A GLUING LEMMA FOR ITERATED FUNCTION SYSTEMS

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Abstract. We obtain a special version of gluing lemma related to the theory of iterated function systems. As an application, we verify that a family of concrete n-dimensional self-affine tiles $\{T_{n,r} : 0 \le r < 3, n \ge 2\}$ are homeomorphic with the unit cube $[0,1]^n$. The tiles $T_{n,r}$ are "nontrivial" in the sense that each of them is neither a self-affine polytope nor the product of an interval with an $(n-1)$-dimensional self-affine tile.

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1. Introduction

Let $\{B_i : i\}$ be a countable family of closed subsets in a topological space X and $g_i : B_i \to Y$ continuous maps. If $g_i(x) = g_j(x)$ for $x \in B_i \cap B_j$, we may "glue together" the maps g_i and obtain a map $F : \bigcup_i B_i \to Y$ by setting $F(x) = g_i(x)$ for any $x \in B_i$ and i. By "gluing lemma" [1, p.70, Theorem 4.8], the map F is continuous if in addition every subset A with $A \cap B_i$ closed in X for each i is also closed in $\bigcup_i B_i$.

We consider the case when $\bigcup_i B_i$ is the attractor with condensation of an iterated function system (IFS) on $\mathbb{R}^n$. Here, an IFS on Euclidean space $\mathbb{R}^n$ is a finite family $\mathcal{F} = \{f_1, \ldots, f_q\}$ of q contractions on $\mathbb{R}^n$. It is well known that there is a unique nonempty compact set $E \subset \mathbb{R}^n$ with $E = \bigcup_j f_j(E)$ [6]. We call E the (Hutchinson) *attractor* of $\mathcal{F}$. Moreover, if $A \subset \mathbb{R}^n$ is a nonempty compact set there is a unique compact set K with $K = A \bigcup \left(\bigcup_{j=1}^q f_j(K)\right)$[3]. We call it the (Barnsley) *attractor of* $\mathcal{F}$ with condensation A.

Denote by Σ_q^* the monoid of words over the alphabet $\{1, 2, \ldots, q\}$. Let $f_\emptyset$ be the identity on $\mathbb{R}^n$ for the empty word $\emptyset$, and $f_\alpha = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}$ for any word $\alpha = i_1 i_2 \cdots i_k$. By uniqueness of Barnsley's attractor with condensation [3], we have

$$K = E \cup \left(\bigcup_{\alpha \in \Sigma_q^*} f_\alpha(A) \right) = \bigcup_{\alpha \in \Sigma_q^*} f_\alpha(A \cup E) = \overline{\bigcup_{\alpha \in \Sigma_q^*} f_\alpha(A)}. \tag{1.1}$$

If $\omega = j_1 j_2 \cdots$ is a word in $\Sigma_q := \{1, 2, \ldots, q\}^\infty$ its prefix of length n is denoted as $\omega(n)$. It is known that $\bigcap_k f_{\omega(k)}(K)$ consists of a single point in E [6], we denote it as $\Pi_{\mathcal{F}}(\omega)$. Moreover, $\Pi_{\mathcal{F}} : \Sigma_q \to E$ is a continuous onto mapping. We we call it the *coding map* of E.

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Let $\mathcal{F}_1 = \{f_{1,j} : 1 \le j \le q\}$ be IFS by injective contractions on an Euclidean space and $\mathcal{F}_2 = \{f_{2,j} : 1 \le j \le l\}$ an IFS consisting of $l \le q$ contractions. Let K_i be the attractor of $\mathcal{F}_i$ with condensation A_i for $i = 1, 2$, and E_i the corresponding Hutchinson attractors.

We want to compare the topology of K_1 with that of K_2, under appropriate assumptions, by finding specific continuous maps from K_1 into K_2. To this end, we may start from a continuous map $g : A_1 \to A_2$ and "glue" together delicately chosen continuous maps $g_\alpha : f_{1,\alpha}(A_1) \to f_{2,\tau(\alpha)}(A_2)$ and a continuous map from E_1 to E_2 described as follows.

For any surjection $\tau : \{1, 2, \ldots, q\} \to \{1, 2, \ldots, l\}$, there is a homomorphism from Σ_q^* onto Σ_l^* given by $\tau(i_1 i_2 \cdots i_k) = \tau(i_1)\tau(i_2)\cdots\tau(i_k)$. Moreover, for an infinite sequence $\omega = i_1 i_2 \cdots \in \Sigma_q$, we put $\tau(\omega) = \tau(i_1)\tau(i_2)\cdots$.

Assume in addition that E_2 **is a factor of** E_1 **with respect to** τ in the sense that $\Pi_{\mathcal{F}_2}(\tau(\omega)) = \Pi_{\mathcal{F}_2}(\tau(\omega'))$ for any $\omega, \omega' \in \Sigma_q$ with $\Pi_{\mathcal{F}_1}(\omega) = \Pi_{\mathcal{F}_1}(\omega')$. Then $\Pi_1(\omega) \xrightarrow{\pi_\tau} \Pi_2(\tau(\omega))$ defines a continuous surjection $\pi_\tau : E_1 \to E_2$, which is called the **factor map** between E_1 and E_2 with respect to τ.

Then the map $g_\alpha := f_{2,\tau(\alpha)} \circ g \circ f_{1,\alpha}^{-1}$ is well defined for each word $\alpha = j_1 \cdots j_k \in \Sigma_q^*$, where $f_{i,\alpha} = f_{i,j_1} \circ \cdots \circ f_{i,j_k}$. Setting $g_\emptyset = g$ for the empty word $\emptyset \in \Sigma_q^*$, we have $K_1 = \overline{\bigcup_{\alpha \in \Sigma_q^*} f_{1,\alpha}(A_1)}$ by Equation 1.1. Unfortunately, the continuous maps in $\{\pi_\tau\} \cup \{g_\alpha : \alpha \in \Sigma_q^*\}$ may not satisfy the conditions in gluing lemma [1, p.70, Theorem 4.8]. So, we need to characterize those maps $g : A_1 \to A_2$ such that the map π_τ and all the maps g_α share a common continuous extension $F_g : K_1 \to K_2$, which automatically satisfies $F_g \circ f_{1,j}|_{K_1} = f_{2,\tau_j} \circ F_g$ for $1 \le j \le q$. Before that, we introduce the following notion of consistence.

Definition 1.1. *A continuous map $g : A_1 \to A_2$ with $g|_{A_1 \cap E_1} = \pi_\tau|_{A_1 \cap E_1}$ is called* consistent with the pair (E_1, E_2) with respect to τ *provided that E_2 is a factor of E_1 with respect to τ and that $f_{2,\tau(\xi)} \circ g \circ f_{1,\xi}^{-1}(y) = f_{2,\tau(\eta)} \circ g \circ f_{1,\eta}^{-1}(y)$ holds for any $y \in f_{1,\eta}(A_1) \cap f_{1,\xi}(A_1)$.*

Now, we are in a good position to state our gluing lemma for IFS.

