Hankel determinants for the Fibonacci word and Padé approximation


Teturo KAMAЕ*Jun-ichi TAMURA†Zhi-Ying WEN‡§

1 Introduction

The aim of the paper is to give a concrete and interesting example of the Padé approximation theory as well as to develop the general theory so as to find a quantitative relation between the Hankel determinant and the Padé pair. Our example is the formal power series related to the Fibonacci word.

The Fibonacci word $\varepsilon(a,b)$ on an alphabet $\{a, b\}$ is the infinite sequence

$$\varepsilon(a, b) \equiv \hat{\varepsilon}_0\hat{\varepsilon}_1\cdots \hat{\varepsilon}_n\cdots := abaababaababab\cdots \quad (\hat{\varepsilon}_n \in \{a, b\})$$

which is the fixed point of the substitution

$$\sigma: \begin{align*}
a & \to ab \\
b & \to a
\end{align*}$$

*Osaka City University, Department of Mathematics, 558-8585 Japan / kamae@sci.osaka-cu.ac.jp
†International Junior College, Faculty of General Education, Ekoda 4-5-1, Nakano-ku, TOKYO, 165 Japan / tamura@rkmath.rikkyo.ac.jp
‡Tsinghua University, Department of Applied Mathematics, Beijing 430072 P. R. China / wenyiz@mail.tsinghua.edu.cn
§This research was partially supported by the Hatori Project of the Mathematical Society of Japan.
The **Hankel determinants** for an infinite word (or sequence) \( \varphi = \varphi_0 \varphi_1 \varphi_2 \cdots \) \((\varphi_n \in K)\) over a field \( K \) are the following

\[
H_{n,m}(\varphi) := \det(\varphi_{n+i+j})_{0 \leq i,j \leq m-1}
\]
\[(n = 0, 1, 2, \cdots; m = 1, 2, \cdots). \tag{3}\]

It is known [2] that the Hankel determinants play an important role in the theory of Padé approximation for the formal Laurent series

\[
\varphi(z) = \sum_{k=0}^{\infty} \varphi_k z^{-k-h}. \tag{4}\]

Let \( K((z^{-1})) \) be the set of formal Laurent series \( \varphi \) as above of \( z \) with coefficients in \( K \) and \( h \in \mathbb{Z} \) providing a non-Archimedean norm \( \| \varphi \| := \exp(-k_0 + h) \) with \( k_0 = \inf \{ k; \varphi_k \neq 0 \} \). Let \( \varphi \) be as above with \( h = -1 \). We say that a pair \((P, Q) \in K[z]^2\) of polynomials of \( z \) over \( K \) is a **Padé pair** of order \( m \) for \( \varphi \) if

\[
\| Q \varphi - P \| < \exp(-m), \quad Q \neq 0, \quad \deg Q \leq m. \tag{5}\]

A Padé pair \((P, Q)\) of order \( m \) for \( \varphi \) always exists and the rational function \( P/Q \in K(z) \) is uniquely determined for each \( m = 0, 1, 2, \cdots \). The element \( P/Q \in K(z) \) with \( P, Q \) satisfying (5) is called the \( m \)-th **diagonal Padé approximation** for \( \varphi \). A number \( m \) is called a normal index if (5) implies \( \deg Q = m \). Note that \( P/Q \) is irreducible if \( m \) is a normal index, although it can be reducible for a general \( m \). A normal Padé pair \((P, Q)\), i.e., \( \deg Q \) is a normal index, is said to be normalized if the leading coefficient of \( Q \) is equal to 1. It is a classical result that \( m \) is a normal index for \( \varphi \) if and only if the Hankel determinant \( \det(\varphi_{i+j})_{0 \leq i,j \leq m-1} \) is nonzero. Note that 0 is always a normal index and the determinant for the empty matrix is considered as 1, so that the above statement remains valid for \( m = 0 \).

We succeed in obtaining a quantitative relation between the Hankel determinant and the normalized Padé pair. Namely,

\[
\det(\varphi_{i+j})_{0 \leq i,j \leq m-1} = (-1)^{[m/2]} \prod_{z \in Q(z) = 0} P(z) \tag{6}\]

for any normal index \( m \) with the normalized Padé pair \((P, Q)\), where \( \prod_{z \in Q(z) = 0} \) indicates a product taken over all zeroes \( z \) of \( Q \) with their multiplicity (Theorem 6).
We are specially interested in the Padé approximation theory applied to the Fibonacci words $\varepsilon := \varepsilon(1, 0)$ and $\overline{\varepsilon} := \varepsilon(0, 1)$, where 0, 1 are considered as elements in the field $\mathbb{Q}$, since we have the following remark.

**Remark 1** Let $M$ be a matrix of size $m \times m$ with entries consisting of two independent variables $a$ and $b$. Then, $\det M = (a - b)^{m-1}(pa + (-1)^{m-1}qb)$, where $p$ and $q$ are integers defined by

$$p = \det M \mid_{a=1, \, \overline{a}=0}, \quad q = \det M \mid_{a=0, \, \overline{a}=1}.$$

*Proof of Remark 1.* Subtracting the first column vector from all the other column vectors of $M$, we see that $\det M$ is divisible by $(a - b)^{m-1}$ as a polynomial in $\mathbb{Z}[a, b]$. Hence, $\det M = (a - b)^{m-1}(xa + yb)$ for integers $x, y$. Setting $(a, b) = (1, 0), \ (0, 1)$, we get the assertion.

In Section 2, we study the structures of the Fibonacci word, in particular, its repetition property. The notion of singular words introduced in Z.-X. Wen and Z.-Y. Wen [5] plays an important role.

In Section 3, we give the value of the Hankel determinants $H_{n,m}(\varepsilon)$ and $H_{n,m}(\overline{\varepsilon})$ for the Fibonacci words in some closed forms. It is a rare case where the Hankel determinants are determined completely. Another such case is for the Thue-Morse sequence $\varphi$ consisting of 0 and 1, where the Hankel determinants $H_{m,n}(\varphi)$ modulo 2 are obtained, and the function $H_{m,n}(\varphi)$ of $(m, n)$ is proved to be 2-dimensionally automatic (J.-P. Allouche, J. Peyrière, Z.-X. Wen and Z.-Y. Wen [1]).

In Section 4, we consider the self-similar property of the values $H_{n,m}(\varepsilon)$ and $H_{n,m}(\overline{\varepsilon})$ for the Fibonacci words. The quarter plane $\{(n, m); n \geq 0, m \geq 1\}$ is tiled by 3 kinds of tiles with the values $H_{n,m}(\varepsilon)$ and $H_{n,m}(\overline{\varepsilon})$ on it with various scales.

In Section 5, we develop a general theory of Padé approximation. We also obtain the admissible continued fraction expansion of $\varphi_{\varepsilon}$ and $\varphi_{\overline{\varepsilon}}$, the formal Laurent series (4) with $h = -1$ for the sequences $\varepsilon$ and $\overline{\varepsilon}$, and determine all the convergents $p_k/q_k$ of the continued fractions. It is known in general that the set of the convergents $p_k/q_k$ for $\varphi$ is the set of diagonal Padé approximations and the set of degrees of $q_k$’s in $\varepsilon$ coincides with the set of normal indices for $\varphi$. 

3
2 Structures of the Fibonacci word

In what follows, $\sigma$ denotes the substitution defined by (2), and

$$\hat{\varepsilon} = \hat{\varepsilon}_0\hat{\varepsilon}_1\hat{\varepsilon}_2\cdots \hat{\varepsilon}_n \cdots \quad (\hat{\varepsilon}_n \in \{a, b\})$$

is the (infinite) Fibonacci word (1). A finite word over $\{a, b\}$ is sometimes considered to be an element of the free group generated by $a$ and $b$ with their inverses $a^{-1}$ and $b^{-1}$. For $n = 0, 1, 2, \cdots$, we define the $n$-th Fibonacci word $F_n$ and the $n$-th singular word $W_n$ as follows:

$$F_n := \sigma^n(a) = \sigma^{n+1}(b)$$
$$W_n := \beta_n F_n a_n^{-1},$$

(7)

where we put

$$\alpha_n = \beta_m = \begin{cases} a & (n : \text{even}, \ m : \text{odd}) \\ b & (n : \text{odd}, \ m : \text{even}), \end{cases}$$

(8)

and we define $W_{-2}$ to be the empty word and $W_{-1} := a$ for convenience.

Let $(f_n; n \in \mathbb{Z})$ be the Fibonacci sequence:

$$f_{n+2} = f_{n+1} + f_n \quad (n \in \mathbb{Z})$$
$$f_{-1} = f_0 = 1.$$  \tag{9}

Then, we have $|F_n| = |W_n| = f_n \ (n \geq 0)$, where $|\xi|$ denotes the length of a finite word $\xi$.

For a finite word $\xi = \xi_0\xi_1\cdots\xi_{n-1}$ and a finite or infinite word $\eta = \eta_0\eta_1\eta_2\cdots$ over an alphabet, we denote

$$\xi \prec_k \eta$$

(10)

if $\xi = \eta_k\eta_{k+1}\cdots\eta_{k+n-1}$. We simply denote

$$\xi \prec \eta$$

(11)

and say that $\xi$ is a subword of $\eta$ if $\xi \prec_k \eta$ holds for some $k$. For a finite word $\xi = \xi_0\xi_1\cdots\xi_{n-1}$ and $i$ with $0 \leq i < n$, we denote the $i$-th cyclic permutation of $\xi$ by $C_i(\xi) := \xi_i\xi_{i+1}\cdots\xi_{n-1}\xi_0\xi_1\cdots\xi_{i-1}$. We also denote $C_i(\xi) := C_{i'}(\xi)$ with $i' := i - n[i/n]$ for any $i \in \mathbb{Z}$.

In this section, we study the structure of the Fibonacci word $\hat{\varepsilon}$ and discuss the repetition property. The following two lemmas are obtained by Z.-X. Wen and Z.-Y. Wen [5] and we omit the proofs.
Lemma 1 We have the following statements (1)-(10):
(1) $\hat{e} = F_n F_{n-1} F_n F_{n+1} F_{n+2} \cdots \ (n \geq 1)$,
(2) $F_n = F_{n-1} F_{n-2} = F_{n-2} F_{n-1}^{-1} \alpha_n^{-1} \beta \alpha_n \ (n \geq 2)$,
(3) $F_n F_n < \hat{e} \ (n \geq 3)$,
(4) $\hat{e} = W_{-1} W_0 W_1 W_2 W_3 \cdots$,
(5) $W_n = W_{n-2} W_{n-3} W_{n-2} \ (n \geq 1)$,
(6) $W_n$ is a palindrome, that is, $W_n$ stays invariant under reading the letters from the end \((n \geq -2)\),
(7) $C_i(F_n) < \hat{e} \ (n \geq 0, \ 0 \leq i < f_n)$,
(8) $C_i(F_n) \neq C_j(F_n)$ for any $i \neq j$, moreover, they are different already before their last places \((n \geq 1, \ 0 \leq i < f_n)\),
(9) $W_n \neq C_i(F_n) \ (n \geq 0, \ 0 \leq i < f_n)$,
(10) $\xi < \hat{e}$ and $|\xi| = f_n$ imply that either $\xi = C_i(F_n)$ for some $i$ with $0 \leq i < f_n$ or $\xi = W_n \ (n \geq 0)$.

Lemma 2 For any $k \geq -1$, we have the decomposition of $\hat{e}$ as follows:
$$\hat{e} = (W_{-1} W_0 \cdots W_{k-1}) W_k \gamma_0 W_k \gamma_1 \cdots W_k \gamma_n \cdots,$$
where all the occurrences of $W_k$ in $\hat{e}$ are picked up and $\gamma_n$ is either $W_{k+1}$ or $W_{k-1}$ corresponding to $\hat{e}_n$ is a or $b$, respectively. That is, any two different occurrences of $W_k$ do not overlap and are separated by $W_{k+1}$ or $W_{k-1}$.

We introduce another method to discuss the repetition property of $\hat{e}$. Let $\mathbb{N}$ be the set of nonnegative integers. For $n \in \mathbb{N}$, let
$$n = \sum_{i=0}^{\infty} \tau_i(n) f_i,$$
$$\tau_i(n) \in \{0, 1\} \quad \text{and} \quad \tau_i(n) \tau_{i+1}(n) = 0 \ (i \in \mathbb{N}) \quad (12)$$
be the regular expression of $n$ in the Fibonacci base due to Zeckendorf. For $m, n \in \mathbb{N}$ and a positive integer $k$, we denote
$$m \equiv_k n \quad (13)$$
if $\tau_i(m) = \tau_i(n)$ holds for all $i < k$.

Lemma 3 It holds that $\hat{e}_n = a$ if and only if $\tau_0(n) = 0$. 

5
Proof. We use induction on $n$. The lemma holds for $n = 0, 1, 2$. Assume that the lemma holds for any $n \in \mathbb{N}$ with $n < f_k$ for some $k \geq 2$. Take any $n \in \mathbb{N}$ with $f_k \leq n < f_{k+1}$. Then, since $0 \leq n - f_k < f_{k-1}$, we have $n = \sum_{i=0}^{k-1} \tau_i(n - f_k) f_i + f_k$, which gives the regular expression if $\tau_{k-1}(n - f_k) = 0$. If $\tau_{k-1}(n - f_k) = 1$, then we have the regular expression $n = \sum_{i=0}^{k-2} \tau_i(n - f_k) f_i + f_{k+1}$. In any case, we have $\tau_0(n) = \tau_0(n - f_k)$. On the other hand, since $\hat{e}$ starts with $F_k F_{k-1}$ by Lemma 1, we have $\hat{e}_n = \hat{e}_{n-f_k}$. Hence, $\hat{e}_n = a$ if and only if $\tau_0(n) = 0$ by the induction hypothesis. Thus, we have the lemma for any $n < f_{k+1}$, and by induction, we complete the proof. \hfill \blacksquare

Lemma 4 Let $n = \sum_{i=0}^{\infty} n_i f_i$ with $n_i \in \{0, 1\}$ ($i \in \mathbb{N}$). Assume that $n_i n_{i+1} = 0$ for $0 \leq i < k$. Then, $n_i = \tau_i(n)$ holds for $0 \leq i < k$.

Proof. If there exists $i \in \mathbb{N}$ such that $n_i n_{i+1} = 1$, take the maximum $i_0$ for such $i$'s. Take the maximum $j$ such that $n_{i_0+1} = n_{i_0+3} = n_{i_0+5} = \cdots = n_j = 1$. Then, replacing $f_{i_0} + f_{i_0+1} + f_{i_0+3} + f_{i_0+5} + \cdots + f_j$ by $f_{j+1}$, we have a new expression of $n$:

$$n = \sum_{i=0}^{\infty} n_i f_i$$

$$:= \sum_{i=0}^{i_0-1} n_i f_i + f_{j+1} + \sum_{i=j+3}^{\infty} n_i f_i.$$

This new expression is unchanged at the indices less than $k$, and is either regular or has a smaller maximum index $i$ with the property $n_i' n_{i+1}' = 1$. By continuing this procedure, we finally get the regular expression of $n$, which is unchanged at the indices less than $k$ from the original expression. Thus, we have $n_i = \tau_i(n)$ for any $0 \leq i < k$. \hfill \blacksquare

Lemma 5 For any $n \in \mathbb{N}$ and $k \geq 0$, $\tau_0(n + f_k) \neq \tau_0(n)$ holds if and only if either $n \equiv_{k+2} f_{k+1} - 2$ or $n \equiv_{k+2} f_{k+1} - 1$. Moreover,

$$\hat{e}_{n+f_k} - \hat{e}_n = \begin{cases} (-1)^{k-1}(a - b) & (n \equiv_{k+2} f_{k+1} - 2) \\ (-1)^k(a - b) & (n \equiv_{k+2} f_{k+1} - 1), \end{cases}$$

where $a$ and $b$ are considered as independent variables.

