N}_b,b)(c-b).$$

Note that $\hat{Y}_c$ is a measurable function of the observation Y_J by Theorem 6. The value can be calculated based on the observation without using any further information on the unknown function H than (H1) and (H2).

Theorem 9. *It holds that*

$$\mathrm{E}[(\hat{Y}_c - Y_c)^2] = o((c-b)^2) + O(\frac{(c-b)^2}{b-a})$$

as $c \downarrow b$ *with* $C(b)$ *in* (2) *in Section 1 as the constant in* $O(\ )$.

Proof. Since

$$\begin{aligned} Y_c &= Y_b + H_x(\mathbf{N}_b,b)G(Y_J)(c) \\ &\quad + \frac{1}{2}H_{xx}(\mathbf{N}_b,b)G(Y_J)(c)^2 + H_s(\mathbf{N}_b,b)(c-b) + o(c-b), \end{aligned}$$

$\hat{Y}_c - Y_c = o(c-b)$ holds if $c - b \leq \tau(Y_J)$. If otherwise, $\hat{Y}_c - Y_c = O((c-b)^{1/2})$ since $|G(Y_J)(c)| \leq (c-b)^{1/2}$, $|\mathbf{N}_c - \mathbf{N}_b| \leq (c-b)^{1/2}$ and

$$|G(Y_J)(c) - (\mathbf{N}_c - \mathbf{N}_b)| = |\mathbf{N}_{\tau(Y_J)} - \mathbf{N}_b| \leq (c-b)^{1/2},$$

so that

$$(\hat{Y}_c - Y_c)^2 \leq (1+\delta) \sup_{|x| \leq |b|^{1/2}} |H_x(x,b)|^2 (c-b)$$

for any $\delta > 0$ as $c \to b$. Since by Theorem 8, $\Pr[\tau(Y_J) < c - b] \leq 48(c-b)/(b-a)$, we have

$$\begin{aligned} & \mathrm{E}[(\hat{Y}_c - Y_c)^2] \\ = \; & \mathrm{E}[(\hat{Y}_c - Y_c)^2 \; | \tau(Y_J) \geq c-b] \; Pr[\tau(Y_J) \geq c-b] \\ & \quad + \mathrm{E}[(\hat{Y}_c - Y_c)^2 \; | \tau(Y_J) < c-b] \; Pr[\tau(Y_J) < c-b] \\ \leq \; & o(c-b)^2 + O(\frac{(c-b)^2}{b-a}) \end{aligned}$$

with $C(b)$ in (2) as the constant in $O(\ )$. □

acknowledgment: The author appreciate the anonymous referee for his suggestions and encouragement.

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