Theorem 1.2. (Gluing Lemma for IFS) *Let $\mathcal{F}_1 = \{f_{1,j} : 1 \le j \le q\}$ be an IFS consisting of injective contractions $f_{1,j} : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{F}_2 = \{f_{2,j} : 1 \le j \le l\}$ an IFS on $\mathbb{R}^m$ with $q \ge l$. Let K_i be the Barnsley attractor of $\mathcal{F}_i$ with condensation A_i, and E_i the Hutchinson attractor. Let $\tau : \{1, 2, \ldots, q\} \to \{1, 2, \ldots, l\}$ be a surjective map. Then a continuous map $g : A_1 \to A_2$ is consistent with the pair (E_1, E_2) with respect to τ if and only if it has a continuous extension $F_g : K_1 \to K_2$ such that $F_g \circ f_{1,j}|_{K_1} = f_{2,\tau(j)} \circ F_g$ for $1 \le j \le q$.*

The extension $F_g : K_1 \to K_2$ is surjective if only $g : A_1 \to A_2$ is. It is even a homeomorphism if g satisfies additional assumptions. Actually, we have the following.

Theorem 1.3. *Let $\mathcal{F}_1 = \{f_{1,j} : 1 \le j \le q\}$ be an IFS consisting of injective contractions $f_{1,j} : \mathbb{R}^n \to \mathbb{R}^n$, and $\mathcal{F}_2 = \{f_{2,j} : 1 \le j \le q\}$ an IFS consisting of injective contractions $f_{2,j} : \mathbb{R}^m \to \mathbb{R}^m$, where it is possible that $m \ne n$. If $g : A_1 \to A_2$ is a homeomorphism such that g is consistent with the pair (E_1, E_2) with respect to τ while g^{-1} is consistent with the pair (E_2, E_1) with respect to τ^{-1} then the extension $F_g : K_1 \to K_2$ is a homeomorphism, too.*

With Theorems 1.2 and 1.3 we consider a class of self-affine tiles $T \subset \mathbb{R}^n$ for $n \ge 3$ and determine their topology. Here, for any $n \times n$ expanding matrix $\mathbb{A}$ with real entries and any finite set $\mathcal{D} \subset \mathbb{R}^n$ with cardinality $\#\mathcal{D} \ge 2$ there is a unique nonempty compact set T with $T = \bigcup_{d\in\mathcal{D}} \mathbb{A}^{-1}(T + d)$, which is called a self-affine tile if further $\#D = |det(\mathbb{A})|$ and $T^o \ne \emptyset$ [7]. In this case, we also call $\mathcal{D}$ the *digit set.* Moreover, the set T may be represented by

$$T = \left\{ \sum_{k=1}^{\infty} \mathbb{A}^{-k} d_k \,\middle|\, d_k \in \mathcal{D} \right\}. \tag{1.2}$$

We wonder whether a concretely constructed T defined as follows is homeomorphic with $[0,1]^n$ for $n \ge 3$. By setting $\mathbb{A} = 3\mathbb{I}_n$ and $\mathcal{D} = \{(i_1, \ldots, i_n)^t : i_j = -1, 0, 1\}$, we easily see that the self-affine tile $T(\mathbb{A}, \mathcal{D})$ is just $[-\frac{1}{2}, \frac{1}{2}]^n$. Now, for any real number $r > 0$ we continuously perturb the digit set $\mathcal{D}$ to the following

$$\mathcal{D}_r = \left\{ \begin{bmatrix} i_1 \\ \vdots \\ i_{n-1} \\ i_n \end{bmatrix} + \chi(i_1)\cdots\chi(i_{n-1}) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ r \end{bmatrix} : \begin{array}{l} i_1, \ldots, i_n \in \{-1, 0, 1\}, \\ \chi(i_j) = 1 \text{ if and only if } i_j = 0, \\ \text{otherwise } \chi(i_j) = 0 \end{array} \right\}. \tag{1.3}$$

Then T_r given by $T_r = \bigcup_{d\in\mathcal{D}_r} \frac{1}{3}\mathbb{I}_n(T_r + d)$ is a self-affine tile. Moreover, T_r is connected if and only if $0 < r \le 3$. For $0 < r < 3$, we will determine the topology of T_r for $0 < r < 3$.

Theorem 1.4. *For $0 < r < 3$ there is a homeomorphism $H : \mathbb{R}^n \to \mathbb{R}^n$ such that $H(T_r) = [-\frac{1}{2}, \frac{1}{2}]^n$.*

Here, we use Theorem 1.3 in constructing the homeomorphism h and do not refer to the generalized Schönflies theorem in dimension $n \ge 3$ [4].

Remark 1.5. The topology of a self-affine tile can be very difficult. When $n = 2$, from Torhorst Theorem (see [11, p.126, Lemma 2]) one may infer that the boundary of a self-affine tile T is a simple closed curve whenever its interior is connected [2, 8]. By Schönflies Theorem in dimension two, one sees that T is a topological disk. When $n \ge 3$, even if the boundary of a self-affine tile $T \subset \mathbb{R}^n$ is known to be a sphere, we still need to verify whether the boundary is a "flat sphere" before we can use the generalized Schönflies theorem [4] to show that the tile itself is homeomorphic with $[0,1]^n$. Very recently, Conner and Thuswaldner [5] develop a deep mathematical model in studying the topology of higher dimensional self-affine tiles and provides algorithms to determine whether a $\mathbb{Z}^n$-tile with $n \geq 3$ is homeomorphic with $[0,1]^n$. To indicate that Schönflies theorem is really needed, the authors also construct a three dimensional self-affine tile T whose boundary is a horned sphere. See [5, Proposition 8.4] for more details. Since the family of self-affine tiles T_r we discuss is arranged by a continuous parameter $r \in (0,3)$, the algorithms suggested in [5] only work for at most countably many of our examples of self-affine tiles.

We arrange the paper as follows. Section 2 provides a proof for Theorem 1.2 based on a topological version of the closed graph theorem (see for instance [9, p.171,Exercise 8]) and then proves Theorem 1.3. Section 3 gives a few applications of Theorem 1.2 to the study on topology of fractal tiles. Section 4 gives the details on how to construct the homeomorphism $H : \mathbb{R}^n \to \mathbb{R}^n$ that sends T_r onto $[-\frac{1}{2}, \frac{1}{2}]^n$. Section 5 is an appendix providing more examples concerning applications of Theorem 1.2.

2. Proof of Gluing Lemma for IFS

Proof of Theorem 1.2.

Assume that $F_g : K_1 \to K_2$ is a continuous extension of g such that $F_g \circ f_{1,j}|_{K_1} = f_{2,\tau(j)} \circ F_g$ for $j \in \{1, \ldots, q\}$. Then, for any $y \in f_{1,\eta}(A_1) \cap f_{1,\xi}(A_1)$ there exist $x, z \in A_1$ with $f_{1,\eta}(x) = y = f_{1,\xi}(z)$ such that

$$\begin{aligned} f_{2,\tau(\xi)} \circ g \circ f_{1,\xi}^{-1}(y) &= f_{2,\tau(\xi)} \circ g(z) = f_{2,\tau(\xi)} \circ F_g(z) = F_g \circ f_{1,\xi}(z) = F_g(y) \\ &= F_g \circ f_{1,\eta}(x) = f_{2,\tau(\eta)} \circ F_g(x) = f_{2,\tau(\eta)} \circ g(x) = f_{2,\tau(\eta)} \circ g \circ f_{1,\eta}^{-1}(y) \end{aligned}$$

Therefore, the proof for "if part" will be completed if we can show that the attractor E_2 is a factor of E_1 with respect to τ and that the map F_g coincides with the factor map π_τ on E_1. In particular, we have $g|_{A_1 \cap E_1} = \pi_\tau|_{A_1 \cap E_1}$.

