Linear expansions, strictly ergodic homogeneous cocycles and fractals

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Abstract

We consider a compact space Ω on which $\mathbf{R}$ acts additively and $\mathbf{R}_+$ acts multiplicatively satisfying the distributive law. Moreover, $\mathbf{R}$-action is strictly ergodic. Such Ω is constructed as a space of colored tilings corresponding to a weighted substitution, which is a kind of natural extension of the f-expansion for a piecewise linear f. We define a homogeneous cocycle F on Ω, which was called a cocycle with the scaling property in [5]. This is a realization of fractal functions which admit the continuous scalings. This also defines a self-similar process with strictly ergodic, stationary increments which has 0 entropy.

1 Introduction

Let Ω be a complete separable metrizable space. Let G be a non-trivial, closed, multiplicative subgroup of $\mathbf{R}_+$, the set of positive real numbers. That is, either $G = \mathbf{R}_+$ or there exists $\lambda > 1$ such that $G = \{\lambda^n; n \in \mathbf{Z}\}$. Assume that $(\mathbf{R}, G)$ **acts** on Ω, that is,

(1) For any $\omega \in \Omega$, $t \in \mathbf{R}$ and $\lambda \in G$, $\omega + t$ and $\lambda\omega$ are defined and belong to Ω so that the mappings $(\omega, t) \mapsto \omega + t$ and $(\omega, \lambda) \mapsto \lambda\omega$ are continuous,

(2) $\cdot + 0 = 1\cdot = \mathrm{id}_\Omega$, and

(3) for any $\omega \in \Omega$, $s, t \in \mathbf{R}$ and $\lambda, \eta \in G$, it holds that

$$(\omega + t) + s = \omega + (t + s), \quad \lambda(\eta\omega) = (\lambda\eta)\omega, \quad \lambda(\omega + t) = \lambda\omega + \lambda t.$$

Let $(\mathbf{R}, G)$ act on Ω. A continuous function $F : \Omega \times \mathbf{R} \to \mathbf{R}$ is called a **cocycle** on Ω if

$$F(\omega, t + s) = F(\omega, t) + F(\omega + t, s)$$

holds for any $\omega \in \Omega$ and $s, t \in \mathbf{R}$. A cocycle F on Ω is called α-G-**homogeneous** if

$$F(\lambda\omega, \lambda t) = \lambda^\alpha F(\omega, t)$$

for any $\omega \in \Omega$, $\lambda \in G$ and $t \in \mathbf{R}$, where α is a given real number with $0 < \alpha < 1$. It is simply called α- **homogeneous** if $G = \mathbf{R}_+$. We remark that the notion of homogeneuos cocycle is equivalent to the notion of cocycle with the scaling property in [5].

Example 1 *Let $\Omega = \mathbf{R}$ and $(\mathbf{R}, \mathbf{R}_+)$ act on $\mathbf{R}$ in the usual sense. Then, a cocycle F on Ω is a* **coboundary***, that is, there exists a continuous function $\varphi : \Omega \to \mathbf{R}$ such that*

$$F(\omega, t) = \varphi(\omega + t) - \varphi(\omega)$$

for any $\omega \in \Omega$ and $t \in \mathbf{R}$. Moreover, if F is α-homogeneous, then the above φ satisfies that

$$\varphi(\omega) = \begin{cases} A|\omega|^\alpha + C & (\omega \geq 0) \\ B|\omega|^\alpha + C & (\omega < 0). \end{cases}$$

In fact, the above φ is defined by $\varphi(\omega) = F(0, \omega)$.

Example 2 *Let $\tilde{\Omega}$ be the space of all continuous function $\omega : \mathbf{R} \to \mathbf{R}$ with $\omega(0) = 0$ with the compact open topology. Let $0 < \alpha < 1$. For any $\omega \in \tilde{\Omega}$, $t \in \mathbf{R}$ and $\lambda \in \mathbf{R}_+$, we define $\omega + t \in \tilde{\Omega}$ and $\lambda\omega \in \tilde{\Omega}$ by*

$$(\omega + t)(s) = \omega(t + s) - \omega(t) \quad \textit{and} \quad (\lambda\omega)(s) = \lambda^\alpha \omega(\lambda^{-1}s)$$

for any $s \in \mathbf{R}$. Then, $(\mathbf{R}, \mathbf{R}_+)$ acts on $\tilde{\Omega}$. Define

$$F(\omega, t) = \omega(t)$$

for any $\omega \in \tilde{\Omega}$ and $t \in \mathbf{R}$. Then, F is a α-homogeneous cocycle. Let μ be an $(\mathbf{R}, \mathbf{R}_+)$- **invariant** *probability Borel measure on $\tilde{\Omega}$, that is,*

$$d\mu(\omega + t) = d\mu(\omega) \quad \textit{and} \quad d\mu(\lambda\omega) = d\mu(\omega)$$

for any $t \in \mathbf{R}$ and $\lambda \in \mathbf{R}_+$. Then, $F(\omega, t)$ is considered as a stochastic process on the probability space $(\tilde{\Omega}, \mu)$ with the time parameter $t \in \mathbf{R}$. This process has stationary increments and is α-selfsimilar. The Wiener process is one of them for $\alpha = 1/2$.

We are interessted in Ω on which $(\mathbf{R}, G)$ acts and which is $\mathbf{R}$-**minimal**. That is,

(4) Ω is compact, and it holds that

$$\overline{\{\omega + t; t \in \mathbf{R}\}} = \Omega$$

for any $\omega \in \Omega$.

We call Ω to be $\mathbf{R}$-**strictly ergodic** if in addition, it is $\mathbf{R}$-**uniquely ergodic**, that is,

(5) there exists a unique **R-invariant** probability Borel measure μ on Ω, that is,

$$d\mu(\omega + t) = d\mu(\omega)$$

for any $t \in \mathbf{R}$.

In this case, μ is also G- **invariant**, that is,
(6)

$$d\mu(\lambda\omega) = d\mu(\omega)$$

for any $\lambda \in G$, since $d\mu(\lambda\omega)$ is $\mathbf{R}$-invariant and by the uniqueness is equal to $d\mu(\omega)$.

We remark that a cocycle on $\mathbf{R}$-minimal Ω is a minimal cocycle in the sense of [5] and vice versa.

Theorem 1 ([5]) *Let $(\mathbf{R}, G)$ act on Ω. Assume that Ω is $\mathbf{R}$-minimal. Then, for a nonzero α-G-homogeneous cocycle F, we have the following results.*
(i) There exists a constant C such that

$$|F(\omega, t) - F(\omega, s)| \leq C|t - s|^{\alpha}$$

for any $\omega \in \Omega$ and $s, t \in \mathbf{R}$. That is, the functions $F(\omega, t)$ on t for $\omega \in \Omega$ are uniformly α- Hölder continuous.
(ii) For any $\omega \in \Omega$ and $t \in \mathbf{R}$,

$$\limsup_{s \downarrow 0} \frac{1}{s^{\alpha}} |F(\omega, t + s) - F(\omega, t)| > 0$$

holds. That is, for any $\omega \in \Omega$ the function $F(\omega, \cdot)$ is nowhere locally β-Hölder continuous for any $\beta > \alpha$. In particular, $F(\omega, \cdot)$ is nowhere differentiable.

Theorem 2 *Let $(\mathbf{R}, G)$ act on Ω. Assume that Ω is $\mathbf{R}$-strictly ergodic with the unique $\mathbf{R}$-invariant probability Borel measure μ. Then, for a nonzero α-G-homogeneous cocycle F on Ω, we have*
(i) $\int F(\omega, t) d\mu(\omega) = 0$, and
(ii) $\int |F(\omega, \lambda t)|^{\frac{1}{\alpha}} d\mu(\omega) = \lambda \int |F(\omega, t)|^{\frac{1}{\alpha}} d\mu(\omega) > 0$
for any $t \in \mathbf{R}$ and $\lambda \in G$ with $t \neq 0$.

PROOF. (i) Without loss of generality, we assume that $t > 0$. Since the $\mathbf{R}$-action on (Ω, μ) is ergodic, we have by Theorem 1,

$$\begin{aligned}
|\int F(\omega,t)d\mu(\omega)| &= |\lim_{N\to\infty} \frac{1}{N}\int_0^N F(\omega+s,t)ds| \\
&= |\lim_{N\to\infty} \frac{1}{N}\int_0^N (F(\omega,s+t)-F(\omega,s))ds| \\
&= |\lim_{N\to\infty} \frac{1}{N}(\int_t^{N+t} F(\omega,s)ds - \int_0^N F(\omega,s))ds| \\
&= |\lim_{N\to\infty} \frac{1}{N}(\int_N^{N+t} F(\omega,s)ds - \int_0^t F(\omega,s))ds| \\
&\le \lim_{N\to\infty} \frac{1}{N}(\int_N^{N+t} |F(\omega,s)|ds + \int_0^t |F(\omega,s)|ds) \\
&\le \lim_{N\to\infty} \frac{tC(N+t)^\alpha + tCt^\alpha}{N} \\
&= 0.
\end{aligned}$$

(ii) Since $d\mu(\lambda\omega) = d\mu(\omega)$, we have

$$\begin{aligned}
\int |F(\omega,\lambda t)|^{\frac{1}{\alpha}} d\mu(\omega) &= \int |F(\lambda\omega,\lambda t)|^{\frac{1}{\alpha}} d\mu(\lambda\omega) \\
&= \int |F(\lambda\omega,\lambda t)|^{\frac{1}{\alpha}} d\mu(\omega) \\
&= \int |\lambda^\alpha F(\omega,t)|^{\frac{1}{\alpha}} d\mu(\omega) \\
&= \lambda \int |F(\omega,t)|^{\frac{1}{\alpha}} d\mu(\omega).
\end{aligned}$$

Moreover, the support of μ is Ω by the minimality. Hence, the above integral is positive since $F(\omega, t)$ is a nonzero continuous function of ω for any $t \neq 0$. ■

There are two important aspects of 'fractal' functions; almost periodicity and self-similarity. Our notion of homogeneous cocycles on minimal Ω is a formulation of 'fractal' functions from these points of view. We are also interested in self-similar processes with strictly ergodic, stationary increments which come from homogeneous cocycles on strictly ergodic Ω. Rudin-Shapiro process defined in [2] is one of them for $\alpha = \frac{1}{2}$ and $G = \{2^n; n \in \mathbf{Z}\}$ if it is restricted on an ergodic component.