Proof. If $k = 0$, we can verify Lemma 5 by a direct calculation.
Assume that \( k \geq 1 \) and \( \tau_k(n) = 0 \), then we have an expression of \( n + f_k \):

\[
n + f_k = \sum_{i=0}^{k-1} \tau_i(n) f_i + f_k + \sum_{i=k+1}^{\infty} \tau_i(n) f_i.
\]

Then by Lemma 4, we have \( \tau_0(n + f_k) = \tau_0(n) \) if \( k \geq 2 \) or if \( k = 1 \) and \( \tau_0(n) = 0 \). In the case where \( k = 1 \), \( \tau_0(n) = 1 \) and \( \tau_2(n) = 0 \), since

\[
n + f_k = 1 + 2 + \sum_{i=3}^{\infty} \tau_i(n) f_i = f_2 + \sum_{i=3}^{\infty} \tau_i(n) f_i,
\]

we have \( \tau_0(n + f_k) = 0 \) by Lemma 4. On the other hand, in the case where \( k = 1 \), \( \tau_0(n) = 1 \) and \( \tau_2(n) = 1 \), since

\[
n + f_k = 1 + 2 + 3 + \sum_{i=4}^{\infty} \tau_i(n) f_i = f_0 + f_3 + \sum_{i=4}^{\infty} \tau_i(n) f_i,
\]

we have \( \tau_0(n + f_k) = 1 \) by Lemma 4.

Thus, in the case where \( k \geq 1 \) and \( \tau_k(n) = 0 \), \( \tau_0(n + f_k) \neq \tau_0(n) \) if and only if \( k = 1 \), \( \tau_0(n) = 1 \) and \( \tau_2(n) = 0 \), or equivalently, if and only if \( n \equiv k + 2 f_{k+1} - 2 \). Note that \( n \equiv k + 1 f_{k+1} - 1 \) does not happen in this case.

Now assume that \( k \geq 1 \) and \( \tau_k(n) = 1 \). Take the minimum \( j \geq 0 \) such that \( \tau_k(n) = \tau_{k-2}(n) = \tau_{k-4}(n) = \cdots = \tau_j(n) = 1 \). Then since \( 2f_i = f_{i+1} + f_{i-2} \) for any \( i \in \mathbb{N} \), we have an expression of \( n + f_k \):

\[
n + f_k = \sum_{i=0}^{j-3} \tau_i(n) f_i + f_{j-2} + f_{j+1} + f_{j+3} + f_{j+5} + \cdots + f_{k+1} + \sum_{i=k+2}^{\infty} \tau_i(n) f_i,
\]

where the first term in the right-hand side vanishes if \( j = 0, 1, 2 \). Hence by Lemma 4, \( \tau_0(n + f_k) = \tau_0(n) \) if \( j \geq 4 \).

In the case where \( j = 3 \), \( \tau_0(n + f_k) = \tau_0(n) \) holds if \( \tau_0(n) = 0 \) by (14) and Lemma 4. If \( \tau_0(n) = 1 \), then by (14) and Lemma 4, \( \tau_0(n + f_k) = 0 \). Thus, in the case where \( j = 3 \), \( \tau_0(n + f_k) \neq \tau_0(n) \) if and only if \( \tau_0(n) = 1 \).

If \( j = 2 \), then by the assumption on \( j \), we have \( \tau_0(n) = 0 \). On the other hand, since \( f_0 = 1 \), by (14) and Lemma 4, we have \( \tau_0(n + f_k) = 1 \). Thus, \( \tau_0(n + f_k) \neq \tau_0(n) \).
If \( j = 1 \), then we have \( \tau_0(n) = 0 \) since \( \tau_1(n) = 1 \) by the assumption on \( j \). On the other hand, since \( f_{-1} = 1 \), we have \( \tau_0(n + f_k) = 1 \) by (14) and Lemma 4. Thus, \( \tau_0(n + f_k) \neq \tau_0(n) \).

If \( j = 0 \), then by the assumption on \( j \), \( \tau_0(n) = 1 \). On the other hand, since \( f_{-2} = 0 \), we have \( \tau_0(n + f_k) = 0 \) by (14) and Lemma 4. Thus, \( \tau_0(n + f_k) \neq \tau_0(n) \).

By combining all the results as above, we get the first part.

The second part follows from Lemma 3 and the fact that for any \( k \geq 0 \),

\[
f_{k+1} - 1 = f_k + f_{k-2} + \cdots + f_i
\]

with \( i = 0 \) if \( k \) is even and \( i = 1 \) if \( k \) is odd. Hence,

\[
\tau_0(f_{k+1} - 1) = \tau_0(f_{h+1} - 2) = \begin{cases} a & (k: \text{odd}, h: \text{even}) \\ b & (k: \text{even}, h: \text{odd}) \end{cases}
\]

Lemma 6

For any \( k \geq 0 \), \( W_k \prec_n \mathcal{E} \) if and only if \( n \equiv_{k+2} f_{k+1} - 1 \).

Proof. By Lemma 2, the smallest \( n \in \mathbb{N} \) such that \( W_k \prec_n \mathcal{E} \) is

\[
f_{-1} + f_0 + f_1 + \cdots + f_{k-1} = f_{k+1} - 1,
\]

which is the smallest \( n \in \mathbb{N} \) such that \( n \equiv_{k+2} f_{k+1} - 1 \). Let \( n_0 := f_{k+1} - 1 \).

Then, the regular expression of \( n_0 \) is

\[
n_0 = f_k + f_{k-2} + f_{k-4} + \cdots + f_d,
\]

where \( d = (1 - (-1)^k)/2 \). The next \( n \) with \( n \equiv_{k+2} n_0 \) is clearly

\[
n = f_{k+2} + f_k + f_{k-2} + \cdots + f_d,
\]

which is, by Lemma 2, the next \( n \) such that \( W_k \prec_n \mathcal{E} \) since \( f_k + f_{k+1} \) = \( f_{k+2} \).

For \( i = 1, 2, 3, \ldots \), let

\[
n_i = n_0 + \sum_{j=0}^{\infty} \tau_j(i) f_{k+2+j}.
\]
Then, it is easy to see that $n_i$ is the $i$-th $n$ after $n_0$ such that $n \equiv_{k+2} f_{k+1} - 1$. We prove by induction on $i$ that $n_i$ is the $i$-th $n$ after $n_0$ such that $W_k \prec_n \hat{e}$. Assume that it is so for $i$. Then by Lemma 4, $W_k \gamma_i; W_k \prec_{n_i} \hat{e}$. Hence, the next $n$ after $n_i$ such that $W_k \prec_n \hat{e}$ is $n_i + f_k + |\gamma_i|$. Thus, we have

$$n_i + f_k + |\gamma_i| = n_i + f_k + f_{k+1}1_{\gamma_i=a} + f_{k-1}1_{\gamma_i=b} = n_i + f_{k+1}1_{\tau_0(i)=0} + f_{k+1}1_{\tau_0(i)=1} = n_{i+1},$$

which completes the proof.

**Lemma 7** Let $k \geq 0$ and $n, i \in \mathbb{N}$ satisfy that $n \equiv_{k+1} i$.

1. If $0 \leq i < f_k$, then, $\tau_0(n+j) = \tau_0(i+j)$ holds for any $j = 0, 1, \cdots, f_{k+2} - i - 3$.
2. If $f_k \leq i < f_{k+1}$, then, $\tau_0(n+j) = \tau_0(i+j)$ holds for any $j = 0, 1, \cdots, f_{k+3} - i - 3$.

**Proof.** (1) We prove the lemma by induction on $k$. The assertion holds for $k = 0$. Let $k \geq 1$ and assume that the assertion is valid for $k - 1$. For $j = 0, 1, \cdots, f_k - i$, $n + j \equiv_k i + j$ holds and hence, $\tau_0(n+j) = \tau_0(i+j)$ holds. Let $j_0 = f_k - i$. Then, since $n + j_0 \equiv_k i + j_0 \equiv_k 0$, we have $\tau_0(n+j_0 + j) = \tau_0(i+j_0 + j) = \tau_0(j)$ for any $j = 0, 1, \cdots, f_{k+1} - 3$ by the induction hypothesis. Thus, $\tau_0(n+j) = \tau_0(i+j)$ holds for any $j = 0, 1, \cdots, f_{k+2} - i - 3$. This proves (1).

(2) In this case, $\tau_{k+1}(n) = 0$ holds. Hence, we have $n \equiv_{k+2} i$. Therefore, we can apply (1) with $k + 1$ for $k$. Thus, we get (2).

Let $n, m, i \in \mathbb{N}$ with $m \geq 2$ and $0 < i < m$. We call $n$ an $(m, i)$-shift invariant place in $\hat{e}$ if

$$\hat{e}_n \hat{e}_{n+1} \cdots \hat{e}_{n+m-1} = \hat{e}_{n+i} \hat{e}_{n+i+1} \cdots \hat{e}_{n+i+m-1}.$$

We call $n$ an $m$-repetitive place in $\hat{e}$ if there exist $i, j \in \mathbb{N}$ with $i > 0$ and $i + j < m$ such that $n + j$ is an $(m, i)$-shift invariant place in $\hat{e}$. Let $\mathcal{R}_m$ be the set of $m$-repetitive places in $\hat{e}$.
Lemma 8  (1) Let \( n \equiv_{k+1} 0 \) for some \( k \geq 1 \). Then, \( n \) is an \((f_{k+1} - 2, f_k)\)-shift invariant place in \( \hat{\mathcal{E}} \).
(2) Let \( n \equiv_{k+1} f_k \) for some \( k \geq 2 \). Then, \( n \) is an \((f_{k+1} - 2, f_{k-1})\)-shift invariant place in \( \hat{\mathcal{E}} \).

**Proof.** (1) Since the least \( i \geq n \) such that either \( i \equiv_{k+2} f_{k+1} - 1 \) or \( i \equiv_{k+2} f_{k+1} - 2 \) is not less than \( n + f_{k+1} - 2 \), by Lemma 5, we have

\[
\hat{\mathcal{E}}_n \hat{\mathcal{E}}_{n+1} \cdots \hat{\mathcal{E}}_{n+f_{k+1}-3} = \hat{\mathcal{E}}_{n+f_k} \hat{\mathcal{E}}_{n+f_k+1} \cdots \hat{\mathcal{E}}_{n+f_k+f_{k+1}-3}.
\]

(2) Since the minimum \( i \geq n \) such that either \( i \equiv_{k+1} f_k - 1 \) or \( i \equiv_{k+1} f_k - 2 \) is \( n + f_{k+1} - 2 \), by Lemma 5, we have

\[
\hat{\mathcal{E}}_n \hat{\mathcal{E}}_{n+1} \cdots \hat{\mathcal{E}}_{n+f_{k+1}-3} = \hat{\mathcal{E}}_{n+f_{k-1}} \hat{\mathcal{E}}_{n+f_{k-1}+1} \cdots \hat{\mathcal{E}}_{n+f_{k-1}+f_{k+1}-3}.
\]

\( \square \)

**Theorem 1** The pair \((n, m)\) of nonnegative integers satisfies \( n \in \mathcal{R}_m \) if one of the following two conditions holds:
(1) \( f_{k+1} - 1 \leq m \leq f_{k+1} - 2 \), \( n - i \equiv_{k+1} 0 \) and \( i \leq n \) for some \( k \geq 1 \) and \( i \in \mathbb{Z} \) with \( f_k + 1 \leq m + i \leq f_{k+1} - 2 \).
(2) \( f_{k-1} + 1 \leq m \leq f_{k+1} - 2 \), \( i \leq n \) and \( n - i \equiv_{k+1} f_k \) for some \( k \geq 2 \) and \( i \in \mathbb{Z} \) with \( f_{k-1} + 1 \leq m + i \leq f_{k+1} - 2 \).

**Remark 2** The “if and only if” statement actually holds in Theorem 1 in place of “if” since we will prove later that \( H_{n,m} \neq 0 \) if none of the conditions (1) and (2) hold.

**Proof of Theorem 1.** Assume (1) and \( i \geq 0 \). By (1) of Lemma 8, \( n - i \) is an \((f_{k+1} - 2, f_k)\)-shift invariant place. Then, \( n \) is an \((m, f_k)\)-shift invariant place since \( i + m \leq f_{k+1} - 2 \). Thus, \( n \in \mathcal{R}_m \) as \( f_k < m \).

Assume (1) and \( i < 0 \). Then, since \( n - i \) is an \((f_{k+1} - 2, f_k)\)-shift invariant place and \( m \leq f_{k+2} - 2 \), it is an \((m, f_k)\)-shift invariant place. Moreover, since \( f_k - i < m \), \( n \) is a \( m \)-repetitive place.

Assume (2) and \( i \geq 0 \). Then, \( n - i \) is an \((f_{k+1} - 2, f_{k-1})\)-shift invariant place by (2) of Lemma 8. Then, \( n \) is an \((m, f_{k-1})\)-shift invariance place since \( i + m \leq f_{k+1} - 2 \). Thus, \( n \) is an \( m \)-repetitive place as \( f_{k-1} < m \).

Assume (2) and \( i < 0 \). Then, since \( n - i \) is an \((f_{k+1} - 2, f_{k-1})\)-shift invariant place and \( m \leq f_{k+1} - 2 \), it is an \((m, f_{k-1})\)-shift invariant place. Then, \( n \) is an \( m \)-repetitive place, since \( f_{k-1} - i < m \). Thus, \( n \in \mathcal{R}_m \). \( \square \)
Corollary 1 The place 0 is $m$-repetitive for an $m \geq 2$ if $m \not\in \cup_{k=1}^{\infty} \{f_k - 1, f_k\}$.

Remark 3 The “if and only if” statement actually holds in Corollary 1 in place of “if” since we will prove later that $H_{0,m} \neq 0$ if $m \in \cup_{k=1}^{\infty} \{f_k - 1, f_k\}$.

Proof of Corollary 1. Let $i = 0$ in (1) of Theorem 1. Then, 0 is $m$-repetitive if $f_k + 1 \leq m \leq f_{k+1} - 2$ for some $k \geq 1$.

Corollary 2 Let $k \geq 2$. The place $n$ is $f_k$-repetitive if

$$W_k \prec \hat{\varepsilon}_{n+1} \hat{\varepsilon}_{n+2} \cdots \hat{\varepsilon}_{n+2f_k-3}.$$  

Proof. By (2) of Theorem 1, for any $k \geq 2$, $n$ is an $f_k$-repetitive place if $n - i \equiv_{k+1} f_k$ for some $i$ with $i \leq n$ and $-f_{k-2} + 1 \leq i \leq f_{k-1} - 2$. Since the condition $n - i \equiv_{k+1} f_k$ is equivalent to $n - i \equiv_{k+2} f_k$ and there is no carry in addition of $-i$ to both sides of $n \equiv_{k+2} f_k + i$, the condition $n - i \equiv_{k+1} f_k$ is equivalent to $n \equiv_{k+2} f_k + i$. Hence, the place $n$ is $f_k$-repetitive if $n \equiv_{k+2} j$ for some $j$ with $f_{k-1} + 1 \leq j \leq f_{k+1} - 2$. By Lemma 6, this condition is equivalent to that $W_k$ starts at one of the places in $\{n+1, n+2, \cdots, f_k - 2\}$, which completes the proof.

3 Hankel determinants

The aim of this section is to find the value of the Hankel determinants

$$H_{n,m} := H_{n,m}(\varepsilon) = \det(\varepsilon_{n+i+j})_{0 \leq i, j \leq m-1}$$

$$\overline{H}_{n,m} := H_{n,m}(\overline{\varepsilon}) = \det(\overline{\varepsilon}_{n+i+j})_{0 \leq i, j \leq m-1}$$

\hspace{.5cm} \hspace{.5cm} (n = 0, 1, 2, \cdots; \hspace{.1cm} m = 1, 2, 3, \cdots)

for the Fibonacci word $\varepsilon(a, b)$ at $(a, b) = (1, 0)$ and $(a, b) = (0, 1)$:

$$\varepsilon := \varepsilon(1, 0) = 10110101101101 \cdots,$$

$$\overline{\varepsilon} := \varepsilon(0, 1) = 01001010010010 \cdots.$$  

It is clear that $H_{n,m}(\varepsilon(a, b)) = 0$ if $n$ is the $m$-repetitive place in $\varepsilon(a, b)$, where $a, b$ are considered to be two independent variables, so that, in general, $H_{n,m}(\varepsilon(a, b))$ becomes a polynomial in $a$ and $b$ as is stated in Remark 1.

11
In the following lemmas, theorems and corollary, we give statements for \( \varepsilon \) and \( \overline{\varepsilon} \) parallely, while we give the proofs only for \( \varepsilon \) since the proofs for \( \overline{\varepsilon} \) are similar to those for \( \varepsilon \). The only difference between them is the starting point, Lemma 5, where \( a - b \) in the right-hand side is 1 for \( \varepsilon \) and \(-1\) for \( \overline{\varepsilon} \).

