Hankel determinants for the Fibonacci word and Padé approximation

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1 Introduction

The aim of the paper is to give a concrete and interesting example of the Padé approximation theory as well as to develop the general theory so as to find a quantitative relation between the Hankel determinant and the Padé pair. Our example is the formal power series related to the Fibonacci word.

The **Fibonacci word** $\varepsilon(a,b)$ on an alphabet $\{a,b\}$ is the infinite sequence

$$\begin{aligned} \varepsilon(a,b) &= \hat{\varepsilon}_0\hat{\varepsilon}_1\cdots\hat{\varepsilon}_n\cdots \\ &:= abaababaabaab\cdots \end{aligned} \quad (\hat{\varepsilon}_n \in \{a,b\}) \tag{1}$$

which is the fixed point of the substitution

$$\sigma : \begin{array}{ccc} a & \to & ab \\ b & \to & a \end{array} \tag{2}$$

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The **Hankel determinants** for an infinite word (or sequence) $\varphi = \varphi_0\varphi_1\varphi_2\cdots$ ($\varphi_n \in \mathbf{K}$) over a field $\mathbf{K}$ are the following

$$\begin{array}{rl} H_{n,m}(\varphi) & := \det(\varphi_{n+i+j})_{0\le i,j\le m-1} \\ & (n = 0,1,2,\cdots;\ m = 1,2,\cdots). \end{array} \tag{3}$$

It is known [2] that the Hankel determinants play an important role in the theory of Padé approximation for the formal Laurent series

$$\varphi(z) = \sum_{k=0}^{\infty} \varphi_k z^{-k+h}. \tag{4}$$

Let $\mathbf{K}((z^{-1}))$ be the set of formal Laurent series φ as above of z with coefficients in $\mathbf{K}$ and $h \in \mathbf{Z}$ providing a non-Archimedean norm $\| \varphi \| := \exp(-k_0 + h)$ with $k_0 = \inf\{k; \varphi_k \neq 0\}$. Let φ be as above with $h = -1$. We say that a pair $(P,Q) \in \mathbf{K}[z]^2$ of polynomials of z over $\mathbf{K}$ is a **Padé pair** of order m for φ if

$$\| Q\varphi - P \| < \exp(-m), \quad Q \neq 0, \quad \deg Q \le m\ . \tag{5}$$

A Padé pair (P,Q) of order m for φ always exists and the rational function $P/Q \in \mathbf{K}(z)$ is uniquely determined for each $m = 0,1,2,\cdots$. The element $P/Q \in \mathbf{K}(z)$ with P,Q satisfying (5) is called the m-th **diagonal Padé approximation** for φ. A number m is called a **normal index** if (5) implies $\deg Q = m$. Note that P/Q is irreducible if m is a normal index, although it can be reducible for a general m. A normal Padé pair (P,Q), i.e., $\deg Q$ is a normal index, is said to be **normalized** if the leading coefficient of Q is equal to 1. It is a classical result that m is a normal index for φ if and only if the Hankel determinant $\det(\varphi_{i+j})_{0\le i,j\le m-1}$ is nonzero. Note that 0 is always a normal index and the determinant for the empty matrix is considered as 1, so that the above statement remains valid for $m = 0$.

We succeed in obtaining a quantitative relation between the Hankel determinant and the normalized Padé pair. Namely,

$$\det(\varphi_{i+j})_{0\le i,j\le m-1} = (-1)^{[m/2]} \prod_{z;Q(z)=0} P(z) \tag{6}$$

for any normal index m with the normalized Padé pair (P,Q), where $\prod_{z;Q(z)=0}$ indicates a product taken over all zeroes z of Q with their multiplicity (Theorem 6).

We are specially interested in the Padé approximation theory applied to the Fibonacci words $\varepsilon := \varepsilon(1,0)$ and $\overline{\varepsilon} := \varepsilon(0,1)$, where 0, 1 are considered as elements in the field $\mathbf{Q}$, since we have the following remark.