Fixing two elements $\omega = i_1 i_2 \cdots$ and $\omega' = i'_1 i'_2 \cdots$ of Σ_q with $\Pi_{\mathcal{F}_1}(\omega) = \Pi_{\mathcal{F}_1}(\omega')$, we have $\lim_{k\to\infty} f_{1,i_1\cdots i_k}(K_1) = \Pi_{\mathcal{F}_1}(\omega)$ and $\lim_{k\to\infty} f_{1,i'_1\cdots i'_k}(K_1) = \Pi_{\mathcal{F}_1}(\omega')$ under Hausdorff distance. Then $\Pi_{\mathcal{F}_1}(\omega) = \lim_{k\to\infty} f_{1,i_1\cdots i_k}(x) = \lim_{k\to\infty} f_{1,i'_1\cdots i'_k}(x) = \Pi_{\mathcal{F}_1}(\omega')$ for any fixed point $x \in A_1$. Since we assume that $F_g \circ f_{1,j}|_{K_1} = f_{2,\tau(j)} \circ F_g$, we have $F_g \circ f_{1,i_1\cdots i_k}(x) = f_{2,\tau(i_1\cdots i_k)} \circ g(x)$ and $F_g \circ f_{1,i'_1\cdots i'_k}(x) = f_{2,\tau(i'_1\cdots i'_k)} \circ g(x)$. Therefore, we further have

$$\begin{cases} F_g \circ \Pi_{\mathcal{F}_1}(\omega) = \lim_{k\to\infty} F_g\left(f_{1,i_1\cdots i_k}(x)\right) = \lim_{k\to\infty} f_{2,\tau(i_1\cdots i_k)} \circ g(x) = \Pi_{\mathcal{F}_2}(\tau(\omega)) \\ F_g \circ \Pi_{\mathcal{F}_1}(\omega') = \lim_{k\to\infty} F_g\left(f_{1,i'_1\cdots i'_k}(x)\right) = \lim_{k\to\infty} f_{2,\tau(i'_1\cdots i'_k)} \circ g(x) = \Pi_{\mathcal{F}_2}(\tau(\omega')) \end{cases}$$

Consequently, we have $\Pi_{\mathcal{F}_2}(\tau(\omega)) = \Pi_{\mathcal{F}_2}(\tau(\omega'))$ and $F_g|_{E_1} = \pi_\tau$.

In the following, we continue to prove the "only if part".

Combining Equation (1.1) and consistency of g with (E_1, E_2) with respect to τ, we may define a map $F_g^*: \bigcup_{\alpha\in\Sigma_q^*} f_{1,\alpha}(A_1) \to \bigcup_{\beta\in\Sigma_l^*} f_{2,\beta}(A_2)$ by putting $F_g^*\big|_{f_{1,\alpha}(A_1)} = f_{2,\tau(\alpha)} \circ g \circ f_{1,\alpha}^{-1}\big|_{f_{1,\alpha}(A_1)}$ for each word $\alpha \in \Sigma_q^*$. As $g: A_1 \to A_2$ coincides with the factor map $\pi_\tau: E_1 \to E_2$ on $A_1 \cap E_1$, we have $F_g^*(x) = \pi_\tau(x)$ for each $x \in A_1 \cap E_1$.

Let $X_n = \bigcup_{|\alpha|\le n} f_{1,\alpha}(A_1)$ and $F_n = F_g^*\big|_{X_n}$. Then $\{X_n\}$ is an increasing sequence of compact subsets of K_1 with $\overline{\bigcup_n X_n} = K_1$. Moreover, consistency of g with (E_1, E_2) also implies that

$$F_n|_{X_n\cap E_1} = \pi_\tau|_{X_n\cap E_1}. \tag{2.1}$$

The topological closed graph theorem states that a map from a topological space X into a compact Hausdorff space Y is continuous if and only if its graph is a closed subset in the product space $X \times Y$ [9, p.171,Exercise 8]. From this we see that the graphs of F_n form an increasing sequence of compact subsets in the product space $K_1 \times K_2$, which necessarily converge to a compact subset Γ of $K_1 \times K_2$. If only we can show that Γ is the graph of a map from K_1 into K_2 then this map is a continuous extension of $F_g^*: \bigcup_n X_n \to K_2$. Therefore, we may complete our proof by the following claim.

Claim. The intersection of Γ with $\{x\} \times K_2$ is a single point for each $x \in K_1$.

Given a point $(x, y) \in \Gamma$, we consider two cases: $x \in E_1$ and $x \in K_1 \setminus E_1$.

If $x \in E_1$ there exists a sequence $\{x_k\}$ in $\bigcup_n X_n$ such that $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} F_{n_k}(x_k) = y$, where n_k is the least integer with $x_k \in X_{n_k}$. For each $k \ge 1$, fix a word u_k with length $|u_k| = n_k$ such that $x_k \in f_{1,u_k}(A_1)$. By choosing an appropriate subsequence, we may assume that ω is the limit of $\{u_k\}$. If ω is a word of finite length $|\omega| = n$, then $\{n_k\}$ is eventually equal to n and $y = \lim_{k\to\infty} F_{n_k}(x_k) = \lim_{k\to\infty} F_n(x_k) = F_n(x)$, which is equal to $\pi_\tau(x)$ by equation (2.1). If ω is an infinite sequence in Σ_q then $x = \Pi_{\mathcal{F}_1}(\omega)$. Moreover, the common prefix of u_k and ω, denoted as v_k, has a length converging to infinity. Therefore, under Hausdorff distance we have $\lim_{n\to\infty} f_{2,\tau(\omega(n))}(K_2) = \lim_{k\to\infty} f_{2,\tau(v_k)}(K_2) = \Pi_{\mathcal{F}_2}(\tau(\omega))$. Since $F_{n_k}(x_k) = f_{2,\tau(u_k)} \circ g \circ f_{1,u_k}^{-1}(x_k)$ lies in $f_{2,\tau(u_k)}(A_2)$ which is a subset of $f_{2,\tau(v_k)}(K_2)$, we have $y = \lim_{k\to\infty} F_{n_k}(x_k) = \lim_{k\to\infty} f_{2,\tau(v_k))}(K_2) = \Pi_{\mathcal{F}_2}(\tau(\omega)) = \pi_\tau(\Pi_{\mathcal{F}_1}(\omega)) = \pi_\tau(x)$.

If $x \in K_1 \setminus E_1$ there is a number $r > 0$ and an integer $n \ge 1$ such that $X_n \setminus E_1$ contains the ball $B_r = \{y \in K_1 : |x - y| \le r\}$. So the graph of $F_k|_{B_r}$ coincides with that of $F_n|_{B_r}$ for $k \ge n$; and the intersection $(B_r \times K_2) \cap \Gamma$ is equal to the graph of $F_n|_{B_r}$. This indicates that $(\{x\} \times K_2) \cap \Gamma = \{(x, F_n(x))\}$. □

Proof of Theorem 1.3.