We use the following notation: for any subset \( S \) of \( \{0, 1, 2, 3, 4, 5\} \), \( \chi(k : S) \) is a function on \( k \in \mathbb{Z} \) such that

\[
\chi(k : S) = \begin{cases} 
-1 & \text{(if } k \equiv s \pmod{6} \text{ for some } s \in S) \\
1 & \text{(otherwise).}
\end{cases}
\]

The following corollary follows from Theorem 1.

**Corollary 3** \( H_{n,m} = 0 \) if one of the conditions (1), (2) in Theorem 1 is satisfied. The same statement holds for \( \overline{\Pi}_{n,m} \).

**Lemma 9** For any \( k \geq 2 \), we have

\[
H_{0,f_k} = \chi(k : 2, 3) \left( H_{0,f_{k-1}} - (-1)^{f_k-1} H_{f_{k-1}, f_{k-1}} \right)
\]

\[
\overline{\Pi}_{0,f_k} = \chi(k : 1, 3, 4, 5) \left( \overline{\Pi}_{0,f_{k-1}} - (-1)^{f_k-1} \overline{\Pi}_{f_{k-1}, f_{k-1}} \right).
\]

**Proof.** The matrix \((\varepsilon_{i,j})_{0\leq i,j<f_k}\) is decomposed into three parts:

\[
(\varepsilon_{i,j})_{0\leq i,j<f_k} = \begin{pmatrix} A \\ A' \\ B \end{pmatrix},
\]

where

\[
A = (\varepsilon_{i,j})_{0\leq i<j<f_k, 0\leq i<j\leq f_k},
\]

\[
A' = (\varepsilon_{f_k-i+j, j})_{0\leq i<j<f_k, 0\leq i<j\leq f_k},
\]

\[
B = (\varepsilon_{i+j, f_k-i})_{0\leq i<j<f_k, 0\leq i<j\leq f_k}.
\]

By Lemma 5, the following two subwords of \( \varepsilon \):

\[
\varepsilon_0 \varepsilon_1 \cdots \varepsilon_{f_k-2} \quad \text{and} \quad \varepsilon_{f_k-3} \varepsilon_{f_k-2+1} \cdots \varepsilon_{f_k-1} \varepsilon_{f_k-2} \varepsilon_{f_k-2} \varepsilon_{f_k-2+1} \varepsilon_{f_k-1} \varepsilon_{f_k-2} \varepsilon_{f_k-2+1} \varepsilon_{f_k-1} \varepsilon_{f_k-2} \varepsilon_{f_k-2+1} \varepsilon_{f_k-1} \varepsilon_{f_k-2}
\]

differ only at two places, namely, \( \varepsilon_{f_k-2} \neq \varepsilon_{f_k-1} \varepsilon_{f_k-2} \) and \( \varepsilon_{f_k-1} \neq \varepsilon_{f_k-1} \varepsilon_{f_k-2} \). Thus, we get

\[
B - A = \begin{pmatrix} 0 & (-1)^k & \ldots & (-1)^{k-2} \\ (-1)^k & (-1)^{k-1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{k-2} & (-1)^{k-1} & \ldots & 0 \end{pmatrix}.
\]
Let $A_0, A_1, \ldots, A_{f_k-1}$ be the columns of the matrix $\begin{pmatrix} A \\ A' \end{pmatrix}$ in order from the left. Since

$$
(A_0 A_1 \cdots A_{f_k-2}) = (\epsilon_{i+j})_{0 \leq i < f_{k-1}, 0 \leq j < f_{k-2} + 1} \\
(A_{f_{k-1}} A_{f_{k-1}+1} \cdots A_{f_k-2}) = (\epsilon_{f_{k-1}+i+j})_{0 \leq i < f_{k-1}, 0 \leq j < f_{k-2} + 1}
$$

and

$$
\epsilon_0 \epsilon_1 \cdots \epsilon_{f_k - 2} + \epsilon_1 \epsilon_{f_k - 3} = \epsilon_{f_{k-1}} \epsilon_{f_{k-1}+1} \cdots \epsilon_{f_{k-1}+f_{k-2}} \epsilon_{f_{k-3} + 3}
$$

by Lemma 5, we get

$$
(A_0 A_1 \cdots A_{f_k-2}) = (A_{f_{k-1}} A_{f_{k-1}+1} \cdots A_{f_k-2}). \quad (16)
$$

Thus, from (15) and (16) we obtain

$$
H_{0, f_k} = \det \begin{pmatrix}
A_0 & \cdots & A_{f_{k-1}} & \cdots & A_{f_k-2} & A_{f_k-1} \\
& & & & (-1)^k & (-1)^{k-1} \\
0 & \cdots & & & (-1)^k & (-1)^{k-1} \\
& & & & 0 & \\
(-1)^k & (-1)^{k-1} & & & 0 \\
& & & & & \\
\end{pmatrix}
$$

$$
= \det \begin{pmatrix}
A_0 & \cdots & A_{f_{k-1}} & 0 & \cdots & 0 & A_{f_k-1} \\
& & & & (-1)^k & (-1)^{k-1} & \cdots & \\
0 & \cdots & & & (-1)^k & (-1)^{k-1} \\
& & & & 0 & \cdots & & \\
(-1)^k & (-1)^{k-1} & & & 0 \\
& & & & & & & \\
\end{pmatrix}
$$

$$
= (-1)^{(k-1)f_{k-2}} (-1)^{\left[ \frac{f_{k-2}}{2} \right]} \det(A_0 A_1 \cdots A_{f_{k-1}-1}) \\
+ (-1)^k_{f_{k-2}} (-1)^{\left[ \frac{f_{k-2}}{2} \right] + f_{k-1}} \det(A_{f_k-1} A_0 A_1 \cdots A_{f_{k-1}-2}).
$$

Since

$$
\epsilon_0 \epsilon_1 \cdots \epsilon_{2f_{k-1}} - 3 = \epsilon_{f_k} \epsilon_{f_k + 1} \cdots \epsilon_{f_k + 2f_{k-1} - 3}
$$
by Lemma 5, we get
\[
\det(A_{f_k-1}A_0A_1 \cdots A_{f_k-2}) = \det(\varepsilon_{f_k-1+i+j})_{0 \leq i,j < f_k} = H_{f_k-1,f_{k-1}}.
\]
Thus we get
\[
H_{0,f_k} = (-1)^{k-1}f_k - 2((-1)^{\frac{k-2}{2}}) H_{0,f_{k-1}} + (-1)^{k-2}((-1)^{\frac{k-1}{2}}) f_k \cdot H_{f_k-1,f_{k-1}}
\]
\[
= \chi(k : 2, 3) \left( H_{0,f_{k-1}} - (-1)^{f_k-1} H_{f_k-1,f_{k-1}} \right),
\]
where we have used the fact that
\[
(-1)^{k-1}f_k - 2((-1)^{\frac{k-2}{2}}) = \chi(k : 2, 3).
\]

\begin{lemma}
For \( k \geq 2 \), we have
\[
H_{f_{k+1}-1,f_k} = \chi(k : 1, 3, 4, 5) H_{f_{k+1}-1,f_{k-1}}
\]
\[
\overline{P}_{f_{k+1}-1,f_k} = \chi(k : 2, 3) \overline{P}_{f_{k+1}-1,f_{k-1}}
\]

\textbf{Proof.} Just like the proof of Lemma 9, we decompose the matrix \((\varepsilon_{f_{k+1}-1+i+j})_{0 \leq i,j < f_k}\) into three parts:
\[
(\varepsilon_{f_{k+1}-1+i+j})_{0 \leq i,j < f_k} = \begin{pmatrix} A & A' \\ A' & B \end{pmatrix},
\]
where
\[
A = (\varepsilon_{f_{k+1}-1+i+j})_{0 \leq i < f_{k-2}, 0 \leq j < f_k}
\]
\[
A' = (\varepsilon_{f_{k+1}-1+f_{k-2}+i+j})_{0 \leq i < f_{k-3}, 0 \leq j < f_k}
\]
\[
B = (\varepsilon_{f_{k+1}-1+f_{k-1}+i+j})_{0 \leq i < f_{k-2}, 0 \leq j < f_k}.
\]
By Lemma 5, the following two subwords of \( \varepsilon \):
\[
\varepsilon_{f_{k+1}-1} \varepsilon_{f_{k+1}} \cdots \varepsilon_{f_{k+1}+f_{k-2}+f_k-3} \quad \text{and} \quad 
\varepsilon_{f_{k+1}-1+f_{k-1}} \varepsilon_{f_{k+1}+f_{k-1}} \cdots \varepsilon_{f_{k+1}+f_{k-1}+f_{k-2}+f_k-3}
\]

differ only at two places. Namely, \( \varepsilon \in f_{k+1} + f_k - 2 \neq \varepsilon \in f_{k+1} + f_{k-1} + f_k - 2 \) and \( \varepsilon \in f_{k+1} + f_{k-1} \neq \varepsilon \in f_{k+1} + f_{k-1} + f_k - 1 \). Therefore, we get

\[
B - A = \begin{pmatrix}
0 & (1)^k & (1)^k & \\
& \ldots & \ldots & \\
& & (1)^k & (1)^k - 1 \\
& & & \ldots \\
(1)^k & (1)^k - 1 & & 0
\end{pmatrix}.
\]

Thus, we have

\[
\det(\varepsilon \in f_{k+1} + i + j)_{0 \leq i, j < f_k} = \det(\begin{pmatrix} \ldots & A_{f_{k-1}} & A_{f_k} & \ldots & A_{f_k - 2} & A_{f_k - 1} \\ & \ldots & \ldots & \ldots & \ldots & \ldots \\ & 0 & \ldots & \ldots & \ldots & \ldots \\ & \ldots & \ldots & \ldots & \ldots & \ldots \\ & (1)^k & (1)^k - 1 & & \ldots \\
\end{pmatrix}) = (-1)^k f_k - 2 (1)^\left[\frac{f_k - 2}{2}\right] \det(A_0 A_1 \cdots A_{f_{k-1} - 1})
\]

\[
= \chi(k : 1, 3, 4, 5) H_{f_{k+1} - 1, f_{k-1}},
\]

which completes the proof for \( H_{f_{k+1} - 1, f_{k-1}} \).

\section*{Lemma 11}

For any \( k \geq 2 \), we have

\[
H_{f_{k+1} - 1, f_{k-1}} = \chi(k : 2, 5) H_{0, f_{k-1}},
\]

\[
H_{f_{k+1} - 1, f_{k-1}} = \chi(k : 2, 5) H_{0, f_{k-1}}.
\]

\section*{Proof}

Since by Lemma 5,

\[
\varepsilon \in f_{k+1} + i + j + i \in f_{k+1} + f_{k-1} - 2 = \varepsilon \in f_{k+1} + f_{k-1} + f_{k+1} + f_{k-1} - 1 - 2 \\
\]

\[
\varepsilon \in f_{k+1} + f_{k-1} + f_{k+1} + f_{k-1} - 2, \]

\[
\varepsilon \in f_{k+1} + f_{k-1} + f_{k+1} + f_{k-1} - 2.
\]
we get
\[
(\varepsilon_{f_{k+1}-1+i+j})_{0 \leq i, j < f_{k-1}} = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
& & \ddots \\
0 & & \end{pmatrix}
\begin{pmatrix}
\varepsilon_{f_{k+1}+i+j})_{0 \leq i, j < f_{k-1}}.
\end{pmatrix}
\]

Also, by Lemma 5,
\[
(\varepsilon_{f_{k+1}+i+j})_{0 \leq i, j < f_k} = (\varepsilon_{i+j})_{0 \leq i, j < f_k}.
\]

Thus we obtain
\[
H_{f_{k+1}-1, f_{k-1}} = \det(\varepsilon_{f_{k+1}-1+i+j})_{0 \leq i, j < f_{k-1}}
\]
\[
= (-1)^{f_{k-1}-1} \det(\varepsilon_{f_{k+1}+i+j})_{0 \leq i, j < f_{k-1}}
\]
\[
= \chi(k: 2, 5) H_{0, f_{k-1}},
\]
which completes the proof.

Lemma 12 For any \( k \geq 3 \), we have
\[
H_{0, f_k} = \chi(k: 2, 3) H_{0, f_{k-1}} + \chi(k: 2, 4) H_{0, f_{k-2}}
\]
\[
\mathcal{P}_{0, f_k} = \chi(k: 1, 3, 4, 5) \mathcal{P}_{0, f_{k-1}} + \chi(k: 0, 1, 2, 3) \mathcal{P}_{0, f_{k-2}}.
\]

Proof. Clear from Lemmas 9–11.

Lemma 13 For any \( k \geq 0 \), we have
\[
H_{0, f_k} = \chi(k: 2) f_{k-1}
\]
\[
\mathcal{P}_{0, f_k} = \chi(k: 1, 2, 4) f_{k-2}
\]

Proof. It holds that
\[
H_{0, f_0} = 1, \quad H_{0, f_1} = 1, \quad H_{0, f_2} = -2
\]
\[
\mathcal{P}_{0, f_0} = 0, \quad \mathcal{P}_{0, f_1} = -1, \quad \mathcal{P}_{0, f_2} = -1.
\]
Thus, the lemma holds for \( k = 0, 1, 2 \). For \( k \geq 3 \), we can prove it by induction on \( k \) using Lemma 12.

16
Lemma 14 For any $k \geq 1$, we have

\[
\begin{align*}
H_{0,f_k-1} &= \chi(k : 0, 4)f_{k-2} \\
\mathbf{T}_{0,f_k-1} &= \chi(k : 2, 3, 4, 5)f_{k-3}.
\end{align*}
\]

Proof. Since the matrix $(\varepsilon_{i+j})_{0 \leq i,j \leq f_k-1}$ is obtained from the matrix $(\varepsilon_{i+j})_{0 \leq i,j \leq f_k}$ by removing the last row and the last column, for any $k \geq 2$ we have by (17),

\[
H_{0,f_k-1} = \det \left( \begin{array}{cccccc}
A_0 & A_1 & \cdots & A_{f_k-2} & 0 & 0 \\
0 & \cdots & 0 & (-1)^k & 0 \\
& \cdots & & (-1)^k & 0 \\
& & & (-1)^k & 0
\end{array} \right)
\]

\[
= (-1)^{k(f_k-2-1)}(-1)^\left[ \frac{f_k-2-1}{2} \right] \det(A_0A_1\cdots A_{f_k-2})
\]

Hence, in view of Lemma 13, we obtain the formula for $H_{0,f_k-1}$. \qed

Theorem 2 For any $m, k \geq 1$ with $f_{k-1} < m \leq f_k$ and $n \in \mathbb{N}$ with $n \equiv k+1 \ 0$, we have

\[
H_{n,m} = \begin{cases} 
\chi(k : 2)f_{k-1} & (\text{if } m = f_k) \\
\chi(k : 0, 4)f_{k-2} & (\text{if } m = f_k - 1) \\
0 & (\text{otherwise})
\end{cases}
\]

\[
\mathbf{T}_{n,m} = \begin{cases} 
\chi(k : 1, 2, 4)f_{k-2} & (\text{if } m = f_k) \\
\chi(k : 2, 3, 4, 5)f_{k-3} & (\text{if } m = f_k - 1) \\
0 & (\text{otherwise})
\end{cases}
\]

Proof. By Lemma 3 and 7, the matrix for $H_{n,m}$ coincides with that for $H_{0,m}$ so that $H_{n,m} = H_{0,m}$. Then, the first two cases follow from Lemma 13.
and 14. For the last case, by Corollary 1, there exist two identical rows in the matrix $(\varepsilon_{i+j})_{0 \leq i,j < m}$, so that $H_{0,m} = 0$.