Remark 1 *Let M be a matrix of size $m \times m$ with entries consisting of two independent variables a and b. Then,* $\det M = (a-b)^{m-1}(pa + (-1)^{m-1}qb)$, *where p and q are integers defined by*

$$p = \det M \mid_{a=1,\ b=0} \quad , \quad q = \det M \mid_{a=0,\ b=1} .$$

Proof of Remark 1. Subtracting the first column vector from all the other column vectors of M, we see that $\det M$ is divisible by $(a-b)^{m-1}$ as a polynomial in $\mathbf{Z}[a,b]$. Hence, $\det M = (a-b)^{m-1}(xa+yb)$ for integers x, y. Setting $(a,b) = (1,0),\ (0,1)$, we get the assertion.

In Section 2, we study the structures of the Fibonacci word, in particular, its repetition property. The notion of singular words introduced in Z.-X. Wen and Z.-Y. Wen [5] plays an important role.

In Section 3, we give the value of the Hankel determinants $H_{n,m}(\varepsilon)$ and $H_{n,m}(\overline{\varepsilon})$ for the Fibonacci words in some closed forms. It is a rare case where the Hankel determinants are determined completely. Another such case is for the Thue-Morse sequence φ consisting of 0 and 1, where the Hankel determinants $H_{m,n}(\varphi)$ modulo 2 are obtained, and the function $H_{m,n}(\varphi)$ of (m,n) is proved to be 2-dimensionally automatic (J.-P. Allouche, J. Peyrière, Z.-X. Wen and Z.-Y. Wen [1]).

In Section 4, we consider the self-similar property of the values $H_{n,m}(\varepsilon)$ and $H_{n,m}(\overline{\varepsilon})$ for the Fibonacci words. The quarter plane $\{(n,m); n \geq 0, m \geq 1\}$ is tiled by 3 kinds of tiles with the values $H_{n,m}(\varepsilon)$ and $H_{n,m}(\overline{\varepsilon})$ on it with various scales.

In Section 5, we develop a general theory of Padé approximation. We also obtain the admissible continued fraction expansion of φ_ε and $\varphi_{\overline{\varepsilon}}$, the formal Laurent series (4) with $h = -1$ for the sequences ε and $\overline{\varepsilon}$, and determine all the convergents p_k/q_k of the continued fractions. It is known in general that the set of the convergents p_k/q_k for φ is the set of diagonal Padé approximations and the set of degrees of q_k's in z coincides with the set of normal indices for φ.

2 Structures of the Fibonacci word

In what follows, σ denotes the substitution defined by (2), and

$$\hat{\varepsilon} = \hat{\varepsilon}_0\hat{\varepsilon}_1\hat{\varepsilon}_2\cdots\hat{\varepsilon}_n\cdots \quad (\hat{\varepsilon}_n \in \{a,b\})$$

is the (infinite) Fibonacci word (1). A finite word over $\{a,b\}$ is sometimes considered to be an element of the free group generated by a and b with their inverses a^{-1} and b^{-1}. For $n = 0, 1, 2, \cdots$, we define the n-th **Fibonacci word** F_n and the n-th **singular word** W_n as follows:

$$\begin{aligned} F_n &:= \sigma^n(a) = \sigma^{n+1}(b) \\ W_n &:= \beta_n F_n \alpha_n^{-1}, \end{aligned} \tag{7}$$

where we put

$$\alpha_n = \beta_m = \begin{cases} a & (n : \text{even},\ m : \text{odd}) \\ b & (n : \text{odd},\ \ m : \text{even}), \end{cases} \tag{8}$$

and we define W_{-2} to be the empty word and $W_{-1} := a$ for convenience. Let $(f_n; n \in \mathbf{Z})$ be the **Fibonacci sequence**:

$$\begin{aligned} &f_{n+2} = f_{n+1} + f_n \quad (n \in \mathbf{Z}) \\ &f_{-1} = f_0 = 1. \end{aligned} \tag{9}$$

Then, we have $|F_n| = |W_n| = f_n$ $(n \geq 0)$, where $|\xi|$ denotes the **length** of a finite word ξ.

For a finite word $\xi = \xi_0\xi_1\cdots\xi_{n-1}$ and a finite or infinite word $\eta = \eta_0\eta_1\eta_2\cdots$ over an alphabet, we denote

$$\xi \prec_k \eta \tag{10}$$

if $\xi = \eta_k\eta_{k+1}\cdots\eta_{k+n-1}$. We simply denote

$$\xi \prec \eta \tag{11}$$

and say that ξ is a **subword** of η if $\xi \prec_k \eta$ holds for some k. For a finite word $\xi = \xi_0\xi_1\cdots\xi_{n-1}$ and i with $0 \leq i < n$, we denote the i-th **cyclic permutation** of ξ by $C_i(\xi) := \xi_i\xi_{i+1}\cdots\xi_{n-1}\xi_0\xi_1\cdots\xi_{i-1}$. We also denote $C_i(\xi) := C_{i'}(\xi)$ with $i' := i - n[i/n]$ for any $i \in \mathbf{Z}$.