Since F_g is a surjection whenever $g: A_1 \to A_2$ is, we only need to show that F_g is also an injection. To this end, we will verify that $F_{g^{-1}} \circ F_g: K_1 \to K_1$ is the identity map on K_1.

As $\tau : \Sigma_q^* \to \Sigma_q^*$ is an isomorphism, the factor maps $\pi_\tau : E_1 \to E_2$ and $\pi_{\tau^{-1}} : E_2 \to E_1$ are each a homeomorphism. By the construction of F_g and $F_{g^{-1}}$, we have $F_g|_{E_1} = \pi_\tau$ and $F_{g^{-1}}\big|_{E_2} = \pi_{\tau^{-1}}$. Then it suffices to show that $F_{g^{-1}} \circ F_g(x) = x$ for every x in $K_1 \setminus E_1$.

Given a point $x \in K_1 \setminus E_1$, there exists a word $\alpha \in \Sigma_q^*$ with $x \in f_{1,\alpha}(A_1)$. Since all the maps $f_{1,j}$ are assumed to be injective, there exists a unique point $z \in A_1$ with $x = f_{1,\alpha}(z)$. Therefore, the following formula will end our proof:

$$x \xrightarrow{F_g} f_{2,\tau(\alpha)} \circ g \circ f_{1,\alpha}^{-1}(x) = f_{2,\tau(\alpha)} \circ g(z) \xrightarrow{F_{g^{-1}}} f_{1,\tau^{-1}\circ\tau(\alpha)} \circ g^{-1} \circ f_{2,\tau(\alpha)}^{-1} \circ f_{2,\tau(\alpha)} \circ g(z) = f_{1,\alpha}(z) = x.$$

□

3. Basic Applications of Theorems 1.2 and 1.3

Throughout this section, we fix an integer $n \geq 3$, a real number $0 < r < 3$, and a set $\mathcal{D}_{n-1} = \{(i_1, \ldots, i_{n-1}) : i_k = -1, 0, 1\}$; and will use the following two IFS. The first one is $\mathcal{F}_1^* = \{f_{1,d}^* : d \in \mathcal{D}_{n-1}\}$ on $\mathbb{R}^{n-1}$ with $f_{1,d}^*(x) = \frac{x+d}{3}$ for each d in $\mathcal{D}_{n-1} \setminus \{(0, \ldots, 0)^t\}$ and $f_{1,d}^*(x) = \frac{x}{4}$ for $d = (0, \ldots, 0)^t \in \mathcal{D}_{n-1}$; the second one is $\mathcal{F}_2^* = \left\{f_{2,1}^*(x) = \frac{x}{3},\ f_{2,2}^* = \frac{x+r}{3}\right\}$ on $\mathbb{R}$.

If we choose $A_1^* = \left[-\frac{1}{6}, \frac{1}{6}\right]^{n-1} \setminus \left(-\frac{1}{8}, \frac{1}{8}\right)^{n-1}$ and $A_2^* = \left[0, \frac{r}{3}\right]$, then the attractor K_1^* of $\mathcal{F}_1^*$ with condensation A_1^* is exactly the unit square $[-\frac{1}{2}, \frac{1}{2}]^{n-1}$, while the attractor K_2^* of $\mathcal{F}_2^*$ with condensation A_2^* is just the interval $\left[0, \frac{r}{2}\right]$.

Let $\tau^* : \mathcal{D}_{n-1} \to \{1, 2\}$ be the unique surjection with $\tau^{*-1}(2) = (0, \ldots, 0)^t$. Let E_i^* be the attractor of $\mathcal{F}_i^*$. Then E_2^* is a factor of E_1^* with respect to τ^*.

Recall that A_1^* is the union of all the line segments $L_x = \{tx : \frac{3}{4} \leq t \leq 1\}$ with x running through $\partial\left[-\frac{1}{6}, \frac{1}{6}\right]^{n-1}$. We may define a continuous function $g^* : A_1^* \to A_2^*$ by requiring that the restriction $g^*|_{L_x}$ to each L_x is an affine map with $g^*(x) = 0$ and $g^*\left(\frac{3}{4}x\right) = \frac{r}{3}$. In particular, we have $g^{*-1}(0) = \partial\left[-\frac{1}{6}, \frac{1}{6}\right]^{n-1}$ and $g^{*-1}(\frac{r}{3}) = \partial\left[-\frac{1}{8}, \frac{1}{8}\right]^{n-1}$.

Example 3.1. The continuous map $g^* : A_1^* \to A_2^*$ is consistent with the pair (E_1^*, E_2^*) with respect to τ^*. By Theorem 1.2 it has a continuous extension $F_{g^*} : K_1^* \to K_2^*$ such that $F_{g^*} \circ f_{1,d}^* = f_{2,\tau^*(d)}^* \circ F_{g^*}$ for each $d \in \mathcal{D}_{n-1}$.

Lemma 3.2. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be as in Theorem 1.2. Let $f_d = f_{1,d} \times f_{2,\tau(d)}$ for any $d = 1, 2, \ldots, q$. Let the graph of g be A and the graph of F_g be K. Then K is the attractor of $\{f_d : d = 1, 2, \ldots, q\}$ with condensation A.*

Proof. Clearly, $A \subset K$. Since $F_g \circ f_{1,d} = f_{2,\tau(d)} \circ F_g$ for $1 \le d \le q$, we may check that for any $z = (x, F_g(x)) \in K$ the following holds

$$f_d(z) = \begin{bmatrix} f_{1,d}(x) \\ f_{2,\tau(d)} \circ F_g(x) \end{bmatrix} = \begin{bmatrix} f_{1,d}(x) \\ F_g \circ f_{1,d}(x) \end{bmatrix} \in K.$$

This means that $F_d(K) \subset K$ for $1 \le d \le q$. On the other hand, for each point $(x, F_g(x))$ in $K \setminus A$ there exist a point $y \in K \setminus A$ and $d \in \{1, 2, \ldots, q\}$ such that $f_{1,d}(y) = x$, which indicates that

$$\begin{bmatrix} x \\ F_g(x) \end{bmatrix} = \begin{bmatrix} f_{1,d}(y) \\ F_g \circ f_{1,d}(y) \end{bmatrix} = \begin{bmatrix} f_{1,d}(y) \\ f_{2,\tau(d)} \circ F_g(y) \end{bmatrix} = f_d\left(\begin{bmatrix} y \\ F_g(y) \end{bmatrix}\right) \in f_{2,d}(K).$$

Therefore, $K = A \cup \left(\bigcup_{d=1}^{q} f_d(K)\right)$ hence K is the attractor of $\mathcal{F}_2$ with condensation A. □

In the following, the maps $g^* : A_1^* \to A_2^*$ and $F_{g^*} : K_1^* \to K_2^*$ are defined as in Example 3.1; moreover, let $\mathcal{F}_2 = \{f_{2,d} : d \in \mathcal{D}_{n-1}\}$ be an IFS with $f_{2,d} = f_{1,d}^* \times f_{2,\tau^*(d)}^*$, let $A_2^\sharp$ denote the graph of $g^* : A_1^* \to A_2^*$ and $K_2^\sharp$ the attractor of $\mathcal{F}_2$ with condensation $A_2^\sharp$.