We will construct such Ω and homogeneous cocycles on it.

2 Colored tiling

Let $\mathcal{R}$ be the set of nonempty rectangles $(a, b] \times [c, d)$ in $\mathbf{R}^2$ such that
(7)
$$e^{-b} = d - c.$$

Let Σ be a finite set with at least 2 elements, which will be called the set of **colors**.

A mapping $\omega: dom(\omega) \to \Sigma$ is called a **colored tiling** if $dom(\omega) \subset \mathcal{R}$ and $\bigcup_{S \in dom(\omega)} S$ gives a partition of $\mathbf{R}^2$. For $S \in dom(\omega)$, we call $\omega(S)$ the **color** of the **tile** S. In addition, if $S = (a, b] \times [c, d)$, then the point $(b, c) \in \mathbf{R}^2$ is called the **corner** of S. For $x \in \mathbf{R}^2$, we define the **color** $\tilde{\omega}(x)$ of ω at **point** $x \in \mathbf{R}^2$ to be $\tilde{\omega}(x) := \omega(S)$ for the tile S with $x \in S \in dom(\omega)$. Let $\Omega(\Sigma)$ be the set of all colored tilings with the colors Σ. It is considered as a topological space in the sense that a net $\{\omega_n\}_{n \in I} \subset \Omega(\Sigma)$ converges to $\omega \in \Omega(\Sigma)$ if for every $S \in dom(\omega)$, there exist $S_n \in dom(\omega_n)$ $(n \in I)$ such that
(8)
$$\omega(S) = \omega_n(S_n) \text{ for any } n \in I \text{ and } \lim \rho(S, S_n) = 0,$$

where ρ is the Hausdorff metric:

$$\rho(S, S_n) = \max\{\sup_{x \in S} \inf_{y \in S_n} \parallel x - y \parallel, \sup_{x \in S_n} \inf_{y \in S} \parallel x - y \parallel\}.$$

For $\omega \in \Omega(\Sigma)$, $t \in \mathbf{R}$ and $\lambda \in \mathbf{R}_+$, we define $\omega + t \in \Omega(\Sigma)$ and $\lambda\omega \in \Omega(\Sigma)$ as follows:
For $S := (a, b] \times [c, d)$ and $S' := (a, b] \times [c - t, d - t)$, $S' \in dom(\omega + t)$ if and only if $S \in dom(\omega)$, and in this case $(\omega + t)(S') = \omega(S)$. Also, for $S := (a, b] \times [c, d)$ and $S' := (a - \log \lambda, b - \log \lambda] \times [\lambda c, \lambda d)$, $S' \in dom(\lambda\omega)$ if and only if $S \in dom(\omega)$, and in this case $(\lambda\omega)(S') = \omega(S)$.

Then, it is easy to see that $(\mathbf{R}, \mathbf{R}_+)$ acts on $\Omega(\Sigma)$. We are interested in compact metrizable subsets of $\Omega(\Sigma)$ which are invariant under the action of $(\mathbf{R}, G)$ for some G.

Example 3 *Let* $\Sigma = \{0,1\}$ *and*

$$\begin{array}{rl} B_2 := \{ & \omega \in \Omega(\Sigma);\ \textit{for any } S := (a,b] \times [c,d) \in dom(\omega) \\ & \textit{it holds that } b = a + \log 2 \in (\log 2)\mathbf{Z} \textit{ and} \\ & S_i := (b, b + log2] \times [c + \frac{i}{2}(d-c), c + \frac{i+1}{2}(d-c)) \\ & \in dom(\omega) \textit{ with } \omega(S_i) = i \textit{ for } i = 0,1\}. \end{array}$$

Then, $(\mathbf{R}, \{2^n; n \in \mathbf{Z}\})$ *acts an* B_2*. We can consider* B_2 *as the set of 2-sided, 2-adic expansions in the sense that* $\omega \in B_2$ *is identified with*

$$\begin{array}{rl} & \sum_{i \in \mathbf{Z}} \tilde{\omega}(i \log 2, 0) 2^{-i} \\ = & \sum_{i \leq 0} \tilde{\omega}(i \log 2, 0) 2^{-i} \oplus \sum_{i > 0} \tilde{\omega}(i \log 2, 0) 2^{-i} \end{array}$$

where the convergence is in $\mathbf{Z}_2 \times [0,1]$ *with the identification of* $x \oplus 1$ *with* $(x+1) \oplus 0$ *for any* $x \in \mathbf{Z}_2$*.*

A **substitution** φ on a set Σ is a mapping $\Sigma \to \Sigma^+$, where $\Sigma^+ = \bigcup_{n=1}^{\infty} \Sigma^n$. For $\xi \in \Sigma^+$, we denote $L(\xi) := n$ if $\xi \in \Sigma^n$, and ξ with $L(\xi) = n$ is usually denoted by $\xi_0 \xi_1 \cdots \xi_{n-1}$. We can extend φ to be a homomorphism $\Sigma^+ \to \Sigma^+$ as follows:

$$\varphi(\xi) := \varphi(\xi_0)\varphi(\xi_1) \cdots \varphi(\xi_{n-1})$$

for $\xi \in \Sigma^n$, where the right-hand side is the concatenations of $\varphi(\xi_i)$'s. We can define $\varphi^2, \varphi^3, \cdots$ as the compositions of $\varphi : \Sigma^+ \to \Sigma^+$.

A **weighted substitution** (φ, η) on Σ is a mapping $\Sigma \to \Sigma^+ \times (0,1)^+$ such that $L(\varphi(\sigma)) = L(\eta(\sigma))$ and $\sum_{i < L(\eta(\sigma))} \eta(\sigma)_i = 1$ for any $\sigma \in \Sigma$. Note that φ is a substitution on Σ. We call η the **weight** on φ. We define $\eta^n : \Sigma \to (0,1)^+ \quad (n = 2, 3, ...)$ inductively by

$$\eta^n(\sigma)_k = \eta(\sigma)_i \eta^{n-1}(\varphi(\sigma)_i)_j$$

for any $\sigma \in \Sigma$ and i, j, k with

$$0 \leq i < L(\varphi(\sigma)), 0 \leq j < L(\varphi^{n-1}(\varphi(\sigma)_i)), k = \sum_{h<i} L(\varphi^{n-1}(\varphi(\sigma)_h)) + j$$

In this sense, (φ^n, η^n) is also a weighted substitution for $n = 2, 3, \cdots$.

A substitution φ on Σ is called **mixing** if there exists a positive integer n such that for any $\sigma, \sigma' \in \Sigma$ there exists i with $0 \le i < L(\varphi^n(\sigma))$ and $\varphi^n(\sigma)_i = \sigma'$.

For a weighted substitution (φ, η) on Σ, we always assume that

(9) **the substitution φ is mixing**.

We define the **base set** $B(\varphi, \eta)$ as the closed, multiplicative subgroup of $\mathbf{R}_+$ generated by the set

$$\{ \ \eta^n(\sigma)_i\ ; \ \ \sigma \in \Sigma, \ \ n = 0, 1, \cdots \text{ and } \ 0 \le i < L(\varphi^n(\sigma)) \text{ such that } \varphi^n(\sigma)_i = \sigma \}.$$

It is called **continuous** if $B(\varphi, \eta) = \mathbf{R}_+$, otherwise, **discrete**.

Let (φ, η) be a weighted substitution on a finite set Σ with $\sharp\Sigma \ge 2$. Let $G := B(\varphi, \eta)$. Then, there exists a function $g : \Sigma \to \mathbf{R}_+$ such that

(10)
$$g(\varphi(\sigma)_i)G = g(\sigma)\eta(\sigma)_i G$$

for any $\sigma \in \Sigma$ and $0 \le i < L(\varphi(\sigma))$. Note that if $G = \mathbf{R}_+$, then we can take $g \equiv 1$. In the discrete case, we can define g by $g(\sigma) := \eta^n(\sigma_0)_i$ for some n and i such that $\varphi^n(\sigma_0)_i = \sigma$, where σ_0 is a fixed element in Σ. For another g' satisfying (10), there exists a constant $C > 0$ such that $g'(\sigma)G = Cg(\sigma)G$ for any $\sigma \in \Sigma$.

Let $\Omega(\varphi, \eta, g)'$ be the set of all elements ω in $\Omega(\Sigma)$ such that

(i) if $(a, b] \times [c, d) \in dom(\omega)$, then $e^{-b} = d - c \in g(\omega((a, b] \times [c, d)))G$,

and

(ii) if $(a, b] \times [c, d) \in dom(\omega)$ and $\omega((a, b] \times [c, d)) = \sigma$, then for $i = 0, 1, \cdots, L(\varphi(\sigma)) - 1$, $S_i \in dom(\omega)$ and $\omega(S_i) = \varphi(\sigma)_i$, where

$$S_i := (b, b - \log \eta(\sigma)_i] \times [c + (d - c)\sum_{j=0}^{i-1} \eta(\sigma)_j, c + (d - c)\sum_{j=0}^{i} \eta(\sigma)_j).$$

We call the tile S_i as above the **i-th child** of the tile S, and S the **mother** of S_i. Let $\Omega(\varphi, \eta, g)''$ be the set of all $\omega \in \Omega(\varphi, \eta, g)'$ such that for any N, there exists $(a, b] \times [c, d) \in dom(\omega)$ with $[c, d) \supset [-N, N)$. Finally, we define $\Omega(\varphi, \eta, g)$ to be the closure of $\Omega(\varphi, \eta, g)''$. Then, $(\mathbf{R}, G)$ acts on $\Omega(\varphi, \eta, g)$. We denote $\Omega(\varphi, \eta, 1)$ simply by $\Omega(\varphi, \eta)$ in the continuous case.

