Two dimensional word with 2k maximal pattern complexity

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1 Introduction

For an infinite 1-dimensional word $\alpha = \alpha_0\alpha_1\alpha_2\cdots$ over a finite alphabet A, Teturo Kamae and Luca Zamboni [1] introduced the maximal pattern complexity as

$$p^*_\alpha(k) := \sup_\tau \sharp\{\alpha_{n+\tau(0)}\alpha_{n+\tau(1)}\cdots\alpha_{n+\tau(k-1)};\ n = 0, 1, 2, \cdots\}$$

where the supremum is taken over all sequences of integers $0 = \tau(0) < \tau(1) < \cdots < \tau(k-1)$ of length k, and $\sharp S$ denotes the cardinality of the set S. They proved that α is eventually periodic if and only if $p^*_\alpha(k)$ is bounded in k, while otherwise, $p^*_\alpha(k) \geq 2k$ $(k = 1, 2, \cdots)$.

Teturo Kamae, Rao Hui and Xue Yu-Mei [3] considered the maximal pattern complexity for 2-dimensional words defined on $\mathbb{Z}^2$ and proved that either $p^*_\alpha(k)$ is bounded in k or $p^*_\alpha(k) \geq 2k$ $(k = 1, 2, \cdots)$ if α satisfies a 2-dimensional recurrence condition.

In this paper, we consider the maximal pattern complexity for 2-dimensional words defined on

$$\Omega := \mathbb{N}^2 \setminus \{(0, 0)\}.$$

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Let $\alpha = (\alpha(x,y)_{(x,y)\in\Omega}) \in A^{\Omega}$ be a 2-dimensional word over $\mathbf{A} = \{0,1\}$ defined on Ω. Let τ be a finite set in $\mathbb{Z}^2$ with $(0,0) \in \tau$ and $\sharp\tau = k$, which is called a *k-window*. For any $i \in \Omega$ with $i + \tau \subset \Omega$, we denote

$$\alpha[i+\tau] := (\alpha(i+j))_{j\in\tau} \in A^{\tau}.$$

We also denote

$$\begin{aligned} F_{\tau}(\alpha) &:= \{(\alpha[i+\tau];\ i \in \Omega \text{ with } i+\tau \subset \Omega\} \\ p^*_{\alpha}(k) &:= \sup\{\sharp F_{\alpha}(\tau);\ \tau : k\text{-window}\} \quad (k = 1, 2, \cdots). \end{aligned}$$

Definition 1: α is called *eventually* 2*-periodic* if there exist $p, q \in \mathbb{Z}_+$ and $a, b \in \mathbb{N}$ such that for any $(x,y) \in \Omega$, $\alpha(x,y) = \alpha(x+p,y)$ holds if $x \geq a$ and $\alpha(x,y) = \alpha(x,y+q)$ holds if $y \geq b$.

Definition 2: α is called *minimal* if for any positive integer L, there exists N such that for any $(n,m) \in \Omega$ there exists $(n',m') \in \Omega$ with $|n-n'| \leq N,\ |m-m'| \leq N$ such that $\alpha(x+n', y+m') = \alpha(x,y)$ holds for any $(x,y) \in \Omega$ with $x < L,\ y < L$.

Definition 3: α is called *sectionally periodic* if for any $(a,b),\ (p,q) \in \Omega$, the word β on $n \in \mathbb{N}$ defined by $\beta(n) = \alpha(a+np, b+nq)$ is periodic.

In this paper, we characterize the words with bounded maximal pattern complexity. We give an example of word α with $p^*_{\alpha}(k) = 2k$ $(k = 1, 2, \cdots)$ which is minimal and sectionally periodic.

2 Words with bounded maximal pattern complexity

Theorem 1. *α is eventually 2-periodic if and only if $p^*_{\alpha}(k)$ is bounded in k.*

Proof. Assume that α is eventually 2-periodic. Take $p, q \in \mathbb{Z}_+$ and $a, b \in \mathbb{N}$ such that for any $(x,y) \in \Omega$, $\alpha(x,y) = \alpha(x+p,y)$ holds if $x \geq a$ and $\alpha(x,y) = \alpha(x,y+q)$ holds if $y \geq b$.