In this section, we study the structure of the Fibonacci word $\hat{\varepsilon}$ and discuss the repetition property. The following two lemmas are obtained by Z.-X. Wen and Z.-Y. Wen [5] and we omit the proofs.

Lemma 1 *We have the following statements (1)–(10):*
(1) $\hat{\varepsilon} = F_n F_{n-1} F_n F_{n+1} F_{n+2} \cdots \ (n \geq 1)$,
(2) $F_n = F_{n-1}F_{n-2} = F_{n-2}F_{n-1}\beta_n^{-1}\alpha_n^{-1}\beta_n\alpha_n \ (n \geq 2)$,
(3) $F_nF_n \prec \hat{\varepsilon} \ (n \geq 3)$,
(4) $\hat{\varepsilon} = W_{-1}W_0W_1W_2W_3\cdots$,
(5) $W_n = W_{n-2}W_{n-3}W_{n-2} \ (n \geq 1)$,
(6) W_n *is a* **palindrome**, *that is,* W_n *stays invariant under reading the letters from the end* $(n \geq -2)$,
(7) $C_i(F_n) \prec \hat{\varepsilon} \ (n \geq 0,\ 0 \leq i < f_n)$,
(8) $C_i(F_n) \neq C_j(F_n)$ *for any* $i \neq j$, *moreover, they are different already before their last places* $(n \geq 1,\ 0 \leq i < f_n)$,
(9) $W_n \neq C_i(F_n) \ (n \geq 0,\ 0 \leq i < f_n)$,
(10) $\xi \prec \hat{\varepsilon}$ *and* $|\xi| = f_n$ *imply that either* $\xi = C_i(F_n)$ *for some* i *with* $0 \leq i < f_n$ *or* $\xi = W_n \ (n \geq 0)$.

Lemma 2 *For any* $k \geq -1$, *we have the decomposition of* $\hat{\varepsilon}$ *as follows:*

$$\hat{\varepsilon} = (W_{-1}W_0\cdots W_{k-1})W_k\gamma_0W_k\gamma_1\cdots W_k\gamma_n\cdots,$$

where all the occurrences of W_k *in* $\hat{\varepsilon}$ *are picked up and* γ_n *is either* W_{k+1} *or* W_{k-1} *corresponding to* $\hat{\varepsilon}_n$ *is* a *or* b, *respectively. That is, any two different occurrences of* W_k *do not overlap and are separated by* W_{k+1} *or* W_{k-1}.

We introduce another method to discuss the repetition property of $\hat{\varepsilon}$. Let $\mathbf{N}$ be the set of nonnegative integers. For $n \in \mathbf{N}$, let

$$\begin{aligned} &n = \sum_{i=0}^{\infty} \tau_i(n) f_i\ , \\ &\tau_i(n) \in \{0,1\} \quad \text{and} \quad \tau_i(n)\tau_{i+1}(n) = 0 \quad (i \in \mathbf{N}) \end{aligned} \tag{12}$$

be the **regular expression** of n in the Fibonacci base due to Zeckendorf. For $m, n \in \mathbf{N}$ and a positive integer k, we denote

$$m \equiv_k n \tag{13}$$

if $\tau_i(m) = \tau_i(n)$ holds for all $i < k$.

Lemma 3 *It holds that* $\hat{\varepsilon}_n = a$ *if and only if* $\tau_0(n) = 0$.