**Theorem 3** For any $k, n, i \in \mathbb{N}$ with $n \equiv_{k+1} i$ and $0 \leq i < f_{k+1} - 1$, we have

\[
H_{n,f_k} = \begin{cases} 
\chi(k : 2)\chi(k : 1, 4)^i f_{k-1} \\
\quad \left( \begin{array}{l}
\text{if either } \tau_{k+1}(n) = 0 \text{ and } 0 \leq i < f_{k-1} \\
\quad \text{or } \tau_{k+1}(n) = 1 \text{ and } 0 \leq i < f_k 
\end{array} \right) \\
\chi(k : 1, 2, 4)f_{k-2} \\
\quad \left( \begin{array}{l}
\text{if either } \tau_{k+1}(n) = 0 \text{ and } i = f_{k-1} \\
\quad \text{or } i = f_{k+1} - 1 
\end{array} \right) \\
0 \quad (\text{otherwise})
\end{cases}
\]

\[
\Pi_{n,f_k} = \begin{cases} 
\chi(k : 1, 2, 4)\chi(k : 1, 4)^i f_{k-2} \\
\quad \left( \begin{array}{l}
\text{if either } \tau_{k+1}(n) = 0 \text{ and } 0 \leq i < f_{k-1} \\
\quad \text{or } \tau_{k+1}(n) = 1 \text{ and } 0 \leq i < f_k 
\end{array} \right) \\
\chi(k : 2)f_{k-3} \\
\quad \left( \begin{array}{l}
\text{if either } \tau_{k+1}(n) = 0 \text{ and } i = f_{k-1} \\
\quad \text{or } i = f_{k+1} - 1 
\end{array} \right) \\
0 \quad (\text{otherwise}).
\end{cases}
\]

**Proof.** The theorem holds for $k = 0$. Let $k \geq 1$.

Assume that either $\tau_{k+1}(n) = 0$ and $0 \leq i < f_{k-1}$ or $\tau_{k+1}(n) = 1$ and $0 \leq i < f_k$. Then we have by Lemma 3 and 7

\[
\varepsilon_{i+j} = \varepsilon_{n+j} \quad (j = 0, 1, \ldots, f_k - i - 1) \\
\varepsilon_{i+j-f_k} = \varepsilon_{n+j} \quad (j = f_k - i, f_k, \ldots, 2f_k - 2) \\
\varepsilon_j = \varepsilon_{j+f_k} \quad (j = 0, 1, \ldots, f_k - 1).
\]

Hence, the columns of the matrix $(\varepsilon_{n+h+j})_{0 \leq h,j \leq f_k}$ coincide with those of the matrix $(\varepsilon_{h+j})_{0 \leq h,j \leq f_k}$. The $j$-th column of the former is the $(i+j)(mod f_k)$-th column of the latter for $j = 0, \ldots, f_k - 1$. Therefore, we get $H_{n,f_k} = (-1)^{i(f_k-i)}H_{0,f_k}$, which leads to the first case of our theorem by Theorem 2.
Assume that \( i = f_{k+1} - 1 \). Then we have \( H_{n,f_k} = H_{f_{k+1} - 1,f_k} \) by Lemmas 3 and 7. Thus, by Lemmas 10–12 we get

\[
H_{n,f_k} = \chi(k : 1, 2, 4)f_k.
\]

Assume that \( \tau_{k+1}(n) = 0 \) and \( i = f_{k-1} \). Then, since \( n \equiv_{k+2} i \), we have \( H_{n,f_k} = H_{f_k-1,f_k} \) by Lemmas 3 and 7. By Lemma 1,

\[
\xi := \varepsilon_{f_k-1}\varepsilon_{f_k+1}^1 \varepsilon_{f_k-3}^1 W_{k-2}W_{k-1}W_kW_{k-1}W_{k-2},
\]

\[
\eta := \varepsilon_{f_k+1}^1 \varepsilon_{f_k+1}^1 \varepsilon_{f_k+1}^1 \varepsilon_{f_k-2}^1 \varepsilon_{f_k-2}^1 W_{k-2}W_{k-1}W_kW_{k-1}W_{k-2}
\]

holds. Since the last letter of \( \eta \) comes one letter before the last letter of the palindrome word \( W_{k-2}W_{k-1}W_kW_{k-1}W_{k-2} \). Hence, \( \xi \) is the mirror image of \( \eta \), so that

\[
(\varepsilon_{f_k+i+j})_{0 \leq i, j < f_k} = \\
\begin{pmatrix}
0 & 1 & 0 \\
\vdots & \ddots & \vdots \\
1 & 0 & 0
\end{pmatrix}
(\varepsilon_{f_k+i+j})_{0 \leq i, j < f_k} \\
\begin{pmatrix}
0 & 1 & 0 \\
\vdots & \ddots & \vdots \\
1 & 0 & 0
\end{pmatrix}.
\]

Thus, we obtain \( H_{f_{k-1},f_k} = H_{f_{k+1} - 1,f_k} \) and

\[
H_{n,f_k} = \chi(k : 1, 2, 4)f_k.
\]

Assume that \( n \) does not belong to the above two cases. Then, since \( \tau_{k+1}(n) = 1 \) implies \( i < f_k \), we have the following condition:

\[
\tau_{k+1}(n) = 0 \text{ and } f_k - 1 + 1 \leq i \leq f_{k+1} - 2.
\]

This condition is nonempty only if \( k \geq 2 \), which we assume. Then, the condition (2) of Theorem 1 is satisfied with \( f_k \) (resp. \( i - f_k \)) in place of \( m \) (resp. \( i \)). Thus, by Corollary 3, \( H_{n,f_k} = 0 \).

\[ \blacksquare \]
Lemma 15 For any \( k, n, i \in \mathbb{N} \) with \( k \geq 1 \) and \( n \equiv_{k+1} i \), assume that either \( \tau_{k+1}(n) = 0 \) and \( 0 \leq i < f_{k-1} \) or \( \tau_{k+1}(n) = 1 \) and \( 0 \leq i < f_k \). Then we have

\[
H_{n, f_{k-1}} = \begin{cases} 
\chi(k : 0, 4) f_{k-2} & (i = 0) \\
\chi(k : 2, 3) \chi(k : 1, 2, 4, 5)^i H_{i+f_k, f_{k-1}-1} + \chi(k : 1, 2, 3, 5) \chi(k : 1, 4)^i f_{k-2} & (0 < i \leq f_{k-2}) \\
\chi(k : 2, 3) \chi(k : 1, 2, 4, 5)^i H_{i+f_k, f_{k-1}-1} & (f_{k-2} < i \leq f_{k-1}) \\
\chi(k : 0, 4) \chi(k : 1, 4)^i f_{k-2} & (f_{k-1} < i < f_k) 
\end{cases}
\]

\[
\prod_{n, f_{k-1}} = \begin{cases} 
\chi(k : 2, 3, 4, 5) f_{k-3} & (i = 0) \\
\chi(k : 1, 3, 4, 5) \chi(k : 1, 2, 4, 5)^i \prod_{i+f_k, f_{k-1}-1} + \chi(k : 0, 1) \chi(k : 1, 4)^i f_{k-3} & (0 < i \leq f_{k-2}) \\
\chi(k : 1, 3, 4, 5) \chi(k : 1, 2, 4, 5)^i \prod_{i+f_k, f_{k-1}-1} & (f_{k-2} < i \leq f_{k-1}) \\
\chi(k : 2, 3, 4, 5) \chi(k : 1, 4)^i f_{k-3} & (f_{k-1} < i < f_k) 
\end{cases}
\]

Proof. If \( i = 0 \), then the statement follows from Theorem 2. Let

\[
\begin{align*}
A_j & = \mathbb{B}^i (\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_{j+f_{k-1}-1}) \\
A'_j & = \mathbb{B}^i (\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_{j+f_{k-1}-2}) \\
B'_j & = \mathbb{B}^i (\varepsilon_{j+f_{k-1}}, \varepsilon_{j+f_{k-1}+1}, \ldots, \varepsilon_{j+f_k-1}) \\
& (j = 0, 1, 2, \ldots).
\end{align*}
\]

Then, by the same argument as in the proof of Theorem 3, we obtain

\[
H_{n, f_{k-1}} = \det \begin{pmatrix} A_i \cdots A_{f_{k-1}} & A_0 \cdots A_{i-2} \\
B'_i \cdots B'_{f_{k-1}} & B'_0 \cdots B'_{i-2} \end{pmatrix} = (-1)^{(i-1)(f_k-i)} \det \begin{pmatrix} A_0 \cdots A_{i-2} & A_1 \cdots A_{f_k-1} \\
B'_0 \cdots B'_{i-2} & B'_1 \cdots B'_{f_k-1} \end{pmatrix}.
\]

Therefore, if \( f_{k-2} < i \leq f_{k-1} \), then by the same argument to get (17), we obtain

\[
(-1)^{(i-1)(f_k-i)} H_{n, f_{k-1}} =
\]

20
\[
\begin{vmatrix}
A_0 \cdots A_{i-2} A_i \cdots A_{f_{k-1}} & 0 & \cdots & 0 & A_{f_k-1} \\
0 & \cdots & 0 & 0 & \vdots \\
(-1)^k & (\cdots) & (\cdots) & (\cdots) & (-1)^{k-1}
\end{vmatrix}.
\]

Since by Lemma 5

\[
A_{f_{k-1}} - A_{f_{k-2}-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 & (-1)^k \end{pmatrix},
\]

we get

\[
(-1)^{(i-1)(f_k-i)} H_{n,f_{k-1}} = 
\begin{vmatrix}
A'_0 \cdots A'_{i-2} A'_i \cdots A'_{f_{k-2}-1} & 0 & \cdots & 0 & 0 \\
* & \cdots & * & \cdots & * \\
0 & \cdots & 0 & (-1)^k & (\cdots) \\
(-1)^k & (\cdots) & (\cdots) & (\cdots) & (-1)^{k-1}
\end{vmatrix}
\]

\[
= (-1)^{(i-1)(f_k-i)} \left[ f_{k-2} \right] \det(A'_0 \cdots A'_{i-2} A'_i \cdots A'_{f_{k-2}-1})
\]

\[
= \chi(k : 1, 3, 4, 5)(-1)^{(i-1)(f_k-i)} H_{i+f_k,f_{k-1}-1}.
\]

Thus we obtain

\[
H_{n,f_{k-1}} = \chi(k : 2, 3) \chi(k : 1, 2, 4, 5)^i H_{i+f_k,f_{k-1}-1}.
\]

Assume that \( f_{k-1} < i < f_k \). Then as above we have

\[
(-1)^{(i-1)(f_k-i)} H_{n,f_{k-1}} =
\]

21
\[
\begin{pmatrix}
A_0 \cdots A_{f_{k-1}-1} & 0 & \cdots & 0 & \cdots & 0 & A_{f_k-1} \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(-1)^{k-1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]
\[
\det
\begin{pmatrix}
0 & (-1)^{k-1} & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
(-1)^{k-1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(-1)^k & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]
\[
= (-1)^{k(i - f_{k-1} - 1) + (k-1)(f_k - i) + \left[\frac{f_k - 2}{2}\right]} \det(A_0 \cdots A_{f_{k-1}-1}).
\]

Hence, by Lemma 13
\[
H_{n, f_k-1} = \chi(k : 0, 3, 4) \chi(k : 1, 4)^i H_{0, f_k-1}
\]
\[
= \chi(k : 0, 4) \chi(k : 1, 4)^i f_{k-2}.
\]

Assume that \(0 < i < f_{k-2}\). Then, since \(A_{i-1} + f_{k-1} = A_{i-1}\), by the same arguments as above we get
\[
(-1)^{(i-1)(f_k - i)} H_{n, f_k-1} =
\]
\[
\begin{pmatrix}
A_0' \cdots A_{i-2}' A_{i-1}' \cdots A_{f_{k-1}-1}' & 0 & \cdots & 0 & \cdots & 0 \\
* & \cdots & * & \cdots & * & \cdots \\
0 & \cdots & 0 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(-1)^{k-1} & (-1)^{k-1} & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]
\[
\det
\begin{pmatrix}
(-1)^k \\
\vdots \\
0 \\
\end{pmatrix}
\]
\[
= (-1)^{k(i-1)} (-1)^{\left[\frac{f_k - 2}{2}\right]} \det(A_0' \cdots A_{i-2}' A_{i-1}' \cdots A_{f_{k-1}-1}')
\]
\[
+ (-1)^{k(i-1) + (k-1)(f_{k-2} - i) - 1} \left[\frac{f_k - 2}{2}\right] 
\]
\[
\det(A_0 \cdots A_{i-2} A_{i-1} \cdots A_{f_{k-1}-1} A_{i-1}).
\]

Since
\[
\det(A_0 \cdots A_{i-2} A_{i-1} \cdots A_{f_{k-1}-1} A_{i-1}) = (-1)^{f_k - i - 1} H_{0, f_k-1},
\]

22
Lemma 13

\[ H_{n,f_{k-1}} = \chi(k : 2, 3) \chi(k : 1, 2, 4, 5)^i H_{i+f, f_{k-1}} + \chi(k : 1, 2, 3, 5) \chi(k : 1, 4)^i f_{k-2} \cdot (21) \]

Note that (21) holds also for \( i = f_{k-2} \) since in this case,

\[ H_{n,f_{k-1}} = (-1)^{k(f_{k-2}-1)} \left[ \frac{f_{k-2}-1}{2} \right] \]

\[ \det(A_0 \cdots A_{f_{k-2}-2} A_{f_{k-2}} \cdots A_{f_{k-1}-2} A_{f_{k-1}}) \]

and

\[ A_{f_{k-1}} = A_{f_{k-1}-1} + (-1)^k, \]

which completes the proof for \( H_{n,f_{k-1}} \).

Lemma 16 For any \( k, n, i \in \mathbb{N} \) with \( k \geq 1 \) and \( n \equiv_{i+1} i \), assume that either \( \tau_{k+1}(n) = 0 \) and \( 0 \leq i < f_{k-1} \) or \( \tau_{k+1}(n) = 1 \) and \( 0 \leq i < f_k \). Then we have

\[
H_{n,f_{k-1}} = \begin{cases}
\chi(k : 2, 3, 5) f_{k-3} & (i = 0) \\
\chi(k : 1, 2, 3, 5) \chi(k : 1, 4)^i f_{k-2} & (0 < i \leq f_{k-1}) \\
\chi(k : 0, 4) \chi(k : 1, 4)^i f_{k-2} & (f_{k-1} < i < f_k)
\end{cases}
\]

\[
\overline{H}_{n,f_{k-1}} = \begin{cases}
\chi(k : 2, 3, 4, 5) f_{k-3} & (i = 0) \\
\chi(k : 0, 1) \chi(k : 1, 4)^i f_{k-4} & (0 < i \leq f_{k-1}) \\
\chi(k : 2, 3, 4, 5) \chi(k : 1, 4)^i f_{k-3} & (f_{k-1} < i < f_k)
\end{cases}
\]

Proof: The first and the third cases have been already proved in Lemma 15. Let us consider the second case where \( 0 < i \leq f_{k-1} \). We divide it into two subcases, and use induction on \( k \).

Case 1. \( i = 1 \):

If \( k = 1 \), then

\[ H_{n,f_{k-1}} = H_{n,1} = \varepsilon_n = 0 \]

since \( n \equiv_2 1 \) and \( \tau_0(n) = 1 \). On the other hand, \( f_{k-3} = f_{k-2} = 0 \), and hence, we get the statement. Assume that \( k \geq 2 \) and the assertion holds for \( k - 1 \).
Then, by Lemma 15 and the induction hypothesis, we get
\[
H_{n,f_{k-1}} \\
= \chi(k : 2, 3)\chi(k : 1, 2, 4, 5)^i H_{i+f_{k-1}, f_{k-1}-1} + \chi(k : 1, 2, 3, 5)\chi(k : 1, 4)^i f_{k-2} \\
= \chi(k : 1, 3, 4, 5)H_{i+f_{k-1}, f_{k-1}-1} + \chi(k : 2, 3, 4, 5)f_{k-2} \\
= \chi(k : 1, 3, 4, 5)\chi(k - 1 : 2, 3, 4, 5)f_{k-4} + \chi(k : 2, 3, 4, 5)f_{k-2} \\
= \chi(k : 0, 1)f_{k-4} + \chi(k : 2, 3, 4, 5)f_{k-2} \\
= \chi(k : 2, 3, 4, 5)f_{k-3},
\]

which is the desired statement.