Let τ be a k-window. Let

$$\begin{aligned}
\Omega_1 &:= \{i=(x,y)\in\Omega;\ i+\tau\subset\Omega\cap[a,\infty)\times[b,\infty)\} \\
\Omega_2 &:= \{i=(x,y)\in\Omega\setminus\Omega_1;\ i+\tau\subset\Omega\cap[a,\infty)\times[0,\infty)\} \\
\Omega_3 &:= \{i=(x,y)\in\Omega\setminus\Omega_1;\ i+\tau\subset\Omega\cap[0,\infty)\times[b,\infty)\} \\
\Omega_4 &:= \{i=(x,y)\in\Omega\setminus(\Omega_1\cup\Omega_2\cup\Omega_3);\quad i+\tau\subset\Omega\}.
\end{aligned}$$

For any $i=(x,y)\in\Omega_1$, we have

$$\alpha[i+(np,mq)+\tau]=\alpha[i+\tau]\ \ (\forall n,m=0,1,2,\cdots).$$

Therefore, there exist at most pq different elements among $\alpha[i+\tau]$ with $i=\Omega_1$.

For any $i=(x,y)\in\Omega_2$, we have

$$\alpha[i+(np,0)+\tau]=\alpha[i+\tau]\ \ (\forall n=0,1,2,\cdots).$$

Hence, there exist at most pb different elements among $\alpha[i+\tau]$ with $i=\Omega_2$.

In the same way, there exist at most qa different elements among $\alpha[i+\tau]$ with $i=\Omega_2$. Finally, there exist at most ab elements in Ω_4.

Therefore, we have

$$\sharp F_\alpha(\tau)\le pq+pb+qa+ab=(p+a)(q+b).$$

Thus, $p^*_\alpha(k)\le(p+a)(q+b)$ for $k=1,2,\cdots$, and hence, $p^*_\alpha(k)$ is bounded in k.

Conversely, assume that $\sup_{k=1,2,\cdots} p^*_\alpha(k)=C<\infty$. There exist $k=1,2,\cdots$ and a k-window τ such that $\sharp F_\alpha(\tau)=C$. Take a positive integer L such that τ is contained in a square of size $L\times L$. Let σ be the $(L+1)^2$-window such that

$$\sigma=\{(x,y)\in\Omega;\ 0\le x\le L,\ 0\le y\le L\}$$

and σ' be the $(L+2)^2$-window such that

$$\sigma'=\{(x,y)\in\Omega;\ 0\le x\le L+1,\ 0\le y\le L+1\}.$$

Since

$$C = \sharp F_\alpha(\tau) \le \sharp F_\alpha(\sigma) \le \sharp F_\alpha(\sigma') \le C,$$

we have $\sharp F_\alpha(\sigma) = \sharp F_\alpha(\sigma') = C$. This implies that each element $\xi \in F_\alpha(\sigma)$ has a unique extension in $F_\alpha(\sigma')$. Therefore, there exists a function $h : F_\alpha(\sigma) \to F_\alpha(\sigma')$ such that $h(\alpha[i+\sigma]) = \alpha[i+\sigma']$ for any $i \in \Omega$.

In particular, there exist functions $f, g : F_\alpha(\sigma) \to F_\alpha(\sigma)$ such that

$$\begin{aligned} f(\alpha[i+\sigma]) &= \alpha[i+(1,0)+\sigma] \\ g(\alpha[i+\sigma]) &= \alpha[i+(0,1)+\sigma] \end{aligned} \tag{1}$$

for any $i \in \Omega$.

Since f is a transformation on a finite set, there exist $a \in \mathbb{N}$ and a period $p \in \mathbb{Z}_+$ such that

$$f^{n+p} = f^n \tag{2}$$

any $n = a, a+1, a+2, \cdots$. Since

$$\alpha[(x,y)+\sigma] = f^x(\alpha[(0,y)+\sigma])$$

by (1), it follows from (2) that

$$\alpha[(x,y)+\sigma]) = \alpha[(x+p,y)+\sigma]$$

for any $(x,y) \in \Omega$ with $x \ge a$.