A tile T in $\omega \in \Omega'(\varphi, \eta, g)$ such that the vertical coordinates of the points in it is a proper subset of that of another tile S is called a **descendant** of S. Equivalently, we call S an **ancestor** of T. These notions are continuations of mother and child. A tile S in $\omega \in \Omega'(\varphi, \eta, g)$ together with its color **determines** its descedants in the sence that if for some $\omega' \in \Omega'(\varphi, \eta, g)$, a tile S' in ω' satisfies that $\omega(S) = \omega'(S')$ and $\rho(\partial S, \partial S') < \epsilon$, then for any descendant T of S, there exists a descendant T' of S' such that $\omega(T) = \omega'(T')$ and $\rho(T, T') < \epsilon$, where ρ is the Hausdorff metric and for a tile $S := (a, b] \times [c, d)$, we denote its **right edge** $\{b\} \times [c, d)$ by ∂S.

Theorem 3 *For any weighted substitution (φ, η) satisfying (9) and g with (10), $\Omega(\varphi, \eta, g)$ is $\mathbf{R}$-strictly ergodic. Moreover, the toplogical entropy of the $\mathbf{R}$-action on $\Omega(\varphi, \eta, g)$ is 0.*

PROOF. (compactness and metrizability) We prove the compactness and the metrizability of $\Omega'(\varphi, \eta, g)$. Those of $\Omega(\varphi, \eta, g)$ follow since it is a closed subset of the former.

By (8), the following sets for $\epsilon = \frac{1}{K}$ form a countable open basis of the topological space $\Omega'(\varphi, \eta, g)$:
(11)

$$\begin{aligned}
&U_\epsilon(S_1, \cdots, S_K; \sigma_1, \cdots, \sigma_K) := \\
&\{\omega \in \Omega'(\varphi, \eta, g);\ \text{there exist } S'_k \in dom(\omega) \text{ such that} \\
&\omega(S'_k) = \sigma_k \text{ and } \rho(S_k, S'_k) < \epsilon \text{ for } k = 1, \cdots, K\},
\end{aligned}$$

where $K = 1, 2, \cdots$ and for $k = 1, \cdots, K$, $\sigma_k \in \Sigma$ and S_k is a rectangle in $\mathbf{R}^2$ of type $(a, b] \times [c, d)$ with rational a, b, c and d such that the above set is not empty. It follows that $\Omega'(\varphi, \eta, g)$ is a Hausdorff space with the 2nd countability axiom. Therefore, by the Urison-Tikhonov theorem, the metrizability follows from the compactness.

Let us prove the compactness. Take an arbitrary infinite sequence $\{\xi_n; n = 1, 2, \cdots\}$ in $\Omega(\varphi, \eta, g)'$. We define a sequence

$$\{\xi_{1,n}; n = 1, 2, \cdots\} \supset \{\xi_{2,n}; n = 1, 2, \cdots\} \supset \cdots$$

of subsequences of $\{\xi_n; n = 1, 2, \cdots\}$ such that a latter side is a subsequence of a former side, inductively. For a positive integer N, assume that $\{\xi_{N-1,n}; n = 1, 2, \cdots\}$ is already defined, where we put $\{\xi_{0,n}; n = 1, 2, \cdots\} := \{\xi_n; n = 1, 2, \cdots\}$. For $\omega \in \Omega(\varphi, \eta, g)'$, let $S_{\pm N}(\omega)$ be the corner of the tile in ω containing the point $(-\log(2N) - u_0, \pm N)$ ($\pm$ respectively), where

(12)
$$u_0 := \max_{\substack{\sigma \in \Sigma \\ 0 \le i < L(\varphi(\sigma))}} -\log \eta(\sigma)_i.$$

Note that the tiles of $\omega \in \Omega(\varphi, \eta, g)$ with the corner $S_N(\omega)$ and $S_{-N}(\omega)$ are either identical or neighboring each other, since the vertical length of any tile intersecting with the line segment connecting the 2 points $(-\log(2N) - u_0, -N)$ and $(-\log(2N) - u_0, N)$ is at least $2N$. Therefore, these 1 or 2 tiles together with their colors, determine ω restricted on the region $(-\log(2N), \infty) \times [-N, N)$. There exists a subsequence $\{\xi_{N,n}; n = 1, 2, \cdots\}$ of $\{\xi_{N-1,n}; n = 1, 2, \cdots\}$ such that
(i) $\sigma_{\pm N} := \tilde{\xi}_{N,n}(-\log(2N) - u_0, \pm N) (\in \Sigma)$ is constant in n ($\pm$ respectively), and
(ii) $S_{\pm N} := \lim_{n\to\infty} S_{\pm N}(\xi_{N,n})$ exists ($\pm$ respectively).

This is possible since Σ is a finite set and the set $\{S_{\pm N}(\omega); \omega \in \Omega(\varphi, \eta, g)'\}$ is bounded. Thus we defined a sequence of subsequences of $\{\xi_n; n = 1, 2, \cdots\}$.

Let ω be the colored tiling which has a tile with corner $S_{\pm N}$ and color $\sigma_{\pm N}$ ($\pm$ respectively) for any $N = 1, 2, \cdots$, which is easily seen to exist uniquely. It is also easy to see that $\omega \in \Omega(\varphi, \eta, g)'$ and that $\xi_{N,N}$ converges to ω as $N \to \infty$. This completes the proof of the compactness of $\Omega(\varphi, \eta, g)'$.

(minimality) Let $G = B(\varphi, \eta)$. Take any

$$U := U_{\frac{1}{K}}(S_1'', \cdots, S_K''; \sigma_1, \cdots, \sigma_K)$$

which intersects with $\Omega(\varphi,\eta,g)$. Take arbitrary $\omega' \in \Omega(\varphi,\eta,g)$. To prove the minimality, it is sufficient to prove that there exists $t \in \mathbf{R}$ such that $\omega' + t \in U$. Since $\Omega''(\varphi,\eta,g)$ is dense in $\Omega(\varphi,\eta,g)$, there exists $\omega \in U \cap \Omega''(\varphi,\eta,g)$. Then, for some $S_k \in dom(\omega)(k = 1,\cdots,K)$, it holds that

$$\omega(S_k) = \sigma_k \text{ and } \rho(S_k, S_k'') < \frac{1}{K} \text{ for } k = 1,\cdots,K.$$

Therefore, there exists $\delta > 0$ such that

$$\rho(S_k, S_k'') < \frac{1}{K} - \delta \text{ for } k = 1,\cdots,K.$$

Since $\omega \in \Omega''(\varphi,\eta,g)$, there exists $(a,b] \times [c,d) \in dom(\omega)$ which is a common ancestor of $S_1,\cdots,S_K$. Let $\omega((a,b]\times[c,d)) = \sigma_0$. Take $\epsilon > 0$ with

$$\epsilon^2 + (e^{-b}(e^{\epsilon}-1))^2 < \delta^2.$$

There exist positive numbers A and B such that $e^{-A} = \eta^{n_1}(\sigma_0)_{i_1}$, $\varphi^{n_1}(\sigma_0)_{i_1} = \sigma_0$, $e^{-B} = \eta^{n_2}(\sigma_0)_{i_2}$ and $\varphi^{n_2}(\sigma_0)_{i_2} = \sigma_0$ for some n_1, i_1 and n_2, i_2 together with the property that if $G = \mathbf{R}_+$, then $0 < B - A < \epsilon$, and if $G = \{\lambda^n; n \in \mathbf{Z}\}$ for $\lambda > 1$, then $B - A = \log\lambda$.

Lemma 1 *For any $x > \frac{B^2}{B-A}$, there exists y, p and k such that $e^{-y} = \eta^p(\sigma_0)_k$, $\sigma_0 = \varphi^p(\sigma_0)_k$ and $0 \le x - y < \log\lambda$ (or ϵ) if $G = \{\lambda^n\}$ with $\lambda > 1$ (or $G = \mathbf{R}_+$, respectively).*

PROOF. Let Λ be the set of all (n,i) such that $n = 1,2,\cdots$, $0 \le i < L(\varphi^n(\sigma_0))$ and $\varphi^n(\sigma_0)_i = \sigma_0$. For $(n,i),(n',i') \in \Lambda$, we define $(n,i)(n',i')$ to be $(n+n', i'') \in \Lambda$ such that $i'' = \sum_{0\le j<i} L(\varphi^{n'}(\varphi^n(\sigma_0)_j)) + i'$. Let $(n-1)B \le x < nB$ for some $n \in \mathbf{Z}$. Then, since $x > \frac{B^2}{B-A}$, it holds that $nA \le x < nB$. Therefore, there exists m with $0 \le m < n$ such that $(n-m)A + mB \le x < (n-m-1)A + (m+1)B$. Let

$$(p,k) := \underbrace{(n_1,i_1)\cdots(n_1,i_1)}_{n-m}\underbrace{(n_2,i_2)\cdots(n_2,i_2)}_{m}$$

and $y := -\log\eta^p(\sigma_0)_k$. Then, since $y = (n-m)A + mB$, we have $0 \le x - y < B - A = \log\lambda$ (or $< \epsilon$), which completes the proof. ■

Using this lemma, we can complete the proof of the minimality. In fact, take any $(a'', b''] \times [c'', d'') \in dom(\omega')$ with $\omega'((a'', b''] \times [c'', d'')) = \sigma_0$ and $b'' < b - \frac{B^2}{B-A}$. This is possible by (9). Then, for $x := b - b''$, we apply the lemma and get the conclusion that there exists $(a', b'] \times [c', d') \in dom(\omega')$ such that $\omega'((a', b'] \times [c', d')) = \sigma_0$ and $0 \leq b - b' < \epsilon$ by taking $b' := b'' + y$ with y in the lemma. Moreover, since $b - b' \in G$ and $0 \leq b - b' < \log \lambda$ in the discrete case, we have $b = b'$.