**Case 2.** \(i \geq 2\):

If \(f_{k-2} < i \leq f_{k-1}\), then it follows from the third case and then the fourth case of Lemma 15 that
\[
H_{n,f_{k-1}} \\
= \chi(k : 2, 3)\chi(k : 1, 2, 4, 5)^i H_{i+f_{k-1}, f_{k-1}-1} \\
= \chi(k : 2, 3)\chi(k : 1, 2, 4, 5)^i \chi(k - 1 : 0, 4)\chi(k - 1 : 1, 4)^i f_{k-3} \\
= \chi(k : 1, 2, 3, 5)\chi(k : 1, 4)^i f_{k-3}.
\]

Assume that \(i \leq f_{k-2}\) and the statement holds for \(k - 1\). Then by Lemma 15, we get
\[
H_{n,f_{k-1}} \\
= \chi(k : 2, 3)\chi(k : 1, 2, 4, 5)^i H_{i+f_{k-1}, f_{k-1}-1} + \chi(k : 1, 2, 3, 5)\chi(k : 1, 4)^i f_{k-2} \\
= \chi(k : 2, 3)\chi(k : 1, 2, 4, 5)^i \chi(k - 1 : 1, 2, 3, 5)\chi(k - 1 : 1, 4)^i f_{k-4} \\
+ \chi(k : 1, 2, 3, 5)\chi(k : 1, 4)^i f_{k-2} \\
= \chi(k : 0, 4)\chi(k : 1, 4)^i f_{k-4} + \chi(k : 1, 2, 3, 5)\chi(k : 1, 4)^i f_{k-2} \\
= \chi(k : 1, 2, 3, 5)\chi(k : 1, 4)^i f_{k-3}.
\]

This completes the proof for \(H_{n,f_{k-1}}\).

**Lemma 17** For any \(k, n \in \mathbb{N}\) with \(k \geq 2\) and \(\tau_{k+1}(n) = 0\), we have
\[
H_{n,f_{k-1}} = \begin{cases} 
\chi(k : 2, 3, 4, 5)f_{k-3} & (n \equiv_{k+1} f_{k-1}) \\
\chi(k : 0, 4)f_{k-2} & (n \equiv_{k+1} f_{k-1} + 1)
\end{cases}
\]

24
Proof. Assume that \( n \equiv_{k+1} f_{k-1} \). Then since \( \tau_{k+1}(n) = 0 \), we have \( n \equiv_{k+2} f_{k-1} \). Therefore, by Lemma 3 and 7, we get

\[
H_{n,f_{k-1}} = \det \left( \begin{array}{ccc}
A_{f_{k-1}} \cdots A_{f_{k-1}+1} A_{f_{k}} \cdots A_{f_{k}+1-2} \\
B'_{f_{k-1}} \cdots B'_{f_{k-1}+1} B'_{f_{k}} \cdots B'_{f_{k}+1-2}
\end{array} \right),
\]

where we use the notation (20). By Lemma 5, the following two subwords of \( \varepsilon \):

\[
\varepsilon_{n} \varepsilon_{n+1} \cdots \varepsilon_{n+f_{k-2}+f_{k}-3} \quad \text{and} \quad \varepsilon_{n+f_{k-1}} \varepsilon_{n+f_{k-1}+1} \cdots \varepsilon_{n+f_{k-1}+f_{k-2}+f_{k}-3}
\]

differ only at two places, namely, at the \((f_{k} - 2 - f_{k-1})\)-th and the \((f_{k} - 1 - f_{k-1})\)-th places. Hence, we have

\[
H_{n,f_{k-1}} = \det \left( \begin{array}{ccc}
A_{f_{k-1}} \cdots A_{f_{k}-1} A_{f_{k}} \cdots A_{f_{k}+1-2} \\
B'_{f_{k-1}} \cdots B'_{f_{k}-1} B'_{f_{k}} \cdots B'_{f_{k}+1-2}
\end{array} \right) = \\
\left( \begin{array}{cccc}
A_{f_{k-1}} & \cdots & A_{f_{k}-1} & A_{f_{k}} \cdots A_{f_{k}+1-2} \\
0 & \cdots & 0 & \vdots \vrule \vline \\
(-1)^{k} & \cdots & (-1)^{k} & \vdots \vrule \vline \\
0 & \cdots & 0 & (-1)^{k-1}
\end{array} \right).
\]

By adding the first \( f_{k-2} - 1 \) columns and subtracting the last \( f_{k-2} - 1 \) columns to and from the column beginning by \( A_{f_{k}-1} \), we get the column

\[
^t(A_{f_{k-1}} 0 \cdots 0) + ^t((-1)^{k-1} 0 \cdots 0(-1)^{k} 0 \cdots 0),
\]

where \((-1)^{k}\) is at the \((f_{k-2} - 1)\)-th place. Since, by Lemma 5

\[
(A_{f_{k-1}} \cdots A_{f_{k}-2}) - (A_{f_{k-1}} \cdots A_{f_{k}+1-2}) =
\]

25
\[
\begin{pmatrix}
0 & (-1)^{k-1} & (-1)^k \\
 & \ddots & \ddots \\
(-1)^{k-1} & (-1)^k & 0 \\
(-1)^{k-1} & (-1)^k & 0 \\
& & & 0
\end{pmatrix},
\]

hence, we get

\[
H_{n,f_{k-1}} = (-1)^{k(f_k-2)+1} (-1)^{f_{k-1}(f_{k-2}-1)+1} \left[ \frac{A_{k-1}}{A_{k-1}} \right]
\]

\[
\{ \det(A_{f_{k-1}} A_{f_k} \cdots A_{f_{k+1}-2}) + (-1)^{k-1} \det(A''_{f_k} \cdots A''_{f_{k+1}-2}) \\
+ (-1)^{k+f_{k-2}-1} \det(A''_{f_{k+1}} A''_{f_{k+1}-2}) \},
\]

where

\[
A''_{j} := \left( \varepsilon_{j+1} \cdots \varepsilon_{j+f_{k-1}-1} \right) \\
A''_{j} := \left( \varepsilon_{j+1} \cdots \varepsilon_{j+f_{k-2}-2} \varepsilon_{j+f_{k-2}} \cdots \varepsilon_{j+f_{k-1}-1} \right).
\]

Here, we have

\[
\det(A_{f_{k-1}} A_{f_k} \cdots A_{f_{k+1}-2}) = H_{f_{k-1}, f_{k-1}} \\
\det(A''_{f_k} \cdots A''_{f_{k+1}-2}) = H_{f_{k+1}, f_{k-1}}.
\]

and by Lemma 5

\[
\det(A''_{f_k} \cdots A''_{f_{k+1}-2}) = \]

26
where we put

\[ C_j = (\varepsilon_j \varepsilon_{j+1} \cdots \varepsilon_{j+f_k-2-1}). \]

Since \( C_{f_k+f_k-2+j} = C_{f_k+j} \) (\( j = 0, 1, \cdots, f_k-3 - 2 \)) by Lemma 5, it holds that

\[
\begin{vmatrix}
C_{f_k} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
C_{f_k+f_k-2} & \cdots & C_{f_k+f_k-2-2} \\
\vdots & \ddots & \vdots \\
C_{f_k+1} & \cdots & 0 \\
\end{vmatrix}
= (-1)^{(k-1)(f_k-3-1)+f_k-3-1+\left\lfloor \frac{f_k-3-1}{2} \right\rfloor} \det \begin{pmatrix}
C_{f_k} \\
\vdots \\
C_{f_k+f_k-2} \\
C_{f_k+1} \\
\end{pmatrix}.
\]

Moreover it follows from Lemma 5 that

\[
\begin{vmatrix}
C_{f_k} \\
\vdots \\
C_{f_k+f_k-2} \\
C_{f_k+1} \\
\end{vmatrix} = \det \begin{pmatrix}
C_{f_k+1} \\
\vdots \\
C_{f_k+1+f_k-2} \\
C_{f_k+1} \\
\end{pmatrix} = (-1)^{f_{k-2}-1} H_{f_{k+1}-1, f_{k-2}}.
\]
which implies
\[
\det(A'''_k \cdots A''_{k+1-2}) = \chi(k : 0, 3, 5) H_{f_{k+1-1}, f_{k-2}}.
\]

Thus by (22), (23), Theorem 3 and Lemma 16, we obtain
\[
H_{n, f_k-1} = \chi(k : 4) H_{f_{k-1}, f_{k-1}} + \chi(k : 0, 2) H_{f_{k+1}, f_{k-1}-1} + \chi(k : 1, 3, 4) H_{f_{k+1}, f_{k-2}} = \chi(k : 2, 3, 4, 5) f_{k-3} + \chi(k : 2, 3, 4, 5) f_{k-4} + \chi(k : 0, 1) f_{k-4}
\]
which is the first case of our lemma.

To prove the second case, assume that \( n \equiv_{k+1} f_{k-1} + 1 \). Then, as above we get
\[
H_{n, f_k-1} = \det \left( \begin{array}{cccc}
A_{f_{k-1}+1} & \cdots & A_{f_k} & A_{f_{k+1}-1} \\
B'_{f_{k-1}+1} & \cdots & B'_{f_k} & B'_{f_{k+1}-1}
\end{array} \right)
= \det \left( \begin{array}{cccc}
A_{f_{k-1}+1} & \cdots & A_{f_k} & A_{f_{k+1}-1} \\
0 & \cdots & 0 & 0 \\
(-1)^k & \cdots & (-1)^{k-1} \\
(-1)^k & \cdots & (-1)^{k-1}
\end{array} \right)
= (-1)^{(k-1)(f_{k-2}-1)} (-1)^{(f_{k-2}-1)f_{k-1} + \left[ \frac{f_{k-2}-1}{2} \right]} \det(A_k \cdots A_{f_{k+1}-1}) .
\]
Therefore, we get by Theorem 3
\[
H_{n, f_k-1} = \chi(k : 0, 3, 4) \chi(k - 1 : 2) f_{k-2} = \chi(k : 0, 4) f_{k-2} ,
\]
which completes the proof for \( H_{n, f_k-1} \). \( \blacksquare \)
Theorem 4 For any $k, n, i \in \mathbb{N}$ with $k \geq 1$, $n \equiv_{k+1} i$ and $0 \leq i < f_{k+1}$, we have

$$H_{n,f_{k-1}} = \begin{cases} \chi(k : 0, 4) f_{k-2} & (i = 0) \\ \chi(k : 1, 2, 3, 5) \chi(k : 1, 4)^i f_{k-3} & (0 < i \leq f_{k-1}) \\ \chi(k : 0, 4) \chi(k : 1, 4)^i f_{k-2} & \left( f_{k-1} < i < f_k \right. \\ \text{(and } \tau_{k+1}(n) = 1 \left. \right) \\ 0 \quad \text{ (otherwise) } \end{cases}$$

$$\Pi_{n,f_{k-1}} = \begin{cases} \chi(k : 2, 3, 4, 5) f_{k-3} & (i = 0) \\ \chi(k : 0, 1) \chi(k : 1, 4)^i f_{k-4} & (0 < i \leq f_{k-1}) \\ \chi(k : 2, 3, 4, 5) \chi(k : 1, 4)^i f_{k-3} & \left( f_{k-1} < i < f_k \right. \\ \text{(and } \tau_{k+1}(n) = 1 \left. \right) \\ 0 \quad \text{ (otherwise) } \end{cases}$$

Proof. The first four cases follow from Lemma 16 and 17. Note that for $i = f_{k-1}$, the assertion in these lemmas coincide, so that $H_{n,f_{k-1}}$ is independent of $\tau_{k+1}(n)$. Let us consider the last case, where $\tau_{k+1}(n) = 0$ and $f_{k-1} + 2 \leq i \leq f_{k-1} - 1$. We may assume that $k \geq 2$. Then, with $m = f_k - 1$ and $i - f_k$ in place of $i$ there, the condition (2) of Theorem 1 is satisfied. Therefore by Theorem 1, $n \in \mathcal{R}_m$ which implies that $H_{n,f_{k-1}} = 0$.

Lemma 18 For any $n, m \in \mathbb{N}$ such that $f_{k-2} + 1 \leq m \leq f_k - 2$, $i \leq n$ and $n - i \equiv_{k+1} 0$ for some $i, k \in \mathbb{Z}$ with $k \geq 2$ and $m + i = f_k$. Then, we have

$$H_{n,m} = \chi(k : 2) \chi(k : 3, 4, 5)^i (-1)^{i/2} f_{k-3}$$
\[ \prod_{n,m} = \chi(k : 1, 4)\chi(k : 0, 1, 2)^i(-1)^{[i/2]} f_{k-3} . \]

\textit{Proof.} At first, we consider the case \( i < f_{k-2} \). By arguments similar to those used in the proof of Lemma 15, we get with the notation (20)

\[
H_{n,m} = \begin{pmatrix} A_i A_{i+1} & \cdots & A_{f_k-1} & 0 & \cdots & 0 & A_{f_k-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & (-1)^k & (-1)^{k-1} \\ \end{pmatrix}.
\]

Therefore, by Theorem 3 and 4,

\[
H_{n,m} = (-1)^{(f_{k-2} - i + 1)} + \left[ \frac{f_{k-2} - i + 1}{2} \right] H_{i, f_k-1} + (-1)^{(k-1)(f_{k-2} - i)} + \left[ \frac{f_{k-2} - i}{2} \right] H_{i, f_k-1}.
\]

\[
= \chi(k : 2)\chi(k : 3, 4, 5)^i(-1)^{[i/2]} (-f_{k-4} + f_{k-2})
\]

\[
= \chi(k : 2)\chi(k : 3, 4, 5)^i(-1)^{[i/2]} f_{k-3} .
\]

If \( i = f_{k-2} \), then the lemma follows from Theorem 3.

Finally, we consider the case \( f_{k-2} < i < f_{k-1} \). Then, denoting

\[
A^*_j = (e_j e_{j+1} \cdots e_{i+r-1}),
\]

we obtain by Theorem 3

\[
H_{n,m} = \det(A_i f_k-i A_{i+1} f_k-i \cdots A_{f_k-1} f_k-i) =
\]

\[
\begin{pmatrix} A_i f_k-i & A_{i+1} f_k-i & \cdots & A_{f_k-2} f_k-i & A_{f_k-2} f_k-i & A_{f_k-2} f_k-i & \cdots & A_{f_k-2} f_k-i \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots \\ 0 & \cdots & 0 & (-1)^k & (-1)^k & (-1)^k & \cdots & 0 \\ \end{pmatrix}
\]

\[
= (-1)^{(f_{k-1} - i)} (-1)^{(f_{k-1} - i) f_k-2} + \left[ \frac{f_{k-1} - i}{2} \right] H_{f_{k-1}, f_k-2}
\]

\[
= \chi(k : 2)\chi(k : 3, 4, 5)^i(-1)^{[i/2]} f_{k-3} .
\]

\[
= \chi(k : 1, 4)\chi(k : 0, 1, 2)^i(-1)^{[i/2]} f_{k-3} .
\]
which completes the proof for $H_{n,m}$.

Lemma 19 For any $n, m \in \mathbb{N}$ such that $f_{k-1} + 1 \leq m \leq f_k - 2$, $i \leq n$, $n - i \equiv_k f_{k-1}$ for some $i, k \in \mathbb{Z}$ with $k \geq 2$ and $m + i = f_k$, we have

$$H_{n,m} = \chi(k : 1, 2, 4)\chi(k : 0, 1, 2)^i(-1)^{[i/2]}f_{k-2}$$

$$\overline{P}_{n,m} = \chi(k : 2)\chi(k : 3, 4, 5)^i(-1)^{[i/2]}f_{k-3}.$$  

Proof. By the same arguments and in the same notations as in the second part of the proof of Lemma 18, we obtain

$$H_{n,m} = \det(A_{i+1}^{f_{k-1} - i} \cdots A_{i+1}^{f_{k-1} - i} A_{f_k}^{f_{k-1} - i} \cdots A_{f_{k+1} - 1}^{f_{k-1} - i}) =$$

$$\det\begin{pmatrix}
A_{i+1}^{f_{k-1} - i} & A_{i+1}^{f_{k-1} - i} & \cdots & A_{f_k - 1}^{f_{k-1} - i} & A_{f_k - 1}^{f_{k-1} - i} & \cdots & A_{f_{k+1} - 1}^{f_{k-1} - i} \\
0 & (-1)^{k-1} & \cdots & \cdots & \cdots & \cdots & 0 \\
(-1)^{k} & (-1)^{k-1} & \cdots & \cdots & \cdots & \cdots & 0 \\
\end{pmatrix}$$

$$= (-1)^{(k-1)(f_k - 2 - i)}(-1)^{(f_k - 2 - i)f_k - 1}f_{k-2}^{f_{k-2}}H_{f_k, f_k - 1},$$

which completes the proof for $H_{n,m}$.