In particular, we have

$$\alpha(x,y) = \alpha(x+p,y)$$

for any $(x,y) \in \Omega$ with $x \ge a$. In the same way, we have

$$\alpha(x,y) = \alpha(x,y+q)$$

for any $(x,y) \in \Omega$ with $y \ge b$. Thus, α is eventually 2-periodic. □

3 A word with $2k$ maximal pattern complexity

A window τ' is said to be an *immediate extension* of a window τ if $\tau' \supset \tau$ and $\sharp\tau' = \sharp\tau + 1$.

The following Lemma 1 is proved in [2] (Theorem 3) for words defined on $\mathbb{N}$. It remains true for words defined on Ω.

Lemma 1. *Let $\alpha \in \{0,1\}^\Omega$ be such that $p^*_\alpha(2) = 4$. Assume that for any 2-window τ and for any immediate extension τ' of τ, it holds that $\sharp F_\alpha(\tau') \leq \sharp F_\alpha(\tau) + 2$. Then, we have $p^*_\alpha(k) \leq 2k$ $(k = 1, 2, \cdots)$.*

Define a 2-dimensional word $\alpha \in \{0,1\}^\Omega$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } e_2(x) = e_2(y) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

for any $(x,y) \in \Omega$, where for $x \in \mathbb{N}$, $e_2(x) = n$ if and only if $2^n \mid x$ and $2^{n+1} \nmid x$. We also define $e_2(0) = \infty$.

Remark 1. The word α defined by (3) together with $\alpha((0,0)) = 0$ is the fixed point of the 2-dimensional substitution

$$\sigma: \quad 0 \to \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \quad \text{and} \quad 1 \to \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}, \tag{4}$$

so that $\alpha = \sigma^\infty(0)$.

Theorem 2. *For α defined by (3), we have $p^*_\alpha(k) = 2k$ for any $k = 1, 2, \cdots$.*

Proof. First we prove that $p^*_\alpha(k) \geq 2k$ $(k = 1, 2, \cdots)$. It is clear that $p^*_\alpha(1) = 2$. For any $k = 2, 3, \cdots$, take a k-window $\tau := \{(0,0), (1,1), \cdots, (k-1, k-1)\}$. Then, since

$$\begin{aligned} \alpha[(1,1) + \tau] &= (1, 1, \cdots, 1) \\ \alpha[(2^k - n,\ 2^{k+1} - n) + \tau] &= (1, \cdots, 1, \overset{(n)}{0}, 1, \cdots, 1) \\ & \quad (n = 0, 1, \cdots, k-1), \end{aligned}$$

$F_\alpha(\tau)$ contains $k+1$ elements containing the letter 0 at most once.

Now, let us consider the elements in $F_\alpha(\tau)$ containing the letter 0 at least twice. They are determined by $a \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $0 \le a < 2^n$ and $a + 2^n < k$ since there exists a unique element in $F_\alpha(\tau)$ of the form

$$(1, \cdots, 1, \overset{(a)}{0}, 1, \cdots, 1, \overset{(a+2^n)}{0}, ***)$$

which is realized as $\alpha[(2^n - a,\ 2^{n+1} - a) + \tau]$. There are exactly

$$L := \sum_{n=0}^{\lfloor \log_2 k \rfloor} \min\{2^n, k - 2^n\}$$

number of elements of this type. Since

$$\begin{aligned} L &= \sum_{n=1}^{\lfloor \log_2 k \rfloor - 1} 2^n + k - 2^{\lfloor \log_2 k \rfloor} \\ &= 2^{\lfloor \log_2 k \rfloor} - 1 + k - 2^{\lfloor \log_2 k \rfloor} = k - 1, \end{aligned}$$

we have $\sharp F_\alpha(\tau) = k+1+k-1 = 2k$. Thus, $p^*_\alpha(k) \ge 2k$ $(k = 1, 2, \cdots)$.

To prove that $p^*_\alpha(k) \le 2k$ $(k = 1, 2, \cdots)$, it is sufficient by Lemma 1 to prove that for any 2-window τ and for any immediate extension τ' of τ, it holds that

$$\sharp F_\alpha(\tau') \le \sharp F_\alpha(\tau) + 2. \tag{5}$$

Take an arbitrary 2-window $\tau = \{(0,0) = \tau_0,\ \tau_1\}$ and an arbitrary immediate extension $\tau' = \{(0,0) = \tau_0,\ \tau_1, \tau_2\}$ of τ.