Proof. We use induction on n. The lemma holds for $n = 0, 1, 2$. Assume that the lemma holds for any $n \in \mathbf{N}$ with $n < f_k$ for some $k \geq 2$. Take any $n \in \mathbf{N}$ with $f_k \leq n < f_{k+1}$. Then, since $0 \leq n - f_k < f_{k-1}$, we have $n = \sum_{i=0}^{k-1} \tau_i(n - f_k) f_i + f_k$, which gives the regular expression if $\tau_{k-1}(n - f_k) = 0$. If $\tau_{k-1}(n - f_k) = 1$, then we have the regular expression $n = \sum_{i=0}^{k-2} \tau_i(n - f_k) f_i + f_{k+1}$. In any case, we have $\tau_0(n) = \tau_0(n - f_k)$. On the other hand, since $\hat{\varepsilon}$ starts with $F_k F_{k-1}$ by Lemma 1, we have $\hat{\varepsilon}_n = \hat{\varepsilon}_{n-f_k}$. Hence, $\hat{\varepsilon}_n = a$ if and only if $\tau_0(n) = 0$ by the induction hypothesis. Thus, we have the lemma for any $n < f_{k+1}$, and by induction, we complete the proof. ∎

Lemma 4 *Let* $n = \sum_{i=0}^{\infty} n_i f_i$ *with* $n_i \in \{0, 1\}\, (i \in \mathbf{N})$. *Assume that* $n_i n_{i+1} = 0$ *for* $0 \leq i < k$. *Then,* $n_i = \tau_i(n)$ *holds for* $0 \leq i < k$.

Proof. If there exists $i \in \mathbf{N}$ such that $n_i n_{i+1} = 1$, take the maximum i_0 for such i's. Take the maximum j such that $n_{i_0+1} = n_{i_0+3} = n_{i_0+5} = \cdots = n_j = 1$. Then, replacing $f_{i_0} + f_{i_0+1} + f_{i_0+3} + f_{i_0+5} + \cdots + f_j$ by f_{j+1}, we have a new expression of n:

$$\begin{aligned} n = & \sum_{i=0}^{\infty} n'_i f_i \\ := & \sum_{i=0}^{i_0-1} n_i f_i + f_{j+1} + \sum_{i=j+3}^{\infty} n_i f_i \,. \end{aligned}$$

This new expression is unchanged at the indices less than k, and is either regular or has a smaller maximum index i with the property $n'_i n'_{i+1} = 1$. By continuing this procedure, we finally get the regular expression of n, which is unchanged at the indices less than k from the original expression. Thus, we have $n_i = \tau_i(n)$ for any $0 \leq i < k$. ∎

Lemma 5 *For any* $n \in \mathbf{N}$ *and* $k \geq 0$, $\tau_0(n + f_k) \neq \tau_0(n)$ *holds if and only if either* $n \equiv_{k+2} f_{k+1} - 2$ *or* $n \equiv_{k+2} f_{k+1} - 1$. *Moreover,*

$$\hat{\varepsilon}_{n+f_k} - \hat{\varepsilon}_n = \begin{cases} (-1)^{k-1}(a - b) & (n \equiv_{k+2} f_{k+1} - 2) \\ (-1)^k (a - b) & (n \equiv_{k+2} f_{k+1} - 1), \end{cases}$$

where a *and* b *are considered as independent variables.*

Proof. If $k = 0$, we can verify Lemma 5 by a direct calculation.

Assume that $k \geq 1$ and $\tau_k(n) = 0$, then we have an expression of $n + f_k$:

$$n + f_k = \sum_{i=0}^{k-1} \tau_i(n) f_i + f_k + \sum_{i=k+1}^{\infty} \tau_i(n) f_i \ .$$

Then by Lemma 4, we have $\tau_0(n + f_k) = \tau_0(n)$ if $k \geq 2$ or if $k = 1$ and $\tau_0(n) = 0$. In the case where $k = 1$, $\tau_0(n) = 1$ and $\tau_2(n) = 0$, since

$$n + f_k = 1 + 2 + \sum_{i=3}^{\infty} \tau_i(n) f_i = f_2 + \sum_{i=3}^{\infty} \tau_i(n) f_i \ ,$$

we have $\tau_0(n + f_k) = 0$ by Lemma 4. On the other hand, in the case where $k = 1$, $\tau_0(n) = 1$ and $\tau_2(n) = 1$, since

$$n + f_k = 1 + 2 + 3 + \sum_{i=4}^{\infty} \tau_i(n) f_i = f_0 + f_3 + \sum_{i=4}^{\infty} \tau_i(n) f_i \ ,$$

we have $\tau_0(n + f_k) = 1$ by Lemma 4.