This implies that for $t = c' - c$, the tiles $(a, b] \times [c, d) \in dom(\omega)$ and $(a', b'] \times [c, d' - t) \in dom(\omega' + t)$ have the same color σ_0 and the ρ-distance between their right edges is less than δ. Since the tile $(a, b] \times [c, d) \in dom(\omega)$ determines the tiles $S_1, \cdots, S_K$, there exists tiles $S'_1, \cdots, S'_K \in dom(\omega' + t)$ with

$$\omega'(S'_k) = \sigma_k \text{ and } \rho(S_k, S'_k) < \delta \text{ for } k = 1, \cdots, K.$$

Thus, $\omega' + t \in U$.

(uniquely ergodicity) Since $\Omega := \Omega(\varphi, \eta, g)$ is a nonempty compact metrizable space and the $\mathbf{R}$-action is continuous, there exists an $\mathbf{R}$-invariant probability Borel measure μ on it. We prove that μ is the unique measure as this.

Let $\omega \in \Omega$ and $(a, b] \times [c, d) \in dom(\omega)$ with $\omega((a, b] \times [c, d)) = \sigma_0$. Take $y \in [c, d)$ randomly according to the normalized Lebesgue measure on $[c, d)$. We arrange the tiles intersecting with the half line $[b, \infty) \times \{y\}$ from the left to right as $S_0, S_1, S_2, \cdots$, where $S_0 = (a, b] \times [c, d)$. Let S_k be the i_k-th child of S_{k-1} $(k = 0, 1, 2, \cdots)$, where S_{-1} is the mother of S_0. Let σ_k be the color of the tile S_k. We put $Y_k(y) := i_k$ and $Z_k(y) := \sigma_k$ for $k = 0, 1, 2, \cdots$ and consider them as random variables on the probability space $[c, d)$ with the normalized Lebesgue measure $\frac{dy}{d-c}$. Then, it is easy to see that random process $\{(Y_0, Z_0), (Y_1, Z_1), (Y_2, Z_2), \cdots\}$ is a Markov process with the transition probability

$$p_{(i,\sigma),(i',\sigma')} = \begin{cases} \eta(\sigma)_{i'} & (\text{if } \varphi(\sigma)_{i'} = \sigma') \\ 0 & (\text{else}). \end{cases}$$

Note that the distribution of $\{(Y_1, Z_1), (Y_2, Z_2), \cdots\}$ depends only on

σ_0. We denote this process by $\{(Y_1^{\sigma_0}, Z_1^{\sigma_0}), (Y_2^{\sigma_0}, Z_2^{\sigma_0}), \cdots\}$ to make sure the dependency on σ_0. Of course, $Z_0^{\sigma_0} = \sigma_0$.

Let $\sigma, \sigma' \in \Sigma$. We define random variables $X_n^{\sigma'\sigma}$ $(n = 1, 2, \cdots)$ by

$$X_n^{\sigma'\sigma} = \sum_{i=1}^{\tau_n} - \log \eta(Z_{i-1}^{\sigma'})_{Y_i^{\sigma'}},$$

where τ_n is the n-th $i(\geq 1)$ such that $Z_i^{\sigma'} = \sigma$. That is,

$$\begin{aligned} &\tau_0 := 0 \\ &\tau_n = \min\{i > \tau_{n-1}; Z_i^{\sigma'} = \sigma\} \quad (n = 1, 2, \cdots). \end{aligned}$$

Note that $\tau_n < \infty$ with probability 1 by (9). It is easy to see that the sequence of random variables

$$\{X_1^{\sigma'\sigma}, X_2^{\sigma'\sigma} - X_1^{\sigma'\sigma}, X_3^{\sigma'\sigma} - X_2^{\sigma'\sigma}, \cdots\}$$

is independent and the distribution of $X_{n+1}^{\sigma'\sigma} - X_n^{\sigma'\sigma}$ is identical with that of $X_1^{\sigma\sigma}$ for $n = 1, 2, \cdots$.

Let $F_{\sigma'\sigma}$ be the distribution of the random variable $X_1^{\sigma'\sigma}$. Then, the distribution of $X_n^{\sigma'\sigma}$ is $F_{\sigma'\sigma} * {F_{\sigma\sigma}}^{(n-1)*}$, where $'*'$ implies the convolution of distributions.

Let $S := (a, b] \times [c, d)$ be a tile in $\omega \in \Omega$ with $\omega(S) = \sigma'$. For $u > b$, let E be the number of the tiles in ω with color σ having the corner belonging to $[u, u + \Delta_u) \times [c, d)$, where Δ_u as well as Δ_v stands for a sufficiently small positive number and by $o(1)$, we denote terms which tend to 0 uniformly in the other variables as $\Delta_u \to 0, \Delta_v \to 0$. From the definition of the random variable $X_n^{\sigma'\sigma}$, it holds that
(13)

$$\begin{aligned} \tfrac{Ee^{-u}}{d-c}(1 + o(1)) \quad &= \textstyle\sum_{n=1}^{\infty} P(b + X_n^{\sigma'\sigma} \in [u, u + \Delta_u)) \\ &= \textstyle\sum_{n=0}^{\infty} \int_{u-b \leq x < u-b+\Delta_u} F_{\sigma'\sigma} * {F_{\sigma\sigma}}^{n*}(dx) \end{aligned}$$

since any tile with the corner $[u, u + \Delta_u) \times [c, d)$ has the vertical length $e^{-u}(1 + o(1))$. It is well known by the renewal theory [1] that the above value converges to

$$\left(\int x F_{\sigma\sigma}(dx)\right)^{-1} \Delta_u$$

as $u \to \infty$ if $G = \mathbf{R}_+$ and to

$$(\int xF_{\sigma\sigma}(dx))^{-1} \log \lambda$$

as $u \to \infty$ satisfying that $e^{-u} \in g(\sigma)G$ if $G = \{\lambda^n; n \in \mathbf{Z}\}$ with $\lambda > 1$. Note that by (9), $0 < \int xF_{\sigma\sigma}(dx) < \infty$.

For $\sigma \in \Sigma$ and a Borel subset U of $\mathbf{R}^2$, let $\Pi(\sigma, U)$ be the subset of Ω consisting of ω which has a tile S such that $\omega(S) = \sigma$ and S has the corner belonging to U. Let $\Delta_u\Delta_v := [u, u+\Delta_u)\times[v, v+\Delta_v)$ and $\sigma \in \Sigma$ satisfy that $e^{-u} \in g(\sigma)G$. Since μ is $\mathbf{R}$-invariant, $\mu(\Pi(\sigma, \Delta_u\Delta_v)) = \mu(\Pi(\sigma, \Delta_u\Delta_v + (0, y)))$ for any $y \in \mathbf{R}$. By integrating this equality with dy from $-v$ to $-v + N$, where N is an arbitrary large positive number, and applying Fubini's theorem we have

(14)

$$\begin{aligned}
\mu(\Pi(\sigma, \Delta_u\Delta_v)) &= \tfrac{1}{N}\int_0^N \mu(\Pi(\sigma, [u, u+\Delta_u) \times [y, y+\Delta_v)))dy \\
&= \tfrac{1}{N}\int_0^N \int 1_{\Pi(\sigma,[u,u+\Delta_u)\times[y,y+\Delta_v))} d\mu dy \\
&= \tfrac{1}{N}\int \int_0^N 1_{\Pi(\sigma,[u,u+\Delta_u)\times[y,y+\Delta_v)} dy d\mu \\
&= \tfrac{1}{N}\int \Delta_v 1_{\Pi(\sigma,[u,u+\Delta_u)\times[0,N))} d\mu (1 + o(1)) \\
&= \tfrac{\Delta_v}{N}\int E(\omega)d\mu(\omega)(1 + o(1)),
\end{aligned}$$

where we denote by $E(\omega)$ the number of the tiles in ω with color σ having the corner belonging to $[u, u + \Delta_u) \times [0, N)$.

For any $\epsilon > 0$, take $L > 0$ such that the the value in (13), for any $\sigma, \sigma' \in \Sigma$, any tile $(a, b] \times [c, d)$ in any element in Ω with color σ' and $u \in \mathbf{R}$ with $u - b \geq L$ and $e^{-u} \in g(\sigma)G$, is close to A within ϵ, where

(15)

$$A = \begin{cases} (\int xF_{\sigma\sigma}(dx))^{-1}\Delta_u & \text{if } G = \mathbf{R}_+ \\ (\int xF_{\sigma\sigma}(dx))^{-1}\log\lambda & \text{if } G = \{\lambda^n; n \in \mathbf{Z}\} \quad (\lambda > 1). \end{cases}$$

Take any $u \in \mathbf{R}$ with $e^{-u} \in g(\sigma)G$. For any $\omega \in \Omega$ and $y \in \mathbf{R}$, let $S(y)$ be the tile in ω such that $S(y)$ intersects with the horizontal half line $[u-L-u_0, \infty)\times\{y\}$ but its mother fails to satisfy this condition, where u_0 is defined in (12). Then, the vertical size of $S(y)$ is at most e^{L-u+u_0}. Let $S_1, \cdots, S_k$ be the set of all distinct $S(y)$'s for $y \in [0, N)$ such that the vertical coordinates of the points in $S(y)$ are contained in $[0,N)$. Then, the sets $\tilde{S}_i$ $(i=1,\cdots,k)$ of the vertical coordinates of the points in S_i are disjoint. We take N large enough so that their union covers more than $\frac{1}{1+\epsilon}$ portion of the inteval $[0,N)$. Let $E_i(\omega)$ be the number of the tiles in ω with color σ having the corner belonging to $[u,u+\Delta_u)\times\tilde{S}_i$. Then, by the assumption on L, (13) and (15), we have $\left|E_i(\omega)e^{-u}(1+o(1))-|\tilde{S}_i|A\right|<|\tilde{S}_i|\epsilon$, where $|\tilde{S}_i|$ is the length of $\tilde{S}_i$. By adding the inequalities, we have $|E(\omega)e^{-u}(1+o(1))-NA|<2N\epsilon$. Thus, by integrating it with $d\mu(\omega)$, we have