To prove (5), we divide into 3 cases according to the parity of τ_1

$$\begin{aligned} \text{Case 1} &: \tau_1 \in e \times e \\ \text{Case 2} &: \tau_1 \in e \times o \\ \text{Case 3} &: \tau_1 \in o \times o, \end{aligned}$$

where "e" stands for the set of even numbers, while "o" stands for the set of odd numbers. By symmetry, we can reduce the case $\tau_1 \in o \times e$ to Case 2.

Lemma 2.
(i) *In* Case 1, $F_\alpha(\tau) = \{(0,0),(0,1),(1,0),(1,1)\}$ *holds.*
(ii) *In* Case 2, $F_\alpha(\tau) = \{(0,0),(0,1),(1,0)\}$ *holds.*
(iii) *In* Case 3, $F_\alpha(\tau) = \{(0,0),(0,1),(1,0),(1,1)\}$ *holds.*

Proof. Let $\tau_1 = (u,v)$.

(i) Let $(u,v) \in e\times e$. For $(x,y) \in e\times o$, we have $\alpha[(x,y)+\tau] = (0,0)$. If $u = v$, then by taking integers N and M with $e_2(u) < N < M$, we have $\alpha[(2^N, 2^M) + \tau] = (0,1)$. If $u \neq v$, then assuming that $u < v$ without loss of generality, we have $\alpha[(v-u,0)+\tau] = (0,1)$. If $u \neq v$, then we have $\alpha[(2^N v - u, 2^N v - u) + \tau] = (1,0)$ for a sufficiently large integer N. If $u = v$, then by taking integers N and M with $e_2(u) < N < M$, we have $\alpha[(2^N - u, 2^M - v) + \tau] = (1,0)$. Finally, for $(x,y) \in o\times o$, we have $\alpha[(x,y)+\tau] = (1,1)$.

(ii) Let $(u,v) \in e \times o$. Then, $\alpha[(2,4)+\tau] = (0,0)$, $\alpha[(v,u)+\tau] = (0,1)$, $\alpha[(1,1)+\tau] = (1,0)$, while $\alpha[(x,y)+\tau] = (1,1)$ is impossible since either x and y have different parities or $x+u$ and $y+v$ have different parities.

(iii) Let $(u,v) \in o\times o$. For $(x,y) \in e \times o$, we have $\alpha[(x,y)+\tau] = (0,0)$. We also have $\alpha[(2,4)+\tau] = (0,1)$ and $\alpha[(2^N - u, 2^M - v)+\tau] = (1,0)$ for integers N and M such that $u+v < 2^N < 2^M$. Moreover, $\alpha[(2,2)+\tau] = (1,1)$. □

We divide the above 3 cases into the following 10 subcases according to the parity of τ_2

$$
\begin{aligned}
\text{Case 1-1} &: \tau_1 \in e \times e,\ \tau_2 \in e \times e \\
\text{Case 1-2} &: \tau_1 \in e \times e,\ \tau_2 \in e \times o \\
\text{Case 1-3} &: \tau_1 \in e \times e,\ \tau_2 \in o \times o \\
\text{Case 2-1} &: \tau_1 \in e \times o,\ \tau_2 \in e \times e \\
\text{Case 2-2} &: \tau_1 \in e \times o,\ \tau_2 \in e \times o \\
\text{Case 2-3} &: \tau_1 \in e \times o,\ \tau_2 \in o \times e \\
\text{Case 2-4} &: \tau_1 \in e \times o,\ \tau_2 \in o \times o \\
\text{Case 3-1} &: \tau_1 \in o \times o,\ \tau_2 \in e \times e \\
\text{Case 3-2} &: \tau_1 \in o \times o,\ \tau_2 \in e \times o \\
\text{Case 3-3} &: \tau_1 \in o \times o,\ \tau_2 \in o \times o.
\end{aligned}
$$