Example 3.3. Setting $\mathcal{F}_1 = \{f_{1,d} : d \in \mathcal{D}_{n-1}\}$ to be an IFS on $\mathbb{R}^n$ with

$$f_{1,d}(x) = \begin{cases} \frac{1}{3}\left(x + \begin{bmatrix} d \\ 0 \end{bmatrix}\right) & \textit{if } d \neq (0, \ldots, 0)^t, \\ \\ \frac{1}{3}\left(x + \begin{bmatrix} \mathbb{O} \\ 0 \end{bmatrix}\right) & \textit{if } d = \mathbb{O} := (0, \ldots, 0)^t. \end{cases}$$

and $A_1^\sharp$ the Cartesian product of $\partial\left[-\frac{1}{6}, \frac{1}{6}\right]^{n-1}$ with $[0, \frac{r}{3}]$; let $K_1^\sharp$ be the attractor of $\mathcal{F}_1$ with condensation $A_1^\sharp$, and $g^\sharp : A_1^\sharp \to A_2^\sharp$ the homeomorphism whose restriction to each line segment $L_y := \{y\} \times [0, \frac{r}{3}]$ with $y \in \partial\left[-\frac{1}{6}, \frac{1}{6}\right]^{n-1}$ is an affine map with

$$g^\sharp\left(\begin{bmatrix} y \\ 0 \end{bmatrix}\right) = \begin{bmatrix} y \\ 0 \end{bmatrix} \quad \text{and} \quad g^\sharp\left(\begin{bmatrix} y \\ \frac{r}{3} \end{bmatrix}\right) = \begin{bmatrix} \frac{3}{4}y \\ \frac{r}{3} \end{bmatrix}.$$

Then the map $g^\sharp : A_1^\sharp \to A_2^\sharp$ is a homeomorphism that satisfies all the conditions in Theorem 1.3 hence the extension $F_{g^\sharp} : K_1^\sharp \to K_2^\sharp$ is a homeomorphism.

Recall that the convex hull $C_H(K_1^\sharp)$ of $K_1^\sharp$ (*the smallest compact convex set containing* $K_1^\sharp$) coincides with that of $K_2^\sharp$. More precisely, $C_H(K_1^\sharp)$ is spanned by $\left[-\frac{1}{2}, \frac{1}{2}\right]^{n-1} \times \{0\}$ and the point $v := (0, \ldots, 0, \frac{r}{2}) \in \mathbb{R}^n$.

Theorem 3.4. *For any real number $\epsilon > 0$ let P_ϵ be the convex hull of $C_H(K_1^\sharp) \cup \{v'\}$, where $v' = v + (0, \ldots, 0, \epsilon)$. Then there exists a homeomorphism of $\mathbb{R}^n$ onto itself which sends $K_1^\sharp$ onto $K_2^\sharp$ and keeps each point off P_ϵ.*

Proof. Setting $A_1 = \overline{P_\epsilon \setminus \bigcup_d f_{1,d}(P_\epsilon)}$ and $A_2 = \overline{P_\epsilon \setminus \bigcup_d f_{2,d}(P_\epsilon)}$. Then P_ϵ is the attractor of $\mathcal{F}_i$ with condensation A_i for $i = 1, 2$.

Let $\sigma_1 = P_\epsilon \bigcap \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = \frac{r}{2} + \frac{\epsilon}{3}\}$, $\sigma_2 = P_\epsilon \bigcap \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = \frac{r}{6} + \frac{\epsilon}{3}\}$ and $\sigma_3 = [\frac{-1}{6}, \frac{1}{6}]^{n-1} \times \{0\}$. Let C be the convex hull of $\sigma_1 \cup \sigma_2 \cup \sigma_3$.

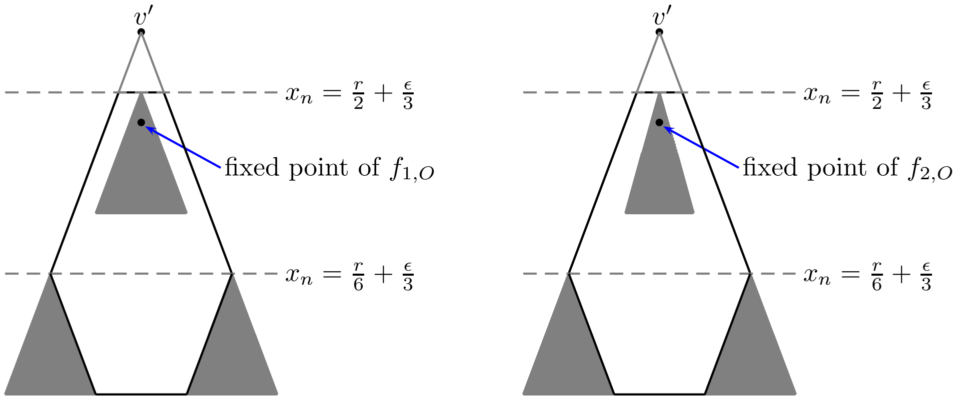

FIGURE 1. A rough depiction of $P_\epsilon, \partial C$ and $f_{1,O}(P_\epsilon)$ when $r = 2$.

Define a homeomorphism $g : A_1 \to A_2$ as follows. Firstly, set $g(x) = x$ for each $x \in A_1 \setminus C^o$. Secondly, for each $x = (x_1, \ldots, x_n)^t$ in $f_{1,O}(\partial P_\epsilon)$, let

$$g(x) = \begin{bmatrix} \frac{3}{4}x_1 \\ \vdots \\ \frac{3}{4}x_{n-1} \\ x_n \end{bmatrix}.$$

Thirdly, for each segment $L_y := \{y\} \times [0, \frac{r}{3}]$ with $y \in [-\frac{1}{6}, \frac{1}{6}]^{n-1}$, let $g|_{L_y}$ be an affine map with

$$g\left(\begin{bmatrix} y \\ 0 \end{bmatrix}\right) = \begin{bmatrix} y \\ 0 \end{bmatrix} \quad \text{and} \quad g\left(\begin{bmatrix} y \\ \frac{r}{3} \end{bmatrix}\right) = \begin{bmatrix} \frac{3}{4}y \\ \frac{r}{3} \end{bmatrix}.$$

Lastly, extend the map g piecewise linearly along $P_\epsilon \bigcap \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n = t\}$ for $0 < t < \frac{r}{2} + \frac{\epsilon}{3}$.

The restriction $g|_{A_1}$ coincides with $g^\sharp : A_1^\sharp \to A_2^\sharp$ in Example 3.3. Moreover, $g : A_1 \to A_2$ is a homeomorphism satisfying the conditions in Theorem 1.3; hence its extension $F_g : K_1 \to K_2$ is a homeomorphism such that $F_g|_{K_1^\sharp}$ coincides with $F_g^\sharp : K_1^\sharp \to K_2^\sharp$ in Example 3.3. As $F_g(x) = x$ for each $x \in \partial P_\epsilon$, we may extend F_g to a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ by setting $h(x) = x$ for $x \notin P_\epsilon$ and $h(x) = F_g(x)$ for $x \in P_\epsilon$. Clearly, h is a homeomorphism we search for. $\square$

4. Structure of T_r for $0 < r < 3$

Throughout in this section, for each fixed $n \geq 3$ and each $r \in (0,3)$, we always set $\mathcal{D}_{n-1}$ as in Example 3.1 and $\mathcal{D}_r$ as in Equation (1.3). Our aim is to study the self-affine tile T_r determined by $T_r = \bigcup_{d\in\mathcal{D}_r} \frac{1}{3}\mathbb{I}_n(T_r + d)$ and construct a proof for Theorem 1.4 as follows.