(16)

$$\left|\int E(\omega)d\mu(\omega)e^{-u}(1+o(1))-NA\right|<2N\epsilon.$$

Conbining (14) and (16), we have

$$\left|\mu(\Pi(\sigma,\Delta_u\Delta_v))e^{-u}(1+o(1))-A\Delta_v\right|<2\epsilon\Delta_v.$$

Since $\epsilon>0$ was arbitrary, we have

$$\mu(\Pi(\sigma,\Delta_u\Delta_v))(1+o(1))=\begin{cases}(\int xF_{\sigma\sigma}(dx))^{-1}e^u\Delta_u\Delta_v \quad (\text{if } G=\mathbf{R}_+)\\ 1_{e^{-u}\in g(\sigma)G}(\int xF_{\sigma\sigma}(dx))^{-1}e^u\log\lambda\Delta_v\\ \qquad\qquad (\text{if } G=\{\lambda^n;n\in\mathbf{Z}\}(\lambda>1)).\end{cases}$$

This holds for any $u\in\mathbf{R}$ not necessarily satisfying $e^{-u}\in g(\sigma)G$, since if $e^{-u}\notin g(\sigma)G$, then $\mu(\Pi(\sigma,\Delta_u\Delta_v))=0$ for any sufficiently small Δ_u. Let $U:=[u',u'')\times[v',v'')$ satisfy that

(17)

$$u''-u'\le\min_{\substack{\sigma\in\Sigma\\0\le i<L(\varphi(\sigma))}}-\log\eta(\sigma)_i \text{ and } v''-v'\le e^{-u''}.$$

Then for any $\omega\in\Omega$, U contains at most 1 corner of the tiles in ω. Therefore, we have

(18)

$$\begin{aligned}\mu(\Pi(\sigma,U)) &= \textstyle\int_{v'}^{v''}\int_{u'}^{u''}(\int xF_{\sigma\sigma}(dx))^{-1}e^u du dv\\ &= (\textstyle\int xF_{\sigma\sigma}(dx))^{-1}(e^{u''}-e^{u'})(v''-v')\end{aligned}$$

if $G=\mathbf{R}_+$, and

(18')

$$\begin{aligned}\mu(\Pi(\sigma,U)) &= \textstyle\int_{v'}^{v''}\sum\limits_{u;e^{-u}\in g(\sigma)G}(\int xF_{\sigma\sigma}(dx))^{-1}e^u\log\lambda dv\\ &= (\textstyle\int xF_{\sigma\sigma}(dx))^{-1}\sum\limits_{u;e^{-u}\in g(\sigma)G}e^u\log\lambda(v''-v')\end{aligned}$$

if $G = \{\lambda^n; n \in \mathbf{Z}\}$ for $\lambda > 1$. This is because by (17), the sets $\Pi(\sigma, U_1)$ and $\Pi(\sigma, U_2)$ are disjoint if U_1 and U_2 are dijoint Borel subsets of U. Note that for general $U := [u', u'') \times [v', v'')$ without (17), we only have the inequalities in (18) and (18'):

(19)

$$\mu(\Pi(\sigma, U)) \leq (\int xF_{\sigma\sigma}(dx))^{-1}(e^{u''} - e^{u'})(v'' - v')$$

(19')

$$\mu(\Pi(\sigma, U)) \leq (\int xF_{\sigma\sigma}(dx))^{-1} \sum_{u; e^{-u} \in g(\sigma)G} e^u \log \lambda (v'' - v').$$

Since any open set in Ω can be written as a countable disjoint union of sets $\Pi(\sigma, U)$ for $\sigma \in \Sigma$ and U with (17), μ is determined by (18) and (18') and is unique, which completes the proof of the strictly ergodicity.

(0 entropy) Since the topological entropy of the $\mathbf{R}$-action on Ω coincides with the measure theoretical entropy of it on the probability space (Ω, μ) by the uniquely ergodcity, it is sufficient to prove that the latter is 0. That is, we prove that $h_\mu(T_1) = 0$ for the transformation $T_t : \Omega \to \Omega$ with $T_t(\omega) = \omega + t$. Then for any $g \in G$, we have $h_\mu(T_g) = h_\mu(T_1)$, since by (3) and (6), the transformations T_g and T_1 are conjugate. On the other hand, since $h_\mu(T_g) = gh_\mu(T_1)$, we have $h_\mu(T_1) = gh_\mu(T_1)$ for any $g \in G$. This implies that either $h_\mu(T_1) = 0$ or ∞. Thus, to prove that $h_\mu(T_1) = 0$ it is sufficient to prove that $h_\mu(T_1) < \infty$. For this purpose, we will show that there exists a countable generator with finite entropy of the transformation T_1 on the measure space (Ω, μ).

Let

$$U_{ij} := [-\sum_{k=1}^{i} \frac{3}{k}, -\sum_{k=1}^{i-1} \frac{3}{k}) \times [\frac{j-1}{i}, \frac{j}{i})$$

for any $i = 1, 2, \cdots$ and $j = 1, 2, \cdots, i$. Then by (19) and (19'),

$$\begin{aligned} &\mu(\Pi(\sigma, U_{ij})) \leq \textstyle\int_{U_{ij}} \Pi(\sigma, dxdy) \\ &\leq (\textstyle\int xF_{\sigma\sigma}(dx))^{-1} e^{-\sum_{k=1}^{i-1} \frac{3}{k}} (\frac{3}{i} + \log \lambda) \frac{1}{i} \\ &\leq Ci^{-4} \end{aligned}$$

for any $\sigma \in \Sigma$, where C is a constant independent of i, j, σ, and $\lambda \geq 1$ is such that $G = \{\lambda^n; n \in \mathbf{Z}\}$ if G is discrete and $\lambda = 1$ if $G = \mathbf{R}_+$. Let i_0 be a positive integer such that

$$\frac{3}{i_0} < \min_{\substack{\sigma \in \Sigma \\ 0 \leq i < L(\varphi(\sigma))}} -\log \eta(\sigma)_i$$

and $C i_0^{-4} \sharp\Sigma < e^{-1}$. For any $i \geq i_0$ and $j = 1, 2, \cdots, i$, let $\mathbf{U}_{ij}$ be the partition of Ω by the sets $\Pi(\sigma, U_{ij})$ for $\sigma \in \Sigma$ and the complement of the union of these sets. Note that in this case, U_{ij} satisfies (17) and the sets $\Pi(\sigma, U_{ij})$ for different σ's are disjoint. Then, the entropy $H_\mu(\mathbf{U}_{ij})$ of the partition $\mathbf{U}_{ij}$ with respect to μ satisfies that

$$\begin{aligned} H_\mu(\mathbf{U}_{ij}) &\leq -\sharp\Sigma C i^{-4} \log(C i^{-4}) - (1 - \sharp\Sigma C i^{-4}) \log(1 - \sharp\Sigma C i^{-4}) \\ &\leq C' i^{-3}, \end{aligned}$$

where C' is a constant independent of i, j. Let $\mathbf{U}$ be the least common refinement of $\mathbf{U}_{ij}$'s for any $i \geq i_0$ and $j = 1, 2, \cdots, i$. Then, we have

$$H_\mu(\mathbf{U}) \leq \sum_{i=i_0}^{\infty} i C' i^{-3} < \infty.$$

To complete the proof, we prove that the partition $\mathbf{U}$ is a generator of the dynamical system (Ω, μ, T_1). Take any $\omega, \omega' \in \Omega$ with $\omega \neq \omega'$. Then, there exists $\epsilon > 0$ and a tile S in ω such that $\rho(S, S') > \epsilon$ for any tile S' in ω' with the same color as S. Take any $i_1 \geq i_0$ with $\frac{\sqrt{10}}{i_1} < \epsilon$. Since any ancestor of S determine S, the same requirement is satisfied for any ancestor of S. Therefore, we may assume that a tile $S = (a, b] \times [c, d)$ in ω as above satisfies that $b < -\sum_{k=1}^{i_1} \frac{3}{k}$. Let n be the integer part of c. Then, there exists $i \geq i_1$, $1 \leq j \leq i$ and $\sigma \in \Sigma$ such that

$$\omega + n \in \Pi(\sigma, U_{ij}).$$

From the assumption on S, $\omega' + n \notin \Pi(\sigma, U_{ij})$. Therefore, any 2 elements in Ω are seperated by the least common refinement of $\mathbf{U} - n$ for $n \in \mathbf{Z}$. This implies that $\mathbf{U}$ is a generator of (Ω, μ, T_1), which completes the proof. ■