Lemma 3.
(i) *In* Case 1-2, $F_\alpha(\tau') \subset \{0,1\}^3 \setminus \{(0,1,1),(1,0,1),(1,1,1)\}$.
(ii) *In* Case 1-3, $F_\alpha(\tau') \subset \{0,1\}^3 \setminus \{(0,1,0),(1,0,0)\}$.
(iii) *In* Case 2-1, $F_\alpha(\tau') \subset F_\alpha(\tau) \times \{0,1\} \setminus \{(0,1,1)\}$.
(iv) *In* Case 2-2, $F_\alpha(\tau') \subset F_\alpha(\tau) \times \{0,1\} \setminus \{(1,0,1)\}$.
(v) *In* Case 2-3, $F_\alpha(\tau') \subset F_\alpha(\tau) \times \{0,1\} \setminus \{(1,0,1)\}$.
(vi) *In* Case 2-4, $F_\alpha(\tau') \subset F_\alpha(\tau) \times \{0,1\} \setminus \{(0,1,1)\}$.
(vii) *In* Case 3-1, $F_\alpha(\tau') \subset \{0,1\}^3 \setminus \{(0,0,1),(1,0,0)\}$.
(viii) *In* Case 3-2, $F_\alpha(\tau') \subset \{0,1\}^3 \setminus \{(0,1,1),(1,0,1),(1,1,1)\}$.
(ix) *In* Case 3-3, $F_\alpha(\tau') \subset \{0,1\}^3 \setminus \{(0,0,1),(0,1,0)\}$.

Proof. Let $\tau_1 = (u,v)$, $\tau_2 = (u',v')$ and $(x,y) \in \Omega$.

(i) Since either x and y have different parities or $x+u'$ and $y+v'$ have different parities, $(1,0,1),(1,1,1)$ do not belong to $F_\alpha(\tau')$. Moreover, since either $x+u$ and $y+v$ have different parities or $x+u'$ and $y+v'$ have different parities, $(0,1,1)$ does not belong to $F_\alpha(\tau')$.

(ii) Note that $\alpha[(x,y)+\tau] \in \{(1,0),(0,1)\}$ implies $(x,y) \in e \times e$. Since $(x,y) \in e \times e$ implies $\alpha((x,y)+(u',v')) = 1$, $(0,1,0)$ and $(1,0,0)$ do not belong to $F_\alpha(\tau')$.

(iii)(iv)(v)(vi)(viii) They follow by applying the parity argument in the proof of (i).

(vii) It follows by the same argument as in the proof of (ii).

(ix) Note that $\alpha((x,y)+(u,v))) \neq \alpha((x,y)+(u',v'))$ implies $(x,y) \in o \times o$. Since $(x,y) \in o \times o$ implies that $\alpha((x,y)) = 1$, $(0,0,1),(0,1,0)$ does not belong to $F_\alpha(\tau')$. □

Lemma 4.
(i) *For any subcase except for* Case 1-1*, we have* (5)*.*
(ii) *For any subcase except for* Case 1-1*, we have*

$$\sharp(F_\alpha(\tau') \setminus \{(0,0,0),(1,1,1)\}) \leq 4. \tag{6}$$

Proof. Clear from Lemma 2 and Lemma 3. □

Now we consider Case 1-1. Assume that $\tau_1 \in e \times e$, $\tau_2 \in e \times e$. Then, we have $\alpha[(x,y)+\tau'] = (1,1,1)$ if $(x,y) \in o \times o$ and $\alpha[(x,y)+\tau'] = (0,0,0)$ if $(x,y) \in e \times o \cup o \times e$. Hence we have

$$F_\alpha(\tau') = \{\alpha[(x,y)+\tau'];\ (x,y) \in e \times e\} \cup \{(0,0,0),(1,1,1)\}.$$

Let $\tau'/2 := \{0, \tau_1/2, \tau_2/2\}$. Since $e_2(x) = e_2(y)$ is equivalent to $e_2(2x) = e_2(2y)$, we have $\alpha[(x,y)+\tau'] = \alpha[(x/2,y/2)+\tau'/2]$ for any $(x,y) \in e \times e$. Therefore, we have

$$F_\alpha(\tau') = F_\alpha(\tau'/2) \cup \{(0,0,0),(1,1,1)\}. \tag{7}$$

If $\tau'/2$ is of Case 1-1, we can apply (7) again.