Proof for Theorem 1.4.

We start from the structure of the convex hull of T_r, denoted as C_r.

Claim 1. Let X_r be the cone spanned by $\left[-\frac{1}{2}, \frac{1}{2}\right]^{n-1} \times \{\frac{1}{2}\}$ and the point $\left(0, \cdots, 0, \frac{1}{2} + \frac{r}{2}\right)^t$. Then $C_r = X_r \cup \left[-\frac{1}{2}, \frac{1}{2}\right]^n$.

Proof. Let $C = X_r \cup [-\frac{1}{2}, \frac{1}{2}]^n$. Then C is spanned by the fixed points of the contractions in the IFS $\mathcal{F}_r := \{x \mapsto \mathbb{A}^{-1}(x + d) : d \in \mathcal{D}_r\}$, thus we have $C \subset C_r$. Moreover, we have $f(C) \subset C$ for each $f \in \mathcal{F}_r$. This indicates that $T_r \subset C$ and hence that $C_r \subset C$. □

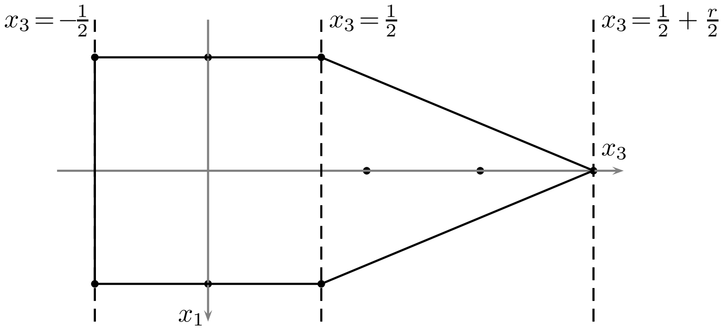

FIGURE 2. A depiction of $T_r \cap \{x_2 = 0\}$ for the case $n = 3$ and $r = 2.4$.

Secondly, we fix a partition of the boundary ∂T_r based on the tiling property of T_r.

Claim 2. The tile T_r does not intersect $C_r + (0, \ldots, 0, s)^t$ for $s \leq -2$, hence its boundary ∂T_r is the union of $B := \left(\partial \left[-\frac{1}{2}, \frac{1}{2}\right]^{n-1}\right) \times [-\frac{1}{2}, \frac{1}{2}]$ and $B_\pm := T_r \cap (T_r + (0, \ldots, 0, \pm 1)^t)$.

Proof. Clearly, $\{T_r + (0, \ldots, 0, k)^t |\ k \in \mathbb{Z}\}$ is a tiling of $[-\frac{1}{2}, \frac{1}{2}]^{n-1} \times \mathbb{R}$. Moreover, $f(C_r)$ is a subset of $C_r \setminus U$ for each $f \in \mathcal{F}_r$, where $U = \left(-\frac{1}{6}, \frac{1}{6}\right)^{n-1} \times [-\frac{1}{2}, -\frac{1}{2} + \frac{r}{3}]$. Now, we can infer that $\bigcup_{f\in\mathcal{F}_r} f(C_r)$ contains T_r and is disjoint from $C_r + (0, \ldots, 0, s)^t$ for any $s \leq -2$. Consequently, we have $\partial T_r = B_+ \cup B \cup B_-$. □

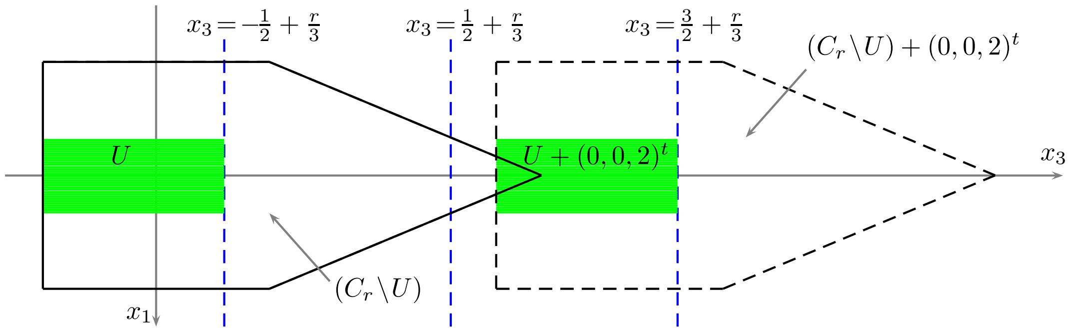

FIGURE 3. A picture for $(C_r \setminus U) \cap \{x_2 = 0\}$ for the case $n = 3$ and $r = 2.4$.

Let $P \subset \mathbb{R}^n$ be the pyramid spanned by $\{(0,\ldots,0,-\frac{1}{2}+\frac{r}{2})^t\} \bigcup [-\frac{1}{2},\frac{1}{2}]^{n-1} \times \{-\frac{1}{2}\}$. Then P is the convex hull of B_-. Moreover, by **Claim 2**, we know that P does not intersect B_+. So we can choose a small number $\varepsilon > 0$ such that $Q = P + (0,\ldots,0,\varepsilon)^t$ does not intersect B_+. Clearly, each of $P \cup (\mathbb{R}^{n-1} \times \{-\frac{1}{2}\})$ and $Q \cup (\mathbb{R}^{n-1} \times \{-\frac{1}{2}+\varepsilon\})$ cuts $\mathbb{R}^n$ into two parts.

Let P_+ be set of points above $P \cup (\mathbb{R}^{n-1} \times \{-\frac{1}{2}\})$ and $P_- = \overline{\mathbb{R}^n \setminus P_+}$. Let Q_+ be set of points above $Q \cup (\mathbb{R}^{n-1} \times \{-\frac{1}{2}+\varepsilon\})$ and $Q_- = \overline{\mathbb{R}^n \setminus Q_+}$. See Figure 4 for relative locations of P, P_+, Q and Q_+. Here we have $B_- \subset P_-$, $B_+ \subset Q_+$, and $Q_+ = P_- + (0,\ldots,0,\varepsilon)^t$.

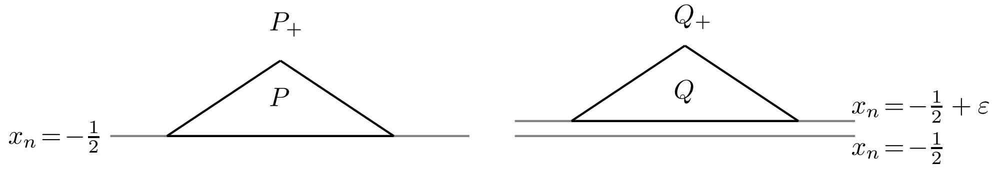

FIGURE 4. A intuitive depiction indicating relative locations of X_r, Q and $Q_\pm$.