Example 4 (Fibonacci expansion) *Let $\Sigma = \{0,1\}$. Let (φ, η) be the weighted substitution on Σ such that*

$$\begin{array}{l} 0 \to (0, \lambda^{-1})(1, \lambda^{-2}) \\ 1 \to (0, \lambda^{-1})(1, \lambda^{-2}), \end{array}$$

where $\lambda = \frac{1+\sqrt{5}}{2}$ and we arranged $(\varphi(\sigma)_i, \eta(\sigma)_i))$ in the order of i after $'\sigma \to'$. Then, $G := B(\varphi, \eta) = \{\lambda^n; n \in \mathbf{Z}\}$. For $g \equiv 1$, (10) is satisfied. Let $\Omega := \Omega(\varphi, \eta, 1)$. Then, by theorem 3, Ω is $\mathbf{R}$-strictly ergodic. Let μ be the unique $\mathbf{R}$-invariant probability Borel measure on Ω. By (18'), μ satisfies that

$$\begin{array}{l} \mu(\Pi(0, dudv) = A^{-1} e^u \log \lambda dv \\ \mu(\Pi(1, dudv)) = B^{-1} e^u \log \lambda dv \end{array}$$

for any $u, v \in \mathbf{R}$ with $e^{-u} \in G$, where

$$\begin{array}{ll} A & = \int x F_{00}(dx) \\ & = \lambda^{-1} \log \lambda + \lambda^{-3} 3 \log \lambda + \lambda^{-5} 5 \log \lambda + \cdots \\ & = \frac{\lambda+2}{\lambda} \log \lambda, \end{array}$$

$$\begin{array}{ll} B & = \int x F_{11}(dx) \\ & = \lambda^{-2} 2 \log \lambda + \lambda^{-3} 3 \log \lambda + \lambda^{-4} 4 \log \lambda + \cdots \\ & = (\lambda + 2) \log \lambda. \end{array}$$

Thus, we have

$$\begin{array}{l} \mu(\Pi(0, dudv)) = \frac{\lambda}{\lambda+2} e^u dv \\ \mu(\Pi(1, dudv)) = \frac{1}{\lambda+2} e^u dv \end{array}$$

for any $u, v \in \mathbf{R}$ with $e^{-u} \in G$. To understand this example as the Fibonacci expansion in the usual way, the symbol $'1'$ should be considered as standing for the block $'10'$. Confer with the next example.

Example 5 *Let φ be a mixing substitution on a set Σ with $\sharp \Sigma \geq 2$. Let $M = (m_{\sigma\sigma'})_{\sigma, \sigma' \in \Sigma}$ be the matrix with entry $m_{\sigma\sigma'} := \sharp\{i; \varphi(\sigma)_i = \sigma'\}$. Let λ be the maximum eigenvalue of M and $\zeta = (\zeta_\sigma)_{\sigma \in \Sigma}$ be the positive eigen column vector of M with eigenvalue λ such that $\max_{\sigma \in \Sigma} |\zeta_\sigma| = 1$. We define a weighted substitution (φ, η) by $\eta(\sigma)_i = (\lambda \zeta_\sigma)^{-1} \zeta_{\varphi(\sigma)_i}$. Then, it holds that $G := B(\varphi, \eta) = \{\lambda^n; n \in \mathbf{Z}\}$.*

Moreover, for $g(\sigma) := \zeta_\sigma$*, we have the equation (10). There is a small difficulty to get* $\Omega(\varphi, \eta, g)$*, namely, for some* $\sigma \in \Sigma$*, it can happen that* $L(\varphi(\sigma)) = 1$ *and* $\eta(\sigma)_0 = 1$*, so that the corresponding 'tile' vanishes. To solve this difficulty, we modify* (φ, η) *so that*

$$(\varphi'(\sigma), \eta'(\sigma)) = (\varphi^n(\sigma), \eta^n(\sigma))$$

with

$$n := \min\{i; L(\varphi^i(\sigma)) \geq 2\}.$$

Thus, we get $\Omega(\varphi', \eta', g)$*. Example 4 is obtained in this way for the Fibonacci substitution* $0 \to 01, 1 \to 0$*. This example will be discussed later.*

Example 6 *Let* (φ, η) *be the weighted substitution on* $\{0, 1\}$ *such that*

$$\begin{array}{l} 0 \to (0, \frac{4}{9})(1, \frac{1}{9})(0, \frac{4}{9}) \\ 1 \to (1, \frac{4}{9})(0, \frac{1}{9})(1, \frac{4}{9}). \end{array}$$

Since $\frac{\log \frac{4}{9}}{\log \frac{1}{9}} = 1 - \frac{\log 2}{\log 3}$ *is irrational, we have* $B(\varphi, \eta) = \mathbf{R}_+$*. Let* $\Omega = \Omega(\varphi, \eta)$*. Then, by theorem 3,* Ω *is* $\mathbf{R}$*-strictly ergodic. Let* μ *be the unique* $\mathbf{R}$*-invariant probability Borel measure on* Ω*. Then,* μ *is also* $\mathbf{R}_+$*-invariant. By (18),* μ *satisfies that*

$$\mu(\Pi(0, dudv)) = \mu(\Pi(1, dudv)) = A^{-1} e^u dudv$$

for any $u, v \in \mathbf{R}$ *with*

$$\begin{aligned} A &= \int x dF_{00}(dx) \\ &= \tfrac{8}{9} \log \tfrac{9}{4} + \textstyle\sum_{n=0}^{\infty} \frac{1}{9} \left(\frac{8}{9}\right)^n \frac{1}{9} \left(\log 9 + n \log \frac{9}{4} + \log 9\right) \\ &= 4 \log 3 - \tfrac{32}{9} \log 2 \end{aligned}$$

This example is a special case of the next example.

Example 7 *Let* $0 < \alpha < 1$*. There exists a unique* $\beta = \beta(\alpha)$ *such that*

$$0 < \beta < \frac{1}{2} \text{ and } 2\beta^\alpha - (1 - 2\beta)^\alpha = 1.$$

Let (φ, η) be the weighted substitution on $\Sigma = \{0, 1\}$ such that

$$\begin{array}{l} 0 \to (0, \beta)(1, 1-2\beta)(0, \beta) \\ 1 \to (1, \beta)(0, 1-2\beta)(1, \beta). \end{array}$$

Since β is a piecewise strictly monotone function of α, it takes rational values only for a countably many α's. Therefore, $B(\varphi, \eta) = \mathbf{R}_+$ except for coutably many α's. In particular, $\alpha = \frac{1}{2}$ satisfies this condition as discussed in Example 6. This example will be also discussed later.

3 Homogeneous cocycle

Let (φ, η) be a weighted substitution on a finite set Σ with $\sharp\Sigma \geq 2$ satisfying (9). Let $G = B(\varphi, \eta)$ and g satisfy (10). Let $\Omega := \Omega(\varphi, \eta, g)$. A cocycle $F(\omega, t)$ on Ω is called **adapted** if ther exists a function $\Xi : \Sigma \times \mathbf{R}_+ \to \mathbf{R}$ such that
(20)

$$\begin{aligned} F(\omega, [c, d)) &:= F(\omega, d) - F(\omega, c) \\ &= \Xi(\omega((a, b] \times [c, d)), d - c) \end{aligned}$$

for any tile $(a, b] \times [c, d)$ in $\omega \in \Omega$. We are going to characterize adapted homogeneous cocycles on Ω.

For $0 < \alpha < 1$, let $M_\alpha = M_\alpha(\varphi, \eta)$ be the matrix $(m_{\sigma\sigma'}{}^{(\alpha)})_{\sigma,\sigma' \in \Sigma}$ such that
(21)

$$m_{\sigma\sigma'}{}^{(\alpha)} = \sum_{\substack{0 \leq i < L(\varphi(\sigma)) \\ \varphi(\sigma)_i = \sigma'}} \eta(\sigma)_i^\alpha.$$

Assume that $F(\omega, t)$ is a nonzero adapted α-G-homogeneous cocycle on Ω. Then, there exists Ξ satisfying (20). For any $\sigma \in \Sigma$ and $h \in g(\sigma)G$, there exist $\omega \in \Omega$ and $(a, b] \times [c, d) \in dom(\omega)$ such that $\omega((a, b] \times [c, d)) = \sigma$ and $d - c = h$. It holds by (20) that

(22)

$$\begin{aligned}\Xi(\sigma,h) &= F(\omega,d)-F(\omega,c)\\ &= \sum_{0\le i<L(\varphi(\sigma))}[F(\omega,c+\sum_{0\le j\le i}\eta(\sigma)_j(d-c))\\ &\qquad -F(\omega,c+\sum_{0\le j<i}\eta(\sigma)_j(d-c))]\\ &= \sum_{0\le i<L(\varphi(\sigma))}\Xi(\varphi(\sigma)_i,\eta(\sigma)_i(d-c)).\end{aligned}$$

Since F is α-G-homogeneous, it holds for any $h'\in G$ that

(23)

$$\begin{aligned}\Xi(\sigma,h'h) &= F(h'\omega,h'd)-F(h'\omega,h'c)\\ &= h'^{\alpha}[F(\omega,d)-F(\omega,c)]\\ &= h'^{\alpha}\Xi(\sigma,h).\end{aligned}$$

Since F is nonzero, Ξ is nonzero. By (23), there exists $\sigma\in\Sigma$ such that $\Xi(\sigma,g(\sigma))\neq 0$. Moreover, for the column vector $\xi=(\xi_\sigma)_{\sigma\in\Sigma}:=(g(\sigma)^{-\alpha}\Xi(\sigma,g(\sigma)))_{\sigma\in\Sigma}$, we have by (22) and (23) that

$$\begin{aligned}(M_\alpha\xi)_\sigma &= \sum_{\sigma'\in\Sigma}m^{(\alpha)}_{\sigma\sigma'}g(\sigma')^{-\alpha}\Xi(\sigma',g(\sigma'))\\ &= \sum_{0\le i<L(\varphi(\sigma))}\eta(\sigma)_i^{\alpha}g(\varphi(\sigma)_i)^{-\alpha}\Xi(\varphi(\sigma)_i,g(\varphi(\sigma)_i))\\ &= \sum_{0\le i<L(\varphi(\sigma))}g(\sigma)^{-\alpha}(g(\sigma)\eta(\sigma)_i g(\varphi(\sigma)_i)^{-1})^{\alpha}\Xi(\varphi(\sigma)_i,g(\varphi(\sigma)_i))\\ &= g(\sigma)^{-\alpha}\sum_{0\le i<L(\varphi(\sigma))}\Xi(\varphi(\sigma)_i,\eta(\sigma)_i g(\sigma))\\ &= g(\sigma)^{-\alpha}\Xi(\sigma,g(\sigma))\\ &= \xi_\sigma,\end{aligned}$$

hence,

(24)

$$\xi\neq 0 \text{ and } M_\alpha\xi=\xi.$$

For a tile $S=(a,b]\times[c,d)$, denote $\tilde{S}:=[c,d)$. Let $\omega\in\Omega$ and $[u,v)$ be a finite interval. A subset $\mathbf{S}$ of $dom(\omega)$ is called the **ω-partition** of $[u,v)$ if it consisits of all elements $S\in\mathbf{S}$ such that $\tilde{S}\subset[u,v)$ and $\tilde{S}$ is maximal among those $\tilde{S'}\subset[u,v)$ such that$S'\in dom(\omega)$. In this case, note that

$$(u,v) \text{ or } [u,v)=\bigcup_{S\in\mathbf{S}}\tilde{S}\ \text{(disjoint)}.$$