Lemma 20 For any $n, m \in \mathbb{N}$ such that $f_{k-1} + 1 \leq m \leq f_k - 2$, $i \leq n$ and $n - i \equiv_{k+1} 0$ for some $i, k \in \mathbb{Z}$ with $k \geq 2$ and $m + i = f_k - 1$, we have

$$H_{n,m} = \chi(k : 0, 4)\chi(k : 3, 4, 5)^i(-1)^{[i/2]}f_{k-2}$$

$$\overline{P}_{n,m} = \chi(k : 2, 3, 4, 5)\chi(k : 0, 1, 2)^i(-1)^{[i/2]}f_{k-3}.$$  

Proof. The proof is similar to the first part of the proof of Lemma 18. With the notation in (20), we get

$$H_{n,m} =$$

31
\[
\begin{pmatrix}
A_i A_{i+1} & \cdots & A_{j_k-i} \cdots & 0 & 0 & \cdots & 0 \\
0 & & & & & & (-1)^k \\
& & & & & \cdots & \\
& & & & & & (-1)^k & (-1)^{k-1} & 0
\end{pmatrix}
\]

\[
= (-1)^{k(f_{k-2} - 1)} \left( -1 \right)^{\left[{\frac{k-2-i}{2}}\right]} \det(A_i A_{i+1} \cdots A_{j_k-i-1}).
\]

Hence, by Theorem 3
\[
H_{n,m} = \chi(k : 0, 4) \chi(k : 3, 4, 5)^i (-1)^{\frac{k-2}{2}} f_{k-2} ,
\]
which completes the proof for \( H_{n,m} \).

**Lemma 21** For any \( n, m \in \mathbb{N} \) such that \( f_{k-2} + 1 \leq m \leq f_k - 2 \), \( i \leq n \) and \( n - i \equiv_k f_{k-1} \) for some \( i, k \in \mathbb{Z} \) with \( k \geq 2 \) and \( m + i = f_k - 1 \), we have
\[
H_{n,m} = \chi(k : 2, 3, 4, 5) \chi(k : 0, 1, 2)^i (-1)^{\frac{k-2}{2}} f_{k-3} \\
\mathcal{H}_{n,m} = \chi(k : 0, 4) \chi(k : 3, 4, 5)^i (-1)^{\frac{k-2}{2}} f_{k-4} .
\]

**Proof.** Since \( i = f_k - 1 - m \), we get \( 1 \leq i \leq f_{k-1} - 2 \)

If \( i = f_{k-2} - 1 \), then \( m = f_{k-1} - 1 \) and \( n \equiv_k f_k - 1 \). Therefore, by Theorem 3, we get
\[
H_{n,m} = \chi(k - 1 : 1, 2, 4) f_{k-3},
\]
which coincides with the required identity since
\[
\chi(k : 0, 1, 2)^{f_{k-2} - 1} = \chi(k : \{0, 1, 2\} \cap \{0, 3\}) = \chi(k : 0), \\
(-1)^{\left[{\frac{k-2}{2}}\right]} = \chi(k : 0, 4).
\]

If \( i = f_{k-2} \), then \( m = f_{k-1} - 1 \) and \( n \equiv_k 0 \). Therefore, by Theorem 4, we get
\[
H_{n,m} = \chi(k - 1 : 0, 4) f_{k-3},
\]
which coincides with the required statement since
\[
\chi(k : 0, 1, 2)^{f_{k-2}} = \chi(k : \{0, 1, 2\} \cap \{1, 2, 4, 5\}) = \chi(k : 1, 2), \\
(-1)^{\left[{\frac{f_{k-2}}{2}}\right]} = \chi(k : 3, 4).
\]
If \( f_{k-2} + 1 \leq i \leq f_{k-1} - 2 \), then \( n - i' \equiv k \) 0 with \( i' := i - f_{k-2} \). Then, since \( m' = f_{k-1} - 1 \) and \( f_{k-2} + 1 \leq m \leq f_{k-1} - 2 \), applying Lemma 20, we obtain

\[
H_{n,m} = \chi(k - 1 : 0, 4)\chi(k - 1 : 3, 4, 5)^{j'}(-1)^{[i'/2]}f_{k-3} \\
= \chi(k : 1, 5)\chi(k : 0, 4, 5)^{i'}\chi(k : \{0, 4, 5\} \cap \{1, 2, 4.5\})(-1)^{[i'/2]}f_{k-3} \\
= \chi(k : 1, 4)^{i'}(-1)^{[i'/2]}(-1)^{\frac{f_{k-2} + 1}{2}}(-1)^{i'f_{k-2}f_{k-3}} \\
= \chi(k : 2, 3, 4, 5)\chi(0, 1, 2)^{j}(-1)^{[i'/2]}f_{k-3}.
\]

Now, we consider the case \( 1 \leq i \leq f_{k-2} - 2 \). Then, with the notations in (24) and in (20), we get

\[
H_{n,m} = \det(A_{f_{k-1}+i}^{f_{k}-i} \cdots A_{f_{k}}^{f_{k}-i} A_{f_{k}}^{f_{k}-i+1} \cdots A_{f_{k+1}}^{f_{k}-i}) = \\
\begin{pmatrix}
A_{f_{k-1}+i}^{f_{k}-i} & A_{f_{k-1}+i+1}^{f_{k}-i} & \cdots & A_{f_{k}}^{f_{k}-i} & A_{f_{k}}^{f_{k}-i} & A_{f_{k+1}}^{f_{k}-i} \\
0 & (\det A_{f_{k}}^{f_{k+1}})^{-1} & \cdots & 0 \\
(\det A_{f_{k}}^{f_{k+1}})^{-1}
\end{pmatrix}.
\]

Therefore, by arguments similar to those used in the first part of the proof of Lemma 17, we get

\[
H_{n,m} = (\det A_{f_{k}}^{f_{k+1}})^{-1}(\det A_{f_{k}}^{f_{k+1}})^{-1}f_{k-2}^{-1-i}(-1)^{i'f_{k-2}f_{k-3}} + (\det A_{f_{k}}^{f_{k+1}})^{-1}f_{k-2}^{-1-i}
\]

where we use the same notations as in the proof of Lemma 17 except for \( A_{j}^{m} \)'s which are defined by

\[
A_{j}^{m} = \varepsilon_{j} \cdots \varepsilon_{j+f_{k-2}+i-2} \varepsilon_{j+f_{k-2}-i} \cdots \varepsilon_{j+f_{k-2}+i-1}.
\]
Then, following the arguments there, we get

\[ H_{n,m} = \chi(k : 4)\chi(k : 0, 1, 2)^i(-1)^{i/2} \left\{ H_{h_{k-1} - 1} + (-1)^{k-1}H_{f_{k+1} - 1} \right\} \]

with

\[ E := \det(A''_{f_k} \cdots A''_{f_{k+1} - 2}) \]
\[ = \det(A'_{f_k} \cdots A'_{f_{k+1} - 2} - A'_{f_{k+1} + f_{k+2} - i - 2} \cdots A'_{f_{k+1} - 1}) \]
\[ = \det(A'_{f_{k+1} + f_{k+2} - i - 2} \cdots A'_{f_{k+1} - 1}) \]
\[ = (-1)^{f_{k+2} - i - 1}(f_{k+2} - i) \det(A'_{f_{k+1} + f_{k+2} - i - 2}) \]
\[ = (-1)^{f_{k+2} - i - 1}(f_{k+2} - i)H_{f_{k+2} - i} \]

where we have used Lemma 5. Therefore, by Theorem 3 and 4, we have

\[ H_{n,m} = \chi(k : 4)\chi(k : 0, 1, 2)^i(-1)^{i/2} \left\{ \chi(k - 1 : 1, 2, 4)_{f_{k-3}} \right. \]
\[ + (-1)^{k-1}\chi(k - 1 : 2, 3, 4, 5)_{f_{k-4}} \]
\[ \left. + (-1)^{k+1}(-1)^{f_{k-2} - i - 1}(f_{k-3} + i) \right\} \]
\[ = \chi(k : 2, 3, 4, 5)\chi(k : 0, 1, 2)^i(-1)^{i/2}f_{k-3}, \]

which completes the proof for \( H_{n,m} \).

\[ \square \]

4 Tiling for \( H_{n,m} \) and \( \mathcal{H}_{n,m} \)

In this section, we collect the values of \( H_{n,m} \) and \( \mathcal{H}_{n,m} \) obtained in the last section and arrange them in the quarter plane \( \Omega := \{0, 1, 2, \cdots\} \times \{1, 2, 3, \cdots\} \). We will tile \( \Omega \) by the following tiles on which the values \( H_{n,m} \) are written in. That is,

\[ U_1 := V_1 := \{(1, -1)\} \]
\[ U_k := \{(i, j) \in \mathbb{Z}^2; 0 \leq i + j \leq f_{k-1} - 1, \ -f_{k-1} \leq j < -1\} \]
\[ V_k := \{(i, j) \in \mathbb{Z}^2; 0 \leq i + j \leq f_{k-2} - 1, \ -f_{k-2} \leq j < -1\} \]
\[ (k = 2, 3, 4, \cdots) \]
with the written-in values $u_k : U_k \to \mathbb{Z}$, $v_k : V_k \to \mathbb{Z}$:
\[ u_1(1, -1) := 0, \quad v_1(1, -1) := 1 \]
\[
u_k(i, j) := \begin{cases} 
\chi(k : 2)\chi(k : 3, 4, 5)^i(-1)^{\lceil i/2 \rceil}f_{k-3} & (i + j = 0) \\
\chi(k : 0, 3, 4)\chi(k : 0, 3)^i f_{k-3} & (j = -f_{k-1}) \\
\chi(k : 3, 5)\chi(k : 2, 3, 4)^i(-1)^{\lceil i/2 \rceil}f_{k-3} & (i + j = f_{k-1} - 1) \\
\chi(k : 1, 2, 3, 5)\chi(k : 1, 4)^i f_{k-3} & (j = -1) \\
0 & (\text{otherwise})
\end{cases}
\]
\[
u_k(i, j) := \begin{cases} 
\chi(k : 1, 2, 4)\chi(k : 0, 1, 2)^i(-1)^{\lceil i/2 \rceil}f_{k-2} & (i + j = 0) \\
\chi(k : 2, 3, 5)\chi(k : 2, 5)^i f_{k-2} & (j = -f_{k-2}) \\
\chi(k : 0, 1, 2, 3)^i(-1)^{\lceil i/2 \rceil}f_{k-2} & (i + j = f_{k-2} - 1) \\
\chi(k : 0, 1)^i f_{k-2} & (j = -1) \\
0 & (\text{otherwise})
\end{cases}
\]
\[
( k = 2, 3, 4, \cdots ) ,
\]
\[
\overline{u}_k : U_k \to \mathbb{Z} \quad \text{and} \quad \overline{v}_k : V_k \to \mathbb{Z} : \]
\[
\overline{u}_1(1, -1) := 1, \quad \overline{v}_1(1, -1) := 0
\]
\[
\overline{u}_k(i, j) := \begin{cases} 
\chi(k : 1, 4)\chi(k : 0, 1, 2)^i(-1)^{\lceil i/2 \rceil}f_{k-4} & (i + j = 0) \\
\chi(k : 4)\chi(k : 0, 3)^i f_{k-4} & (j = -f_{k-1}) \\
\chi(k : 1, 2, 3, 4)^i(-1)^{\lceil i/2 \rceil}f_{k-4} & (i + j = f_{k-1} - 1) \\
\chi(k : 0, 1)^i f_{k-4} & (j = -1) \\
0 & (\text{otherwise})
\end{cases}
\]
\[
\overline{v}_k(i, j) := \begin{cases} 
\chi(k : 2)\chi(k : 3, 4, 5)^i(-1)^{\lceil i/2 \rceil}f_{k-3} & (i + j = 0) \\
\chi(k : 3)\chi(k : 2, 5)^i f_{k-3} & (j = -f_{k-2}) \\
\chi(k : 2, 4)\chi(k : 0, 4, 5)^i(-1)^{\lceil i/2 \rceil}f_{k-3} & (i + j = f_{k-2} - 1) \\
\chi(k : 1, 2, 3, 5)^i(-1)^{\lceil i/2 \rceil}f_{k-3} & (j = -1) \\
0 & (\text{otherwise})
\end{cases}
\]
\[
( k = 2, 3, 4, \cdots ).
\]
Let
\[
\mathcal{U}_k := \{(n, f_k); \ n \in \mathbb{N} \quad \text{and} \quad n \equiv_{k+1} 0\}
\]

35
\( \mathcal{V}_k := \{(n, f_k); n \in \mathbb{N} \text{ and } n \equiv_{k+2} f_{k+1} + f_{k-1}\} \)

\( T_k := (V_k + (-f_{k-2}, f_k)) \cap \Omega \)

\( (k = 1, 2, 3, \ldots) \),

where \( V + (x, y) := \{v + x, w + y\}; (v, w) \in V \) for \( V \subset \mathbb{Z}^2 \), \( (x,y) \in \mathbb{Z}^2 \).

**Theorem 5** It holds that

\[
\Omega = \bigcup_{k=1}^{\infty} \left( \bigcup_{(i,j) \in U_k} (U_k + (i, j)) \cup \bigcup_{(i,j) \in V_k} (V_k + (i, j)) \cup T_k \right),
\]

where the right hand side is a disjoint union, so that \( \Omega \) is tiled by the tiles \( U_k \)'s, \( V_k \)'s and \( T_k \)'s. Moreover, for any \((n, m) \in \mathcal{V}_k\), if \((n, m) = (i, j) + (i', j')\) with \((i, j) \in U_k\) and \((i', j') \in U_k\), then we have \( H_{n,m} = u_k(i, j) \) and \( \overline{H}_{n,m} = u_k(i, j) \). Also, if \((n, m) = (i, j) + (i', j')\) with \((i, j) \in V_k\) and either \((i', j') \in \mathcal{V}_k\) or \((i', j') = (-f_{k-2}, f_k)\), then we have \( H_{n,m} = v_k(i, j) \) and \( \overline{H}_{n,m} = v_k(i, j) \). Furthermore, in this tiling, the tiles \( U_k\), \( V_k\) and \( T_k\) with \( k \geq 2 \) are followed by the sequences of smaller tiles \( U_{k-1}V_{k-1}U_{k-1}, U_{k-1} \) and \( U_{k-1}\), respectively, as shown in Figure 1.

**Proof.** Take an arbitrary point \((n, m) \in \mathcal{V}_k\). Let \( f_{k-1} \leq m < f_k \). If \( n + m - f_k \geq 0\), define \( 0 \leq i < f_{k+2} \) by \( i \equiv_{k+2} n \).

**Case 1** \( n + m - f_k < 0 \): We get \((n, m) \in T_k\).

**Case 2** \( 0 \leq i < f_{k-1} \): We get \((n, m) \in U_k + (n + m - i - f_k, f_k)\).

**Case 3** \( f_{k-1} \leq i < f_{k+1} \): We get \((n, m) \in U_{k+1} + (n + m - i - f_{k+1}, f_{k+1})\).

**Case 4** \( f_{k+1} \leq i < f_{k+1} + f_{k-1} \): We get \((n, m) \in U_k + (n + m - i + f_{k-1}, f_k)\).

**Case 5** \( f_{k+1} + f_{k-1} \leq i < f_{k+2} \): We get \((n, m) \in V_k + (n + m - i + 2f_{k-1}, f_k)\).

The fact that the written-in values coincide with \( H_{n,m} \) and \( \overline{H}_{n,m} \) follows from Lemma 18 (first case in \( u_k \) and \( \overline{u}_k \)), Theorem 3 (second case), Lemma 21 (third case), Theorem 4 (fourth case), Corollary 3 (fifth case), Lemma 19 (first case in \( v_k \) and \( \overline{v}_k \)), Theorem 3 (second case), Lemma 20 (third case), Lemma 20 (fourth case) and Corollary 3 (fifth case). The \( m \) in the preceding lemmas and theorems coincides with \( f_k + j \) in Theorem 5 while the meanings of the symbols \( k, i, n \) are not necessarily the same between them. 

\( \blacksquare \)
Figure 1: Tiling for $H_{m,n}$

5 Padé approximation

Let $\varphi = \varphi_0 \varphi_1 \varphi_2 \cdots$ be an infinite sequence over a field $K$, $\hat{H}_{n,m} := H_{n,m}(\varphi)$ be the Hankel determinant (3), and $\varphi(z)$ the formal Laurent series (4) with $h = -1$. We also denote the **Hankel matrices** by

\[
\hat{M}_{n,m} := (\varphi_{n+i+j})_{i,j=0,1,\ldots,m-1}
\]

so that $\hat{H}_{n,m} = \det \hat{M}_{n,m}$.