Thus, in the case where $k \geq 1$ and $\tau_k(n) = 0$, $\tau_0(n + f_k) \neq \tau_0(n)$ if and only if $k = 1$, $\tau_0(n) = 1$ and $\tau_2(n) = 0$, or equivalently, if and only if $n \equiv_{k+2} f_{k+1} - 2$. Note that $n \equiv_{k+1} f_{k+1} - 1$ does not happen in this case.

Now assume that $k \geq 1$ and $\tau_k(n) = 1$. Take the minimum $j \geq 0$ such that $\tau_k(n) = \tau_{k-2}(n) = \tau_{k-4}(n) = \cdots = \tau_j(n) = 1$. Then since $2f_i = f_{i+1} + f_{i-2}$ for any $i \in \mathbf{N}$, we have an expression of $n + f_k$:

$$\begin{aligned} n + f_k &= \sum_{i=0}^{j-3} \tau_i(n) f_i + f_{j-2} \\ &+ f_{j+1} + f_{j+3} + f_{j+5} + \cdots + f_{k+1} + \sum_{i=k+2}^{\infty} \tau_i(n) f_i \end{aligned} \tag{14}$$

where the first term in the right-hand side vanishes if $j = 0, 1, 2$. Hence by Lemma 4, $\tau_0(n + f_k) = \tau_0(n)$ if $j \geq 4$.

In the case where $j = 3$, $\tau_0(n + f_k) = \tau_0(n)$ holds if $\tau_0(n) = 0$ by (14) and Lemma 4. If $\tau_0(n) = 1$, then by (14) and Lemma 4, $\tau_0(n + f_k) = 0$. Thus, in the case where $j = 3$, $\tau_0(n + f_k) \neq \tau_0(n)$ if and only if $\tau_0(n) = 1$.

If $j = 2$, then by the assumption on j, we have $\tau_0(n) = 0$. On the other hand, since $f_0 = 1$, by (14) and Lemma 4, we have $\tau_0(n + f_k) = 1$. Thus, $\tau_0(n + f_k) \neq \tau_0(n)$.

If $j = 1$, then we have $\tau_0(n) = 0$ since $\tau_1(n) = 1$ by the assumption on j. On the other hand, since $f_{-1} = 1$, we have $\tau_0(n + f_k) = 1$ by (14) and Lemma 4. Thus, $\tau_0(n + f_k) \neq \tau_0(n)$.

If $j = 0$, then by the assumption on j, $\tau_0(n) = 1$. On the other hand, since $f_{-2} = 0$, we have $\tau_0(n + f_k) = 0$ by (14) and Lemma 4. Thus, $\tau_0(n + f_k) \neq \tau_0(n)$.

By combining all the results as above, we get the first part.

The second part follows from Lemma 3 and the fact that for any $k \geq 0$,

$$f_{k+1} - 1 = f_k + f_{k-2} + \cdots + f_i$$

with $i = 0$ if k is even and $i = 1$ if k is odd. Hence,

$$\tau_0(f_{k+1} - 1) = \tau_0(f_{h+1} - 2) = \begin{cases} a & (k : odd,\ h : even) \\ b & (k : even,\ h : odd). \end{cases}$$

■

Lemma 6 *For any $k \geq 0$, $W_k \prec_n \hat{\varepsilon}$ if and only if $n \equiv_{k+2} f_{k+1} - 1$.*

Proof. By Lemma 2, the smallest $n \in \mathbf{N}$ such that $W_k \prec_n \hat{\varepsilon}$ is

$$f_{-1} + f_0 + f_1 + \cdots + f_{k-1} = f_{k+1} - 1,$$

which is the smallest $n \in \mathbf{N}$ such that $n \equiv_{k+2} f_{k+1} - 1$. Let $n_0 := f_{k+1} - 1$. Then, the regular expression of n_0 is

$$n_0 = f_k + f_{k-2} + f_{k-4} + \cdots + f_d\ ,$$

where $d = (1 - (-1)^k)/2$. The next n with $n \equiv_{k+2} n_0$ is clearly

$$n = f_{k+2} + f_k + f_{k-2} + \cdots + f_d\ ,$$

which is, by Lemma 2, the next n such that $W_k \prec_n \hat{\varepsilon}$ since $f_k + f_{k+1} = f_{k+2}$.