Moreover, since for any $x \in [u, v)$ and $\epsilon > 0$ with $v - x > e^{u_0}\epsilon$ and $x - u > e^{u_0}\epsilon$ (refer (12) for u_0), $x \in \tilde{S}$ with $S \in \mathbf{S}$ and $|\tilde{S}| > \epsilon$, it holds that

(25)
$$\sharp\{S \in \mathbf{S}; 2^n \leq |\tilde{S}| < 2^{n+1}\} \leq 4e^{u_0}$$

for any $n \leq \log_2(v - u)$. Moreover if $n > \log_2(v - u)$, then the above set is empty.

For any $\omega \in \Omega$ and $t > 0$, let $\mathbf{S}$ be the ω-partition of $[0, t)$. We can represent F by ξ as follows.

$$\begin{aligned} F(\omega, t) &= \sum_{S \in \mathbf{S}} F(\omega, \tilde{S}) \qquad \text{(refer (20) for the notation)} \\ &= \sum_{S \in \mathbf{S}} \Xi(\omega(S), |\tilde{S}|) \\ &= \sum_{S \in \mathbf{S}} |\tilde{S}|^\alpha g(\omega(S))^{-\alpha} \Xi(\omega(S), g(\omega(S))) \\ &= \sum_{S \in \mathbf{S}} |\tilde{S}|^\alpha \xi_{\omega(S)}, \end{aligned}$$

where by (25) and Theorem 1, the sum in the 2nd side converges to the 1st side in the above equality.

We can prove the converse. Take any ξ satisfying (24) and define F by the above equality. That is,

(26)
$$F(\omega, t) := \sum_{S \in \mathbf{S}} |\tilde{S}|^\alpha \xi_{\omega(S)}$$

for any $\omega \in \Omega$ and $t > 0$, where $\mathbf{S}$ is the ω-partition of the interval $[0, t)$. Let $F(\omega, 0) = 0$ and for a negative t, let

$$F(\omega, t) := -F(\omega + t, -t).$$

For any $\omega \in \Omega$ and $t, s > 0$, denote by $\mathbf{S}, \mathbf{S}', \mathbf{S}''$ the ω-partition of $[0, t)$, the $\omega + t$-partition of $[0, s)$, the ω-partition of $[0, t + s)$, respectively. Then, note that $\mathbf{S}' + (0, t) := \{S + (0, t); S \in \mathbf{S}'\}$ is a ω-partition of the interval $[t, t + s)$, where we denote $S + (0, t) := \{(x, y + t); (x, y) \in S\}$. Since $\widetilde{S + (0, t)} = \tilde{S} + t$, it holds that

$$\begin{aligned} (0, t + s) \text{ or } [0, t + s) &= \bigcup_{S \in \mathbf{S}} \tilde{S} \cup \bigcup_{S \in \mathbf{S}'} \tilde{S} + t \\ &= \bigcup_{S \in \mathbf{S}''} \tilde{S} \end{aligned}$$

and that the 2nd side is a refinement of the 3rd side such that any element, say $\tilde{S}$ in $\{\tilde{S}; S \in \mathbf{S}''\}$ is a disjoint union of at most $\max_{\sigma\in\Sigma} L(\varphi(\sigma))$ number of elements, say $\tilde{S}_1, \cdots, \tilde{S}_K, \tilde{S}_{K+1}+t, \cdots, \tilde{S}_L+t$ in $\{\tilde{S}; S \in \mathbf{S}\} \cup \{\tilde{S}+t; S \in \mathbf{S}'\}$. In this case, since the set of tiles corresponding to the latter is the set of children of the tile S in ω, we have by (24) that

(27)
$$|\tilde{S}|^\alpha \xi_{\omega(S)} = \sum_{i=1}^{L} |\tilde{S}_i|^\alpha \xi_{\omega(S_i)},$$

where we used the fact that $(\omega+t)(S) = \omega(S+(0,t))$. Therefore, by (26),

$$\begin{aligned} F(\omega, t+s) &= \sum_{S\in\mathbf{S}''} |\tilde{S}|^\alpha \xi_{\omega(S)} \\ &= \sum_{S\in\mathbf{S}} |\tilde{S}|^\alpha \xi_{\omega(S)} + \sum_{S\in\mathbf{S}'} |\tilde{S}|^\alpha \xi_{\omega(S)} \\ &= F(\omega,t) + F(\omega+t,s). \end{aligned}$$

The above equality for a general t and s follows easily from the positive case.

Now we prove the continuity of F. By (25) and (26), we have

(28)
$$\begin{aligned} |F(\omega,t) - F(\omega,s)| &\le \max\{|F(\omega+s,t-s)|, |F(\omega+t,s-t)|\} \\ &\le \sum_{n; n\le \log_2|t-s|} 4e^{u_0}(2^{n+1})^\alpha \parallel \xi \parallel_\infty \\ &\le C|t-s|^\alpha \end{aligned}$$

for any $\omega \in \Omega$ and $s, t \in \mathbf{R}$ with a constant C. Hence, $F(\omega, t)$ is a uniformly equicontinuous function of $t \in \mathbf{R}$ with respect to $\omega \in \Omega$.

Therefore, to prove the continuity of $F(\omega, t)$ in 2 variables $\omega \in \Omega$ and $t \in \mathbf{R}$, it is sufficient to prove the continuity of $F(\omega,t)$ in the variable $\omega \in \Omega$ for any fixed $t = t_0$. This is clear for $t_0 = 0$ since $F(\omega, 0) \equiv 0$. Without loss of generality, we may assume that $t_0 > 0$ since $F(\omega, -t_0) = -F(\omega - t_0, t_0)$. Take an arbitrary $\omega_0 \in \Omega$. Let $\mathbf{S}$ be the ω_0-partition of the interval $[0, t_0)$. Take a sufficiently small $\epsilon > 0$. Take ω_0-**rational points** c, d such that $0 < c < d < t_0$ and $d - c > t_0 - \epsilon$, where by a ω_0-rational point, we mean a vertical coordinate of the corner of some tile in ω_0. Then, the interval $[c,d)$ is a finite union of elements in $\mathbf{S}$, say $[c,d) = S_1 \cup \cdots \cup S_K$. Take an open neighborhood U of ω_0 such that for any $\omega \in U$, there exist $S'_k \in dom(\omega)$ $(k = 1, \cdots, K)$ such that $\omega_0(S_k) = \omega(S'_k)$ and $\rho(S_k, S'_k) < \delta$, where $\delta > 0$ is small enough so that $S'_1 \cup \cdots \cup S'_K$ is an interval $[c', d')$ with $0 < c' < d' < t_0$ and $d' - c' > t_0 - \epsilon$ and the union is a disjoint union. Then by taking δ further small, we have

$$\begin{aligned} |F(\omega_0, [c,d)) - F(\omega, [c', d'))| &= \left| \sum_{k=1}^{K} |\tilde{S}_k|^\alpha \xi_{\omega_0(S_k)} - \sum_{k=1}^{K} |\tilde{S}'_k|^\alpha \xi_{\omega(S'_k)} \right| \\ &\leq \sum_{k=1}^{K} ||\tilde{S}_k|^\alpha - |\tilde{S}'_k|^\alpha||\xi_{\omega_0(S_k)}| \\ &\leq \sum_{k=1}^{K} [|\tilde{S}_k|^\alpha - (|\tilde{S}_k| - \delta)^\alpha] \parallel \xi \parallel_\infty \\ &< \epsilon. \end{aligned}$$

Therefore by (28),

$$\begin{aligned} & |F(\omega_0, t_0) - F(\omega, t_0)| \\ \leq\ & |F(\omega_0, t_0) - F(\omega_0, [c,d))| + |F(\omega_0, [c,d)) - F(\omega, [c', d'))| \\ & + |F(\omega, t_0) - F(\omega, [c', d'))| \\ \leq\ & 2C\epsilon^\alpha + \epsilon + 2C\epsilon^\alpha \end{aligned}$$

for any $\omega \in U$. Thus, F is continuous, and hence, is a cocycle on Ω.

Since it is clear that F is adapted and nonzero, to complete the proof, it is sufficient to prove that F is α-G-homogeneous. Take any $\omega \in \Omega$, $t \in \mathbf{R}$ and $\lambda \in G$. Let $\mathbf{S}$ be the ω-partition of the interval $[0,t)$. Then, it is clear that $\lambda\mathbf{S}$ is the $\lambda\omega$-partition of the interval $[0, \lambda t)$. Since we have

$$\begin{aligned} F(\lambda\omega, \lambda t) &= \sum_{S \in \mathbf{S}} |\widetilde{\lambda S}|^\alpha \xi_{\lambda\omega(\lambda S)} \\ &= \lambda^\alpha \sum_{S \in \mathbf{S}} \xi_{\omega(S)} \\ &= \lambda^\alpha F(\omega, t), \end{aligned}$$

F is α-G-homogeneous.