The following proposition is well known ([1], for example). But we give a proof for self-containedness.

**Proposition 1**

(1) For any $m = 1, 2, \cdots$, a Padé pair $(P, Q)$ of order $m$ for $\varphi$ exists. Moreover, for each $m$, the rational function $P/Q \in K(z)$ is determined uniquely for such Padé pairs $(P, Q)$.

(2) For any $m = 1, 2, \cdots$, $m$ is a normal index for $\varphi$ if and only if $\hat{H}_{0,m}(\varphi) \neq 0$.

37
Proof. Let
\[
P = p_0 + p_1 z + p_2 z^2 + \cdots + p_m z^m
\]
\[
Q = q_0 + q_1 z + q_2 z^2 + \cdots + q_m z^m.
\]
Then, the condition \( \| Q \varphi - P \| < \exp(-m) \) is equivalent to
\[
\begin{array}{cccc}
p_m \varphi_0 & -p_m & = & 0 \\
q_m \varphi_0 & -p_m-1 & = & 0 \\
\cdot & \cdot & \cdot & \cdot \\
q_0 \varphi_0+ & q_1 \varphi_1+ & \cdots & q_m \varphi_{m-1} - p_0 = 0 \\
\cdot & \cdot & \cdot & \cdot \\
q_0 \varphi_{m-1}+ & q_1 \varphi_{m-2}+ & \cdots & q_m \varphi_{2m-1} = 0.
\end{array}
\]
Furthermore, Equation (26) for \((q_0q_1 \cdots q_m)\) is equivalent to
\[
(q_0q_1 \cdots q_{m-1}) \tilde{M}_{0,m} + q_m (\varphi_m \varphi_{m+1} \cdots \varphi_{2m-1}) = (00 \cdots 0),
\]
where \((p_0p_1 \cdots p_m)\) is determined by \((q_0q_1 \cdots q_m)\) by the upper half of Equation (26). There are two cases.

Case 1: \( \hat{H}_{0,m} = 0 \). In this case, since \( \det \tilde{M}_{0,m} = \hat{H}_{0,m} = 0 \), there exists a nonzero vector \((q_0q_1 \cdots q_{m-1})\) such that \((q_0q_1 \cdots q_{m-1}) \tilde{M}_{0,m} = 0 \). Then, Equation (27) is satisfied with this \((q_0q_1 \cdots q_{m-1})\) and \( q_m = 0 \).

Case 2: \( \hat{H}_{0,m} \neq 0 \). In this case, since \( \det \tilde{M}_{0,m} = \hat{H}_{0,m} \neq 0 \), there exists a unique vector \((q_0q_1 \cdots q_{m-1})\) such that
\[
(q_0q_1 \cdots q_{m-1}) \tilde{M}_{0,m} = -(\varphi_m \varphi_{m+1} \cdots \varphi_{2m-1}).
\]
Then, (27) is satisfied with this \((q_0q_1 \cdots q_{m-1})\) and \( q_m = 1 \).

Thus, a Padé pair of order \( m \) exists. Moreover, by the above arguments, a Padé pair \((P, Q)\) of order \( m \) with \( \deg Q < m \) exists if and only if \( \hat{H}_{0,m} = 0 \), since if \( \hat{H}_{0,m} \neq 0 \), then by (27), \( q_m = 0 \) implies \((q_0q_1 \cdots q_{m-1}) = (00 \cdots 0) \) and hence, \( Q = 0 \).

Now we prove that for any Padé pairs \((P, Q)\) and \((P', Q')\) of order \( m \), it holds \( P/Q = P'/Q' \). By (5), we have
\[
\| \varphi - P/Q \| < \exp(-n - \deg Q)
\]
and
\[ \| \varphi - P'/Q' \| \leq \exp(-m - \deg Q'). \]

Hence, we have
\[ \| P/Q - P'/Q' \| \leq \exp(-m - \deg Q \land \deg Q'). \]

Therefore,
\[ \| PQ' - P'Q \| \leq \exp(-m + \deg Q \lor \deg Q') \leq 1. \]

Since \( PQ' - P'Q \) is a polynomial of \( z \), \( \| PQ' - P'Q \| \) is either 0 or not less than 1. Hence, the above inequality implies \( PQ' - P'Q = 0 \), which completes the proof.

In view of (26), without loss of generality, we can put
\[
P = p_0 + p_1 z + p_2 z^2 + \cdots + p_{m-1} z^{m-1},
\]
\[
Q = q_0 + q_1 z + q_2 z^2 + \cdots + q_m z^m. \tag{29}
\]

**Theorem 6** Let \((P, Q)\) be the normalized Padé pair for \( \varphi \) with \( \deg Q \) as its normal index \( m \) with \( P, Q \) given by (29). Then, we have
\( (1) \ Q(z) = \tilde{H}_{0,m}^{-1} \det(z \tilde{M}_{0,m} - \tilde{M}_{1,m}). \)
\( (2) \ \det(zI - \tilde{M}_{0,m}) = \)
\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
p_0 & \cdots & p_m & q_0 & \cdots & q_{m-1} \\
p_0 & \cdots & p_{m-2} & p_m & q_1 & \cdots & q_{m-1} \\
p_0 & \cdots & p_{m-3} & p_{m-1} & q_2 & \cdots & q_{m-1} \\
p_0 & \cdots & p_{m-4} & p_{m-2} & q_3 & \cdots & q_{m-1} \\
p_0 & \cdots & p_{m-5} & p_{m-3} & q_4 & \cdots & q_{m-1} \\
p_0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
p_0 & \cdots & p_{m-1} & q_{m-2} & q_{m-1} & 1 \\
p_0 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
p_0 & \cdots & p_{m-2} & q_m & 1 \\
p_0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
p_0 & p_m & q_0 & q_1 & q_2 & \cdots & 1 \\
p_0 & p_m & q_0 & q_1 & q_2 & \cdots & 1 \\
p_0 & p_m & q_0 & q_1 & q_2 & \cdots & 1 \\
p_0 & p_m & q_0 & q_1 & q_2 & \cdots & 1 \\
p_0 & p_m & q_0 & q_1 & q_2 & \cdots & 1 \\
\end{array}
\]
where $I$ is the unit matrix of size $m$.

(3)  
\[ \hat{H}_{0,m} = (-1)^{[m/2]} \prod_{z:Q(z)=0} P(z) = (-1)^{[m/2]} p_k^m \prod_{z:P(z)=0} Q(z), \]

where $\prod_{z:R(z)=0}$ denotes the product over all the roots of the polynomial $R(z)$ with their multiplicity and $p_k$ is the leading coefficient of $P(z)$, that is, $p_{m-1} = \cdots = p_{k+1} = 0$, $p_k \neq 0$ if $P(z)$ is not the zero polynomial, otherwise, $p_k = 0$.

Proof. (1) Note that $q_m = 1$ by the assumption that $(P, Q)$ is the normalized Padé pair. By (28), we have

\[
\begin{pmatrix}
0 & 1 & 0 & 1 \\
0 & 1 & \ddots & 0 & 1 \\
-\phi_0 & -q_1 & \cdots & -q_{m-2} & -q_{m-1} \\
\end{pmatrix}
\hat{M}_{0,m} = \hat{M}_{1,m}.
\]

Since $\hat{H}_{0,m} = \det \hat{M}_{0,m} \neq 0$ by the normality of the index $m$, it follows that

\[
Q(z) = \det \left( zI - \begin{pmatrix}
0 & 1 & 0 & 1 \\
0 & 1 & \ddots & 0 & 1 \\
-\phi_0 & -q_1 & \cdots & -q_{m-2} & -q_{m-1} \\
\end{pmatrix}\right)
= \det(zI - \hat{M}_{1,m} \hat{M}_{0,m}^{-1})
= \hat{H}_{0,m}^{-1} \det(z \hat{M}_{0,m} - \hat{M}_{1,m}).
\]
(2) We define the matrices:

\[ P_m := \begin{pmatrix} p_{m-1} & p_{m-2} & \cdots & p_1 & p_0 \\ p_{m-2} & \cdots & \cdots & p_0 \\ \vdots & \ddots & \ddots & \vdots \\ p_1 & \ddots & 0 \\ p_0 \end{pmatrix} \]

\[ P'_{m-1} := \begin{pmatrix} 0 & p_{m-1} & p_{m-2} \\ p_{m-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ p_{m-1} & p_{m-2} & \ddots & p_2 \\ p_{m-1} & p_{m-2} & \ddots & p_1 \end{pmatrix} \]

\[ Q_m := \begin{pmatrix} 1 & 1 & 0 \\ q_{m-1} & 1 & \vdots \\ \vdots & \ddots & \ddots \\ q_1 & q_2 & \cdots & q_{m-1} & 1 \end{pmatrix} \]

\[ Q'_m := \begin{pmatrix} 0 & 1 & q_{m-1} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & q_{m-1} & \cdots & q_2 & q_1 \end{pmatrix} \]

\[ Q''_{m-1} := \begin{pmatrix} 1 & q_{m-1} & 0 \\ q_{m-1} & 1 & \vdots \\ \vdots & \ddots & \ddots \\ q_2 & q_3 & \cdots & q_{m-1} & 1 \end{pmatrix} \]
\[
Q_{m,m-1} := \begin{pmatrix}
q_1 & q_2 & \cdots & q_{m-2} & q_{m-1} \\
\varphi_0 & q_1 & \cdots & q_{m-3} & q_{m-2} \\
\varphi_0 & q_1 & \cdots & q_{m-3} & \ddots \\
\ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & q_1 & \varphi_0 & q_0
\end{pmatrix}
\]

\[
\phi_{m-1} := \begin{pmatrix}
0 & \varphi_0 & \varphi_1 \\
\ddots & \ddots & \vdots \\
\ddots & \ddots & \ddots & \ddots & \varphi_{m-3} \\
\varphi_0 & \varphi_1 & \cdots & \varphi_{m-3} & \varphi_{m-2}
\end{pmatrix}
\]

We denote by \( O \) the zero matrices of various sizes. We also denote by \( I_n \) the unit matrix of size \( n \). By (26), we have

\[
\det(zI - \hat{M}_{0,m}) = \det(z\begin{pmatrix} O & O \\ O & I_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \\ Q_{m}^{-1}Q_{m,m-1} & \hat{M}_{0,m} \end{pmatrix})
\]

\[
= \det\left(z\begin{pmatrix} O & O \\ O & I_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \\ Q_{m}^{-1}Q_{m,m-1} & \hat{M}_{0,m} \end{pmatrix}\right)
\]

\[
= \det\left(z\begin{pmatrix} O & O \\ O & Q_{m} \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \\ Q_{m,m-1} & Q_{m}\hat{M}_{0,m} \end{pmatrix}\right)
\]

\[
= \det\left(z\begin{pmatrix} O & O \\ O & Q_{m} \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \\ Q_{m,m-1} & Q_{m}\hat{M}_{0,m} \end{pmatrix}\right)\begin{pmatrix} I_{m-1} & O \phi_{m-1} \\ O & I_m \end{pmatrix}
\]

\[
= \det\left(z\begin{pmatrix} O & O \\ O & Q_{m} \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \\ Q_{m,m-1} & P_{m} \end{pmatrix}\right),
\]

where we use (26) to get the last equality. Hence

\[
\det(zI - \hat{M}_{0,m}) = \det\left(z\begin{pmatrix} O & O \\ O & Q_{m} \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \\ Q_{m,m-1} & P_{m} \end{pmatrix}\right)
\]

\[
= \det\left(z\begin{pmatrix} O & O \\ O & Q_{m} \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \\ Q_{m,m-1} & P_{m} \end{pmatrix}\right)
\]

\[
= \det\left(z\begin{pmatrix} Q_{m-1}^{-1} & O \\ O & I_m \end{pmatrix}\begin{pmatrix} O & O \\ O & Q_{m} \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \\ Q_{m,m-1} & P_{m} \end{pmatrix}\right)
\]

42
\[ = \det \left( z \begin{pmatrix} O & O \\ O & Q_m \end{pmatrix} - \begin{pmatrix} -Q'_m & O \\ Q_m & P_m \end{pmatrix} \right) \]
\[ = (-1)^m \det \left( \begin{pmatrix} Q'_m & O \\ Q_m & P_m \end{pmatrix} - z \begin{pmatrix} O & O \\ O & Q_m \end{pmatrix} \right) \]
\[ = (-1)^m \det \left( \begin{pmatrix} I_m & O \\ Q'_m & Q_m \end{pmatrix} - zI_m \right), \]
which implies (2).

(3) By (2), we have
\[ \hat{H}_{0,m} = (-1)^m \det (0I - \hat{M}_{0,m}) \]
\[ = (-1)^{m/2} \begin{bmatrix} p_{m-1} & 1 \\ p_{m-2} & p_{m-1} \\ \vdots & \ddots \\ p_1 & \cdots & p_{m-1} & q_{m-1} \\ p_0 & \cdots & p_{m-2} & p_{m-1} & q_{1} & q_{m-1} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ p_0 & p_1 & \cdots & q_1 \\ p_0 & q_0 \end{bmatrix}, \]
which completes the proof since the determinant in the last-side in the above equality is Sylvester’s determinant for \( P(z) \) and \( Q(z) \).

For a finite or infinite sequence \( a_0(z), a_1(z), a_2(z), \ldots \) of elements in \( K((z^{-1})) \), we use the notation
\[ [a_0(z); a_1(z), a_2(z), \ldots, a_n(z)] := a_0(z) + \frac{1}{a_1(z) + \frac{1}{a_2(z) + \cdots + \frac{1}{a_n(z)}}} \]
and
\[ [a_0(z); a_1(z), a_2(z), \ldots] := \lim_{n \to \infty} [a_0(z); a_1(z), a_2(z), \ldots, a_n(z)] \quad (30) \]
provided that the limit exists, where the limit is taken with respect to the metric induced by the non-Archimedean norm in $\mathbf{K}((z^{-1}))$.

We define

\[
p_0(z) = a_0(z), \quad p_{-1}(z) = 1, \quad q_0(z) = 1, \quad q_{-1}(z) = 0
\]

\[
p_n(z) = a_n(z)p_{n-1}(z) + p_{n-2}(z)
\]

\[
q_n(z) = a_n(z)q_{n-1}(z) + q_{n-2}(z)
\]

\[(n = 1, 2, 3, \cdots)\] (31)

for any given sequence $a_1(z), a_2(z), \cdots \in \mathbf{K}((z^{-1}))$. Then $p_n(z), q_n(z) \in \mathbf{K}((z^{-1}))$, $p_n(z) \neq 0$ if $q_n(z) = 0$, and

\[
\frac{p_n(z)}{q_n(z)} = [a_0(z); a_1(z), a_2(z), \cdots, a_n(z)] \in \mathbf{K}((z^{-1})) \cup \{\infty\} \quad (n \geq 0)
\]

holds, where we mean $\psi/0 := \infty$ for $\psi \in \mathbf{K}((z^{-1})) \setminus \{0\}$, and $\psi + \infty := \infty$, $\psi/\infty := 0$ for $\psi \in \mathbf{K}((z^{-1}))$. By using (31), it can be shown that the limit (30) always exists in the set $\mathbf{K}((z^{-1}))$ as far as

\[
a_n(z) \in \mathbf{K}[z] \quad (n \geq 0), \quad \deg a_n(z) \geq 1 \quad (n \geq 1). \quad (32)
\]

For $\varphi(z) \in \mathbf{K}((z^{-1}))$ given by (4), we denote by $[\varphi(z)]$ the polynomial part of $\varphi(z)$, which is defined as follows:

\[
[\varphi(z)] := \sum_{k=0}^{h} \varphi_h z^{-k+h} \in \mathbf{K}[z].
\]

By $T$, we denote the mapping $T : \mathbf{K}((z^{-1})) \setminus \{0\} \to \mathbf{K}((z^{-1}))$ defined by

\[
T(\psi(z)) := \frac{1}{\psi(z)} - [\frac{1}{\psi(z)}] \quad (\psi(z) \in \mathbf{K}((z^{-1})) \setminus \{0\}).
\]

Then, for any given $\varphi(z) \in \mathbf{K}((z^{-1}))$, we can define the continued fraction expansion of $\varphi(z)$:

\[
\varphi(z) = \begin{cases} 
[ a_0(z); a_1(z), a_2(z), \cdots, a_{N-1}(z) ] & \text{if } \varphi(z) \in \mathbf{K}(z) \\
[ a_0(z); a_1(z), a_2(z), a_3(z), \cdots ] & \text{otherwise}
\end{cases} \quad (33)
\]

with $a_n(z)$ satisfying (32) according to the following algorithm.
Continued Fraction Algorithm:

\[ a_0(z) = \lfloor \varphi(z) \rfloor, \quad a_n(z) = \left\lfloor \frac{1}{T_{n-1}(\varphi(z) - a_0(z))} \right\rfloor \]

\[ N = N(\varphi(z)) := \inf \{ m ; T_{m-1}(\varphi(z)) = 0 \} \quad (\inf \emptyset := \infty). \]

We note that if \( \varphi(z) \in K(z) \), then \( N < \infty \); if \( \varphi(z) \in K((z^{-1})) \setminus K(z) \), then \( N = \infty \) and the continued fraction (33) converges to the given \( \varphi(z) \in K(z) \).