Define a map $h_1 : (P_- \cup Q_+) \to \mathbb{R}^n$ by setting $h_1(x) = x - (0,\ldots,0,\frac{r}{2})^t$ for $x \in P_-$ and $h_1(x) = x$ for $x \in Q_+$. Then, considering each line segments $L_y = \{y\} \times [a_y, a_y+\varepsilon]$ with $y \in \mathbb{R}^{n-1}$, $(y, a_y) \in \partial P_-$, and $(y, a_y+\varepsilon) \in \partial Q_+$. Then, the interior of L_y is disjoint from $Q_+ \cup P_-$. Therefore, we may obtain a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ by setting $h(y, a_y + s\varepsilon) = h_1(y, a_y) + (0,\ldots,0,\frac{r}{2}s)^t$ for each $y \in \mathbb{R}^{n-1}$. Consequently, we have a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(B_+) = B_+$ and that $h(B_-) = B_- - (0,\ldots,0,\frac{r}{2})^t$ is below the hyper plane $\mathbb{R}^{n-1} \times \{-\frac{1}{2}\}$. Moreover, the boundary of $h(T_r)$ is $B_+ \cup B' \cup h(B_-)$, where B' is the product of $\partial[-\frac{1}{2},\frac{1}{2}]^{n-1}$ with $[-\frac{1}{2}-\frac{r}{2},\frac{1}{2}]$.

Recall that we have $B_+ = K_1 + (0,\ldots,0,\frac{1}{2})^t$ and $h(B_-) = K_1 - (0,\ldots,0,-\frac{1}{2}-\frac{r}{2})^t$. Let $K_1^\sharp$ and $K_2^\sharp$ be defined as in Example 3.3. Let P_ϵ be defined as in Theorem 3.4. Then, B_+ is a subset of $P_\epsilon + (0,\ldots,0,\frac{1}{2})^t$ while $B' \cup h(B_-)$ is disjoint from the interior of $P_\epsilon + (0,\ldots,0,\frac{1}{2})^t$.

By Theorem 3.4, there exists a homeomorphism $h' : \mathbb{R}^n \to \mathbb{R}^n$ maintaining each point off $P_\epsilon + (0,\ldots,0,\frac{1}{2})^t$ and sending B_+ onto the graph Γ_1 of a continuous function $F_1 : [-\frac{1}{2},\frac{1}{2}]^{n-1} \to [\frac{1}{2},\frac{1}{2}+\frac{r}{2}]$. Here, $F_1 = F_{g^*} + \frac{1}{2}$ for F_{g^*} defined as in Example 3.1. More precisely, we have $h'(\partial(h(T_r))) = \Gamma_1 \cup B' \cup h(B_-)$.

Using Theorem 3.4 again, we see that there exists another homeomorphism $h'' : \mathbb{R}^n \to \mathbb{R}^n$ maintaining each point off $P_\epsilon + (0,\ldots,0,-\frac{1}{2}-\frac{r}{2})^t$ and sending $h(B_-)$ onto the graph Γ_2 of $F_2 = F_{g^*} - \frac{r}{2}$. Here, we even have $h''(\Gamma_1 \cup B') = \Gamma_1 \cup B'$.

Now, we can see that the composite $h'' \circ h' \circ h : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism sending ∂T_r onto $\Gamma_1 \cup B' \cup \Gamma_2$. Recall that B' is the product of $\partial[-\frac{1}{2},\frac{1}{2}]^{n-1}$ with $[-\frac{1}{2}-\frac{r}{2},\frac{1}{2}]$. Since

$F_2 = F_1 - 1 - \frac{r}{2}$ and $F_1(x) = \frac{1}{2}$ for each x in $\partial[-\frac{1}{2}, \frac{1}{2}]^{n-1}$, the map

$$\begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \xrightarrow{h^*} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \varphi(x_1, \ldots, x_{n-1}) \end{bmatrix}$$

is a homeomorphism sending $\Gamma_1 \cup B' \cup \Gamma_2$ onto the boundary of $[-\frac{1}{2}, \frac{1}{2}]^{n-1} \times [-\frac{1}{2} - \frac{r}{2}, \frac{1}{2}]$. Here

$$\varphi(x_1, \ldots, x_{n-1}) = \begin{cases} \frac{1}{2} - F_1(x_1, \ldots, x_{n-1}) & (x_1, \ldots, x_{n-1})^t \in [-\frac{1}{2}, \frac{1}{2}]^{n-1} \\ 0 & \textit{otherwise} \end{cases}$$

Finally, we have a homeomorphism $H = h^* \circ h'' \circ h' \circ h : \mathbb{R}^n \to \mathbb{R}^n$ sending ∂T_r onto the boundary of $[-\frac{1}{2}, \frac{1}{2}]^{n-1} \times [-\frac{1}{2} - \frac{r}{2}, \frac{1}{2}]$. This ends our proof for Theorem 1.4. □

5. Appendix: More Examples

This section gives a few more examples. The first example indicates that consistency of the map $g : A_1 \to A_2$ in Theorem 1.2 is necessary.

Example 5.1. Let $\mathcal{F}_i = \{f_{i,1}, f_{i,2}\}$ be an IFS on $\mathbb{R}$ for $i = 1, 2$ with $f_{i,1}(x) = \frac{x}{3}, f_{i,2}(x) = \frac{x+2}{3}$. Let $A_1 = [\frac{1}{3}, \frac{2}{3}] = A_2$ and $g(x) = 1 - x$. Then $g : A_1 \to A_2$ does not coincide with the factor map on $A_1 \cap E_1 = \{\frac{1}{3}, \frac{2}{3}\}$ and hence is not consistent with the pair (E_1, E_2). See Figure 5. Therefore,

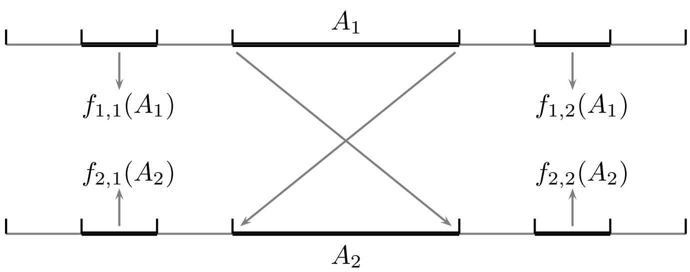

Figure 5. The segments $A_i, f_{i,j}(A_i)$ for $i, j \in \{1, 2\}$ and how g works on $x = \frac{1}{3}, \frac{2}{3}$.

we may glue together all the maps $f_{2,\alpha} \circ g \circ f_{1,\alpha}^{-1} : f_{1,\alpha}(A_1) \to f_{2,\alpha}(A_2)$ for $\alpha \in \Sigma^*$ and obtain a map $F_g^* : \bigcup_{\alpha \in \Sigma^*} f_{1,\alpha}(A_1) \to \bigcup_{\alpha \in \Sigma^*} f_{2,\alpha}(A_2)$; but this map is not continuous at $x = \frac{1}{3}, \frac{2}{3}$. Therefore, g does not have an extension $F_g : K_1 \to K_2$ such that $f_{2,j} \circ F_g = F_g \circ f_{1,j}$ for $j = 1, 2$.