Thus, we have proved the following theorem.

Theorem 4 *A nonzero adapted α-G-homogeneous cocycle on $\Omega(\varphi, \eta, g)$, with (9) and (10), where $G := B(\varphi, \eta)$, is characterized by (26) with some ξ satisfying (24).*

Corollary 1 *If $G = \mathbf{R}_+$ in theorem 4, then F defines a self-similar process with strictly ergodic, stationary increments having 0 entropy.*

PROOF. Let $\zeta(\omega) := F(\omega, \cdot)$ and $F(\omega, \cdot)$ is considered as an element of $\tilde{\Omega}$ in Example 2. Let $\tilde{\Omega}(F)$ be the image of ζ. Then, the dynamical system of the $(\mathbf{R}, \mathbf{R}_+)$-action on $\tilde{\Omega}(F)$ is a factor of that on Ω, where $(\mathbf{R}, \mathbf{R}_+)$-action on $\tilde{\Omega}$ is as in Example 2. Thus, Corollary 1 follows from theorem 4. ■

Corollary 2 *(i) The set of nonzero α-G-homogeneous cocycles on $\Omega := \Omega(\varphi, \eta, g)$ with distinct exponents α is linearly independent.*
(ii) The set of adapted α-G-homogeneous cocycles on Ω, where α can be any number $0 < \alpha < 1$, has only finitely many linearly independent elements.

PROOF. (i) Let F_i $(i = 1, \cdots, K)$ be nonzero α_i-G-homogeneous cocycles on Ω with distinct α_i's. Suppose that they are linearly dependent. Then there exists $(c_1, \cdots, c_K) \neq (0, \cdots, 0)$ such that

$$c_1 F_1(\omega, t) + \cdots + c_K F_K(\omega, t) \equiv 0.$$

We may assume that $c_1 \neq 0$. Since F_1 is nonzero, we can take ω, t such that $F_1(\omega, t) \neq 0$. Let $\lambda \in G$ with $\lambda > 1$. Since F_i's are α_i-G-homogeneous, we have for any $n \in \mathbf{Z}$ that

$$\begin{aligned} 0 &= c_1 F_1(\lambda^n \omega, \lambda^n t) + \cdots + c_K F_K(\lambda^n \omega, \lambda^n t) \\ &= c_1 \lambda^{n\alpha_1} F_1(\omega, t) + \cdots + c_K \lambda^{n\alpha_K} F_K(\omega, t). \end{aligned}$$

Each $\lambda^{n\alpha_i}$ has different order of infinity as $n \to \infty$, so it follows from this equality that $c_i F_i(\omega, t) = 0$ $(i = 1, \cdots, K)$, which contradicts with $c_1 F_1(\omega, t) \neq 0$.
(ii) Let M_α be the matrix as in (21). Let r_α be the dimension of the eigenspace of the matrix M_α with eigenvalue 1. Then by theorem 4,

it is sufficient to prove that $\sum_{0<\alpha<1} r_\alpha < \infty$. Suppose to the contrary that $\sum_{0<\alpha<1} r_\alpha = \infty$. Then, the characteristic equation

(30)
$$\det(m^{(\alpha)}_{\sigma\sigma'} - \delta_{\sigma\sigma'})_{\sigma,\sigma'\in\Sigma} = 0$$

has infitely many solutions in α with $0 < \alpha < 1$, since the dimension of the eigen space for each α is not bigger than $\sharp\Sigma$. Therefore, there exists an accumulation point α_0 with $0 \leq \alpha_0 \leq 1$ of the solution α of (30). Since the equation (30) is of the form

$$f(\alpha) \equiv c_1\eta_1^\alpha + \cdots + c_K\eta_K^\alpha = 0$$

with $0 < \eta_1 < \cdots < \eta_K$, it holds that

$$\begin{aligned} f^{(n)}(\alpha_0) &= c_1(\log\eta_1)^n\eta_1^{\alpha_0} + \cdots + c_1(\log\eta_K)^n\eta_K^{\alpha_0} \\ &= 0 \end{aligned}$$

for $n = 0, 1, \cdots$. Since $\det((\log\eta_j)^{i-1})_{i,j=1,\cdots,K} \neq 0$, it follows that $c_j\eta_j^{\alpha_0} = 0$ $(j = 1, \cdots, K)$. Hence, $c_1 = \cdots = c_K = 0$ and $f(\alpha) \equiv 0$, which is a cotradiction since $f(-\infty) = \pm 1$ by (30). ■

Example 8 *Let us take $\Omega = \Omega(\varphi, \eta)$ in Example 7 for α such that $B(\varphi, \eta) = \mathbf{R}_+$. Then, the matrix M_α in (21) is as follows.*

$$M_\alpha = \begin{pmatrix} 2\beta^\alpha & (1-2\beta)^\alpha \\ (1-2\beta)^\alpha & 2\beta^\alpha \end{pmatrix}.$$

Then $\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of M_α with eigenvalue 1. Let F be the cocycle on Ω defined by (26) for this ξ. Then, F is a self-similar process with stationary increments of order α which has 0 entropy. In particular, we have such a process for $\alpha = \frac{1}{2}$ with

$$M_{\frac{1}{2}} = \begin{pmatrix} \frac{4}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{4}{3} \end{pmatrix}.$$

Example 9 *Let us take* $\Omega := \Omega(\varphi', \eta', g)$ *in Example 5. Assume that the matrix* M *in Example 5 has another eigenvalue* τ *such that* $1 < \tau < \lambda$ *with an eigen column vector* $\pi \neq 0$. *Let* $\alpha := \frac{\log \tau}{\log \lambda}$. *Then, the column vector*

$$\xi := \left(\frac{\pi_\sigma}{\zeta_\sigma^\alpha}\right)_{\sigma \in \Sigma}$$

satisfies (24). Thus, we have an α*-G-homogeneous cocycle* F *on* Ω *by (26) with* $G = \{\lambda^n; n \in \mathbf{Z}\}$.

Example 10 (Rudin-Shapiro cocycle) *Consider a mixing substitution* φ *on* $\{0, 1, 2, 3\}$*:*

$$\begin{array}{l} 0 \to 0 \quad 1 \quad 0 \quad 3 \\ 1 \to 0 \quad 1 \quad 2 \quad 1 \\ 2 \to 2 \quad 3 \quad 2 \quad 1 \\ 3 \to 2 \quad 3 \quad 0 \quad 3. \end{array}$$

Then, the matrix

$$M = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}$$

in Example 5 for this φ *has the maximum eigenvalue 4 with the following eigenvector* ζ. *It also has eigenvalue 2 with the following eigenvector* π. *Then* ξ *in Example 9 for these* ζ *and* π *is equal to* π*:*

$$\zeta = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \xi = \pi = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

We define a $\frac{1}{2}$*-G- cocycle* F *by (26), where* F *is defined on* $\Omega(\varphi, \eta, 1)$ *with* $\eta(\sigma)_i = \frac{1}{4}$ $(\sigma, i \in \{0, 1, 2, 3\})$ *and* $G := B(\varphi, \eta) = \{4^n; n \in \mathbf{Z}\}$. *It is called Rudin-Shapiro cocycle and was discussed in [2] and [4]. It is a rare case where we know something about the distribution of* $F(\omega, t)$ *more than Theorem 2. In fact,*

$$\int (F(\omega, b) - F(\omega, a))(F(\omega, d) - F(\omega, c))d\mu(\omega) = |[a, b] \cap [c, d]|$$

for any $a, b, c, d \in \mathbf{R}$.

Now let us consider generally an α-G-homogeneous cocycle $F(\omega, t)$ on $\Omega := \Omega(\varphi, \eta, g)$ with (9) and (10) which are not necessarily adapted.

4 Remarks and acknowledgment

To represent a nonlinear f-expansion, we need a space of colored tilings with curved tiles S of the shape

$$S = \{(x, y); a(y) < x \leq b(y) \text{ and } c \leq y < d\},$$

where $c < d$ are real numbers and a, b are smooth functions on $[c, d)$ such that $a(y) < b(y)$ for any $y \in [c, d)$ and $\int_c^d e^{b(y)} dy = 1$. It is discussed in [4] in a somewhat different form.

The cocycle in Example 8 has the least possible complexity among the nonzero, α-homogeneous, minimal cocycles [5]. Though it is not the same cocycle in [5], the proof is the same.

The G-action on the probability space (Ω, μ) with the unique $\mathbf{R}$-invariant probability Borel measure μ, where $\Omega := \Omega(\varphi, \eta, g)$ and $G := B(\varphi, \eta)$ with (9) and (10), can be proved to be ergodic. Moreover, for any adapted α-G-homogeneous cocycle F on Ω and for any $\omega \in \Omega$,

$$C = \lim_{\epsilon \downarrow 0} \frac{1}{-\log \epsilon} \int_\epsilon^1 \frac{|F(\omega, t+s) - F(\omega, t)|^{1/\alpha}}{s} \frac{ds}{s}$$

holds for almost all $t \in \mathbf{R}$, where

$$C := \int |F(\omega, 1)|^{1/\alpha} d\mu(\omega).$$

Using this, we can prove Itô's formula for the case $\alpha = 1/2$:

$$\begin{aligned} & f(F(\omega, B)) - f(F(\omega, A)) \\ = & \int_A^B f'(F(\omega, s)) dW(\omega, s) + \tfrac{C}{2} \int_A^B f''(F(\omega, s)) ds \end{aligned}$$

for any $\omega \in \Omega$, where the 'martingale part' $W(\omega, s)$ is defined in a weak sense [3].

It is an interesting question to ask when a quotient space of $\Omega(\varphi, \eta, g)$ admits a nontrivial additive group structure consistent with $(\mathbf{R}, G)$-action. We know only a little about this.

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