We say a continued fraction is admissible if it is obtained by the algorithm. We remark that a continued fraction (33) is admissible if and only if (32) holds.

The following proposition is known [2], but we give a proof for completeness.

**Proposition 2** The set of all \( P/Q \in K(z) \) for Padé pairs \((P,Q)\) for \( \varphi(z) \in K((z)) \) coincides with the set of convergents \( p_n(z)/q_n(z) \) \((0 \leq n < N)\) of the continued fraction expansion of \( \varphi(z) \). Moreover, \( m \) is a normal index if and only if \( m \) is a degree of \( q_n(z) \) for some \( n = 0, 1, 2, \ldots \) (with \( n < N \) if \( \varphi(z) \in K(z) \)).

**Proof.** Note that

\[ \varphi(z) = \frac{(a_n(z) + T^n(\varphi(z) - a_0(z)))p_{n-1}(z) + p_{n-2}(z)}{(a_n(z) + T^n(\varphi(z) - a_0(z)))q_{n-1}(z) + q_{n-2}(z)} \]

\[ (-1)^n = p_{n-1}(z)q_{n-2}(z) - p_{n-2}(z)q_{n-1}(z). \]

Hence, we have

\[ \| q_n(z)\varphi(z) - p_n(z) \| \]

\[ = \left\| \frac{(-1)^n T^n(\varphi(z) - a_0(z))}{q_n(z) + T^n(\varphi(z) - a_0(z))q_{n-1}(z)} \right\| \]

\[ = \exp(-\deg a_{n+1}(n) - \deg q_n(z)), \]

so that

\[ \| q_n(z)\varphi(z) - p_n(z) \| < \exp(-\deg q_n(z)) \quad (n < N). \quad (34) \]

In the case \( N < \infty \), the left-hand side of (34) turns out to be 0 for \( n = N - 1 \). Therefore, \( (p_n(z), q_n(z)) \) is a Padé pair of order \( m = \deg q_n(z) \) for all \( m \in \{ \deg q_n(z) ; 0 \leq n < N \} \).
Conversely, for any $k = 1, 2, \cdots$, let $(P, Q)$ be a Padé pair of order $k$. Let \( \deg q_n(z) \leq k < \deg q_{n+1} \) for some $n = 0, 1, 2, \cdots$ with $n < N$ (\( \deg q_N(z) := \infty \)). Then, since $\deg Q \leq k < \deg q_{n+1}$, it follows from (34) that

\[
\| \varphi(z) - \frac{p_n(z)}{q_n(z)} \| = \exp(-\deg q_n(z) - \deg q_{n+1}(z)) \\
< \exp(-\deg q_n(z) - \deg Q).
\]

Since $(P, Q)$ be a Padé pair of order $k$, we have

\[
\| \varphi(z) - \frac{P}{Q} \| < \exp(-k - \deg Q) \\
\leq \exp(-\deg q_n(z) - \deg Q).
\]

Therefore, we have

\[
\| \frac{P}{Q} - \frac{p_n(z)}{q_n(z)} \| < \exp(-\deg q_n(z) - \deg Q).
\]

On the other hand, if $P/Q \neq p_n(z)/q_n(z)$, then

\[
\| \frac{P}{Q} - \frac{p_n(z)}{q_n(z)} \| = \| \frac{Pq_n(z) - Qp_n(z)}{Qq_n(z)} \| \\
\geq \exp(-\deg q_n(z) - \deg Q),
\]

which is a contradiction. Thus we have $P/Q = p_n(z)/q_n(z)$.

Note that $p_n(z)/q_n(z)$ is irreducible for any $n = 1, 2, \cdots$ with $n < N$, since $p_nq_{n-1} - p_{n-1}q_n = (-1)^{n-1}$. Let $m = \deg q_n(z)$ for some $n = 1, 2, \cdots$ with $n < N$. Take any Padé pair $(P, Q)$ of order $m$. Then $\deg Q \leq m$. On the other hand, by the above argument, we have $P/Q = p_n(z)/q_n(z)$. Since $p_n(z)/q_n(z)$ is irreducible, this implies that $\deg Q \geq \deg q_n(z) = m$. Thus, $m$ is a normal index.

Conversely, let $m \geq 0$ be any normal index. Take any Padé pair $(P, Q)$ of order $m$. Then, by the above argument, there exists $n = 0, 1, 2, \cdots$ with $n < N$ such that $P/Q = p_n(z)/q_n(z)$. Hence the irreducibility of $p_n(z)/q_n(z)$ implies $\deg q_n(z) \leq \deg Q(\leq m)$. Hence, $(p_n(z), q_n(z))$ is a Padé pair of order $m$. Since $m$ is a normal index, $\deg q_n(z) = m$. ■
Let us obtain the continued fraction expansions for
\[
\varphi_\varepsilon(z) = \hat{\varepsilon}_0 z^{-1} + \hat{\varepsilon}_1 z^{-2} + \hat{\varepsilon}_2 z^{-3} + \cdots \in \mathbb{Q}((z^{-1}))
\]
corresponding to the Fibonacci words \( \varepsilon = \varepsilon(a, b) \) with \((a, b) = (1, 0)\) and \((a, b) = (0, 1)\). As in §3, we use the notations \( \varepsilon \) and \( \overset{\sim}{\varepsilon} \) for them. The proofs in the following theorems are given only for \( \varepsilon \), since the proof is similar for \( \overset{\sim}{\varepsilon} \).

In [3], J. Tamura gave the Jacobi-Perron-Parusnikov expansion for a vector consisting of Laurent series with coefficients given by certain substitutions, which contains the following as its special case, cf. the footnote, p. 301 [3]:

**Proposition 3** It holds that
\[
(z - 1)\varphi_\varepsilon(z) = [0; z; z; z + 1; z; z + 1; z; z + 1; \cdots].
\]

**Theorem 7** We have the following admissible continued fraction for \( \varphi_\varepsilon(z) \) and \( \varphi_{\overset{\sim}{\varepsilon}}(z) \):
\[
\begin{align*}
\varphi_\varepsilon(z) &= [0; a_1, a_2, a_3, \cdots] \\
\varphi_{\overset{\sim}{\varepsilon}}(z) &= [0; \overline{a}_1, \overline{a}_2, \overline{a}_3, \cdots]
\end{align*}
\]
with
\[
a_1 = z, \quad a_2 = -z + 1, \quad a_3 = \frac{1}{2}(z + 1)
\]
\[
a_{2n+2} = (-1)^{n-1} f_n^2 (z f_n - 1 + z f_{n-2} + \cdots + 1)
\]
\[
a_{2n+3} = (-1)^{n-1} \frac{1}{f_n f_{n+1}} (z - 1)
\]
\[\quad (n = 1, 2, \cdots),\]
and
\[
\begin{align*}
\overline{a}_1 &= z^2, \quad \overline{a}_2 = -z, \\
\overline{a}_{2n+1} &= (-1)^{n-1} f_n^2 (z f_n - 1 + z f_{n-2} + \cdots + 1) \\
\overline{a}_{2n+2} &= (-1)^{n-1} \frac{1}{f_n f_{n-1}} (z - 1)
\end{align*}\]
\[\quad (n = 1, 2, \cdots).\]
Proof. We put

\[
\begin{align*}
\theta_n &:= [0 ; \ z^{f_n}, z^{f_{n+1}}, z^{f_{n+2}}, \ldots] \quad (n \geq -2) \\
\xi_n &:= (-1)^{n-1} \frac{f_n^2 z^{f_n} + f_{n-1} f_n + f_{n+1}^2 \theta_{n+1}}{z-1} \quad (n \geq 1) \\
\eta_n &:= (-1)^{n-1} \frac{z-1}{f_n f_{n+1} + f_{n+1}^2 \theta_{n+1}} \quad (n \geq 1) \\
c_n &:= (-1)^{n-1} f_n^2 (z^{f_n-1} + z^{f_n-2} + \ldots + 1) \quad (n \geq 1) \\
d_n &:= (-1)^{n-1} \frac{1}{f_n f_{n+1}} (z-1) \quad (n \geq 1).
\end{align*}
\]

Then we have

\[
\xi_n = [c_n ; \eta_n] = c_n + \frac{1}{\eta_n}, \quad \eta_n = [d_n ; \xi_n] \quad \text{(35)}
\]

Using

\[
\theta_n^{-1} = z^{f_n} + \theta_{n+1}
\]

and Proposition 3, we get

\[
\varphi(z) = \frac{z - \theta_{-2}}{z - 1} \quad (\| z^{-2}/(z - 1) \| < 1)
\]

\[
= [0 ; (z - 1) \theta_{-2}^{-1}]
\]

\[
= [0 ; z - 1 + (z - 1) \theta_{-1}] \quad (\| -1 + (z - 1) \theta_{-1} \| < 1)
\]

\[
= [0 ; z, \frac{\theta_{-1}^{-1}}{-\theta_{-1}^{-1} + z - 1}]
\]

\[
= [0 ; z, \frac{z + \theta_0}{-1 - \theta_0}]
\]

\[
= [0 ; z, -z + 1 + \frac{1 + (z + 2) \theta_0}{-1 - \theta_0}]
\]

\[
\quad (\| \frac{1 + (z + 2) \theta_0}{-1 - \theta_0} \| < 1)
\]

\[
= [0 ; z, -z + 1, \frac{-1 - \theta_0^{-1}}{-z + 2 + \theta_0^{-1}}]
\]

\[
= [0 ; z, -z + 1, \frac{-z - 1 - \theta_1}{2 + \theta_1}]
\]
\[
\begin{align*}
= \left[0 ; z, \ -z + 1, \ -\frac{1}{2}(z + 1), \ \frac{4\theta_1^{-1} + 2}{z - 1} \right] \\
= \left[0 ; z, \ -z + 1, \ -\frac{1}{2}(z + 1), \ \frac{4z + 2 + 4\theta_2}{z - 1} \right].
\end{align*}
\]

Hence, we have
\[
f(z) = \left[0 ; z, \ -z + 1, \ -\frac{1}{2}(z + 1), \ \xi_1 \right] \quad \text{(||} \xi_1^{-1} || < 1). \quad (36)
\]

From (35) and (36), it follows that
\[
f(z) = \left[0 ; z, \ -z + 1, \ -\frac{1}{2}(z + 1) \ c_1, \ d_1, \ \cdots, \ c_n, \ d_n, \ \xi_{n+1} \right] \\
= \left[0 ; z, \ -z + 1, \ -\frac{1}{2}(z + 1) \ c_1, \ c_2, \ d_2, \ \cdots \right]
\]
which completes the proof for \( \varphi(z) \).

Starting from the identity \( \varphi(z) = \frac{1-\theta_2}{z-1} \) instead of \( \varphi(z) = \frac{\theta_2}{z-1} \), we can get the admissible continued fraction for \( \varphi(z) \) by the similar fashion as above.

\textbf{Theorem 8} The numerator \( p_n := p_n(z) \) (\( \overline{p}_n := \overline{p}_n(z) \), resp.) and the denominator \( q_n := q_n(z) \) (\( \overline{q}_n := \overline{q}_n(z) \), resp.) of the \( n \)-th convergent of the continued fraction expansion for \( \varphi(z) \) (and \( \varphi(z) \), resp.) are given as follows:

\[
\begin{align*}
p_0 &= 0, \quad p_1 = 1, \quad p_2 = -z + 1 \\
q_0 &= 1, \quad q_1 = z, \quad q_2 = -z^2 + z + 1 \\
p_{2n-1} &= \frac{1}{f_n} \left( \varepsilon_0 z^{f_{n-1}} + \varepsilon_1 z^{f_{n-2}} + \cdots + \varepsilon_{f_{n-1}} \right) \\
p_{2n} &= (-1)^n \left\{ f_n \left( \varepsilon_0 z^{f_{n-1}} + \varepsilon_1 z^{f_{n-2}} + \cdots + \varepsilon_{f_{n-1}} \right) - f_{n-1} \left( \varepsilon_0 z^{f_{n-1}} + \varepsilon_1 z^{f_{n-2}} + \cdots + \varepsilon_{f_{n-1}} \right) \right\} / (z - 1) \\
q_{2n-1} &= \frac{1}{f_n} (z^{f_n} - 1) \\
q_{2n} &= (-1)^n \left\{ f_n \left( z^{f_{n-1}} + z^{f_{n-2}} + \cdots + 1 \right) - f_{n-2} \left( z^{f_{n-1}} + z^{f_{n-2}} + \cdots + 1 \right) \right\} / (z - 1) \\
( & n = 2, 3, \cdots)
\end{align*}
\]
and
\[
\begin{align*}
\overline{p}_0 &= 0, \quad \overline{p}_1 = 1 \\
\overline{q}_0 &= 1, \quad \overline{q}_1 = z^2 \\
\overline{p}_{2n-2} &= -\frac{1}{f_{n-2}} (\overline{p}_0 z^{f_{n-1}} + \overline{p}_1 z^{f_{n-2}} + \cdots + \overline{p}_{f_{n-1}-1}) \\
\overline{p}_{2n-1} &= (-1)^{n-1} \left\{ f_{n-2} z^{f_{n-1}} (\overline{p}_0 z^{f_{n-1}} + \overline{p}_1 z^{f_{n-2}} + \cdots + \overline{p}_{f_{n-1}-1}) \
- f_{n-3} (\overline{p}_0 z^{f_{n-1}} + \overline{p}_1 z^{f_{n-2}} + \cdots + \overline{p}_{f_{n-1}-1}) \right\} \frac{1}{(z - 1) + f_{n-2}} \\
\overline{q}_{2n-2} &= -\frac{1}{f_{n-2}} (z^{f_{n}} - 1) \\
\overline{q}_{2n-1} &= (-1)^{n-1} \left\{ f_{n-2} z^{f_{n-1}} (z^{f_{n-1}} + z^{f_{n-2}} + \cdots + 1) \
- f_{n-3} (z^{f_{n-1}} + z^{f_{n-2}} + \cdots + 1) \right\} \quad (\ n = 2, 3, \cdots ) ,
\end{align*}
\]

where \( p_{2n} \) and \( \overline{p}_{2n-1} \) in the above are polynomials since the numerators are divisible by \( z - 1 \).

**Proof.** The values for \( p_0, p_1, p_2, q_0, q_1, q_2 \) are obtained from Theorem 7 by direct calculations. For a general \( n \), we can prove the formula for \( p_n, q_n \) by induction on \( n \) using (31) and Theorem 7 without difficulty.

**Remark 4** From Proposition 2 and Theorem 8, it follows that the set of normal indices for \( \varphi(z) \) (and \( \varphi(z) \) resp.) is \( \{0, f_0 = f_1 - 1, f_1 = f_2 - 1, f_2 = f_3 - 1, \cdots \} \) (\( \{0, f_1 = f_2 - 1, f_2 = f_3 - 1, \cdots \} \), resp.) which together with Proposition 1 give another proof of the third cases of Theorem 2 with \( n = 0 \).

**Remark 5** In [4], the continued fraction expansion for Laurent series corresponding to infinite words over \( \{a, b\} \) generated by substitutions of “Fibonacci type” are considered, where \( a, b \) will be considered as independent variables.

**References**


Subject classification: 41A21, 11B39