By applying (7) repeatedly, we have

$$F_\alpha(\tau') = F_\alpha(\tau'/2^e) \cup \{(0,0,0),(1,1,1)\}$$

with $\tau'/2^e$ not of Case 1-1. Then, by (ii) of Lemma 4, we have $\sharp F_\alpha(\tau') \leq 6$. Thus, we have (5) by Lemma 2, which complete the proof of Theorem 2. □

Theorem 3. *The word α defined by* (3) *is minimal and sectionally periodic.*

Proof. Take any positive integer L. Let N be a positive integer such that $L < 2^N$. Take any $(n,m) \in \Omega$. Then, there exists $(n',m') \in \Omega$ with $|n-n'| \leq 2^N$ and $|m-m'| \leq 2^N$ such that $e_2(n') \geq N$ and

$e_2(m') \geq N$. Then, since $e_2(x+n') = e_2(x)$ and $e_2(y+m') = e_2(y)$ for any $(x,y) \in \Omega$ with $x < L$ and $y < L$, we have $\alpha(x+n', y+m') = \alpha(x,y)$ for any $(x,y) \in \Omega$ with $x < L$ and $y < L$. Thus, α is minimal.

Take any $(a,b),(p,q) \in \Omega$. Let β be a word on $n \in \mathbb{N}$ defined by $\beta(n) = \alpha(a+np, b+nq)$.

Let us consider the case where $a+p=0$ or $b+q=0$. Without loss of generality, assume $a+p=0$. Then, we have $a=p=0$ and $b>0,\ q>0$. Hence, β is periodic since $\beta(n)=0$ $(n=0,1,2,\cdots)$.

Now assume that $a+p>0$ and $b+q>0$. Let us consider the case where $aq-bp=0$. Suppose that $p=0$. Then, $a>0$ and $q>0$ since $a+p>0$ and $p+q>0$. This contradicts with $aq-bp=0$. Therefore, $p>0$. By the same reason, $q>0$. Since $q(a+np)=p(b+nq)$ for $n=0,1,2,\cdots$, we have $e_2(q)+e_2(a+np) = e_2(p)+e_2(b+nq)$ $(n=0,1,2,\cdots)$. Therefore, either $\beta(n)=1$ $(n=0,1,2,\cdots)$ or $\beta(n)=0$ $(n=0,1,2,\cdots)$ holds according as $e_2(q)=e_2(p)$ or not, and hence, β is periodic.

Now assume that $aq-bp \neq 0$. Let N be a positive integer such that $N > e_2(|aq-bp|)$. Then, since $q(a+np)-p(b+nq) = aq-bp$ $(n=0,1,2,\cdots)$, we have $e_2(|q(a+np)-p(b+nq)|) < N$ $(n=0,1,2,\cdots)$. This implies that $\min\{e_2(q(a+np)), e_2(p(b+nq))\} < N$, and hence, $\min\{e_2(a+np), e_2(b+nq)\} < N$ $(n=0,1,2,\cdots)$. Therefore, if $e_2(a+np)=e_2(b+nq)$, then $e_2(a+np)=e_2(b+nq)<N$ holds, and hence, we have $e_2(a+(n+2^N)p) = e_2(a+np) = e_2(b+nq) = e_2(b+(n+2^N)q)$.

If $e_2(a+np) < e_2(b+nq)$, then either $e_2(a+np) < e_2(b+nq) \leq N$ or $e_2(a+np) < N \leq e_2(b+nq)$ holds, and hence, we have $e_2(a+(n+2^N)p) = e_2(a+np) < \min\{e_2(b+nq), N\} \leq e_2(b+(n+2^N)q)$. In the same way, if $e_2(a+np) > e_2(b+nq)$, then $e_2(a+(n+2^N)p) > e_2(b+(n+2^N)q)$.

Hence, we proved that $e_2(a+np)=e_2(b+nq)$ holds if and only if $e_2(a+(n+2^N)p) = e_2(b+(n+2^N)q)$ holds, so that $\beta(n) = \beta(n+2^N)$ $(n=0,1,2,\cdots)$ and β is periodic.

Thus, α is sectionally periodic. □

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