The second example shows that in Theorem 1.3 the assumption that the inverse of the homeomorphism $g : A_1 \to A_2$ is consistent with the pair (E_2, E_1) can not be removed.

Example 5.2. Let $\mathcal{F}_i = \{f_{i,1}, f_{i,2}\}$ be an IFS on $\mathbb{R}$ for $i = 1, 2$ with $f_{1,1}(x) = \frac{x}{5}, f_{1,2}(x) = \frac{5-x}{5}, f_{2,1}(x) = \frac{x}{4}, f_{2,2}(x) = \frac{4-x}{4}$. Let $A_1 = [1, 2] = A_2$ and $g(x) = x$. Then $g : A_1 \to A_2$ is a homeomorphism consistent with (E_1, E_2) but g^{-1} is not consistent with (E_2, E_1). See Figure 6.

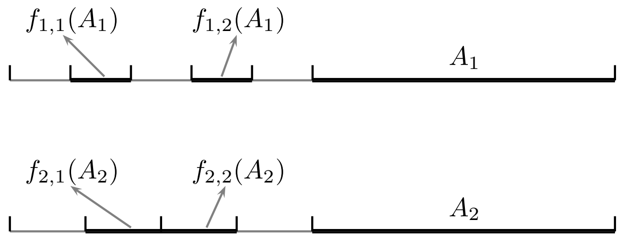

FIGURE 6. A depiction of A_i and $f_{i,j}(A_i)$ for $i, j \in \{1, 2\}$.

Therefore, g has an extension $F_g : K_1 \to K_2$ such that $f_{2,j} \circ F_g = F_g \circ f_{1,j}$ for $j = 1, 2$, but F_g is not a homeomorphism. In fact, K_1 has infinitely many components while $K_2 = [0, 2]$.

The third example provides a general procedure to construct simple arcs that cross a Cantor set, the Hausdorff dimension of which may be as large as we wish. The structure in Example 5.3 may be slightly generalized to construct a sequence of simple arcs that converge to a curve filling the unit square or a general convex polygon. One may compare Figure 7 of this paper with Figure 6 of [10], the latter of which indicates the construct of Lebesgue's space-filling curve.

Example 5.3. Given $v_1, v_2, v_3 \in \mathbb{R}^2$ that span a triangle Δ. Let $\mathcal{F}_1 = \{f_{1,1}, f_{1,2}, f_{1,3}\}$ be an IFS on $\mathbb{R}^2$ by orientation-preserving affine contractions $f_{1,j}$ such that $f_{1,j}(v_j) = v_j, f_{1,j}(\Delta) \subset \Delta$ for $1 \le j \le 3$. See upper part of the following figure. For $s \in (0, \frac{1}{3})$, let $\mathcal{F}_2 = \{f_{2,1}, f_{2,2}, f_{2,3}\}$ be an

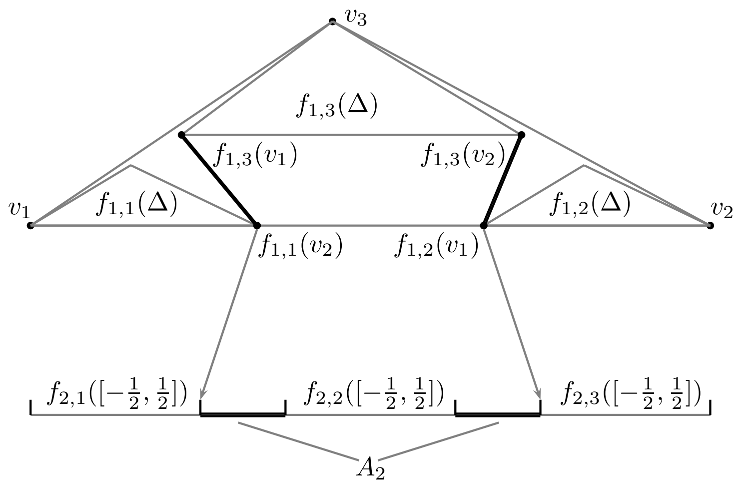

FIGURE 7. Depiction of $f_{1,j}(\Delta)$, $f_{2,j}([-\frac{1}{2}, \frac{1}{2}]$, and how g works on $x = f_{1,1}(v_2), f_{1,2}(v_1)$.

IFS on $\mathbb{R}$ with $f_{2,1}(x) = sx, f_{2,2}(x) = sx + \frac{1-s}{2}, f_{2,3}(x) = sx + 1 - s$. Denote by E_i the Hutchinson attractor of $\mathcal{F}_i$. Then the convex hull of E_1 is Δ and that of E_2 is the interval $[-\frac{1}{2}, \frac{1}{2}]$. Let A_1 be the union of two line segments, one connecting $f_{1,1}(v_2)$ to $f_{1,3}(v_1)$ and the other $f_{1,2}(v_1)$ to $f_{1,3}(v_2)$. Let $A_2 = [s, \frac{1-s}{2}] \cup [\frac{1+s}{2}, 1 - s]$. In Figure 7, we mark A_1 and A_2 as thick line segments. Let $g(f_{1,1}(v_2)) = f_{2,1}(1)$, $g(f_{1,3}(v_1)) = f_{2,2}(0)$, $g(f_{1,3}(v_2)) = f_{2,2}(1)$, and $g(f_{1,2}(v_1)) = f_{2,3}(0)$. Linearly extending g to A_1, we have a homeomorphism $g : A_1 \to A_2$ consistent with (E_1, E_2) such that its inverse $g^{-1} : A_2 \to A_1$ is consistent with (E_2, E_1). Therefore, by Theorem 1.3, g can be extended to a homeomorphism $F_g : K_1 \to K_2$ such that $f_{2,j} \circ F_g = F_g \circ f_{1,j}$ for $1 \le j \le 3$.

References

[1] M. A. Armstrong, *Basic Topology*, 1983 Springer Science+Business Media, Inc.

[2] C. Bandt and Y. Wang, Disk-like self-affine tiles in $\mathbb{R}^2$, Discrete Comput. Geom., 26:591-601, 2001.

[3] M.F. Barnsley, Fractals Everywhere, 2nd edn. Academic Press Professional, Boston,MA(1993). Revised with the assistance of and with a foreword by Hawley Rising, III.

[4] M. Brown, A proof of the generalized Schoenflies Theorem, Bull. Amer. math. Soc., 66(1960): 74-76.

[5] G. R. Conner and J. M. Thuswaldner, Self-affine manifolds, http://arxiv.org/pdf/1402.3000.pdf

[6] J. E. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J., 30:713-747, 1981.

[7] J. Lagarias and Y. Wang. Self-affine tiles in $\mathbb{R}^n$, Adv. Math., 121:21-48, 1996.

[8] J. Luo, H. Rao and B. Tan, Topological structure of self-similar sets, Fractals 10 (2002), no. 2, 223C227.

[9] J. R. Munkres, *Topology*, 2nd edition, Upper Saddle River, NJ: Prentice-Hall, 1999.

[10] H. Sagan, A geometrization of Lebesgue's space-filling curve, Math. Intelligencer 15 (1993), no. 4, 37C43.

[11] G. T. Whyburn and E. Duda, *Dynamic Topology*, Undergraduate Texts in Mathematics, New York: Springer-Verlag, 1979.

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