For $i = 1, 2, 3, \cdots$, let

$$n_i = n_0 + \sum_{j=0}^{\infty} \tau_j(i) f_{k+2+j}\ .$$

Then, it is easy to see that n_i is the i-th n after n_0 such that $n \equiv_{k+2} f_{k+1} - 1$. We prove by induction on i that n_i is the i-th n after n_0 such that $W_k \prec_n \hat{\varepsilon}$. Assume that it is so for i. Then by Lemma 4, $W_k \gamma_i W_k \prec_{n_i} \hat{\varepsilon}$. Hence, the next n after n_i such that $W_k \prec_n \hat{\varepsilon}$ is $n_i + f_k + |\gamma_i|$. Thus, we have

$$\begin{aligned} n_i + f_k + |\gamma_i| &= n_i + f_k + f_{k+1} 1_{\hat{\varepsilon}_i = a} + f_{k-1} 1_{\hat{\varepsilon}_i = b} \\ &= n_i + f_{k+2} 1_{\tau_0(i)=0} + f_{k+1} 1_{\tau_0(i)=1} \\ &= n_{i+1} \ , \end{aligned}$$

which completes the proof. ■

Lemma 7 *Let $k \geq 0$ and $n, i \in \mathbf{N}$ satisfy that $n \equiv_{k+1} i$.*
(1) If $0 \leq i < f_k$, then, $\tau_0(n+j) = \tau_0(i+j)$ holds for any $j = 0, 1, \cdots, f_{k+2} - i - 3$.
(2) If $f_k \leq i < f_{k+1}$, then, $\tau_0(n+j) = \tau_0(i+j)$ holds for any $j = 0, 1, \cdots, f_{k+3} - i - 3$.

Proof. (1) We prove the lemma by induction on k. The assertion holds for $k = 0$. Let $k \geq 1$ and assume that the assertion is valid for $k - 1$. For $j = 0, 1, \cdots, f_k - i$, $n + j \equiv_k i + j$ holds and hence, $\tau_0(n + j) = \tau_0(i + j)$ holds. Let $j_0 = f_k - i$. Then, since $n + j_0 \equiv_k i + j_0 \equiv_k 0$, we have $\tau_0(n + j_0 + j) = \tau_0(i + j_0 + j) = \tau_0(j)$ for any $j = 0, 1, \cdots, f_{k+1} - 3$ by the the induction hypothesis. Thus, $\tau_0(n + j) = \tau_0(i + j)$ holds for any $j = 0, 1, \cdots, f_{k+2} - i - 3$. This proves (1).
(2) In this case, $\tau_{k+1}(n) = 0$ holds. Hence, we have $n \equiv_{k+2} i$. Therefore, we can apply (1) with $k + 1$ for k. Thus, we get (2) ■

Let $n, m, i \in \mathbf{N}$ with $m \geq 2$ and $0 < i < m$. We call n an (m, i)-**shift invariant place** in $\hat{\varepsilon}$ if

$$\hat{\varepsilon}_n \hat{\varepsilon}_{n+1} \cdots \hat{\varepsilon}_{n+m-1} = \hat{\varepsilon}_{n+i} \hat{\varepsilon}_{n+i+1} \cdots \hat{\varepsilon}_{n+i+m-1} \ .$$

We call n an m-**repetitive** place in $\hat{\varepsilon}$ if there exist $i, j \in \mathbf{N}$ with $i > 0$ and $i + j < m$ such that $n + j$ is an (m, i)-shift invariant place in $\hat{\varepsilon}$. Let $\mathcal{R}_m$ be the set of m-repetitive places in $\hat{\varepsilon}$.

Lemma 8 *(1) Let* $n \equiv_{k+1} 0$ *for some* $k \geq 1$*. Then,* n *is an* $(f_{k+1} - 2, f_k)$*-shift invariant place in* $\hat{\varepsilon}$*.*
(2) Let $n \equiv_{k+1} f_k$ *for some* $k \geq 2$*. Then,* n *is an* $(f_{k+1} - 2, f_{k-1})$*-shift invariant place in* $\hat{\varepsilon}$*.*

Proof. (1) Since the least $i \geq n$ such that either $i \equiv_{k+2} f_{k+1} - 1$ or $i \equiv_{k+2} f_{k+1} - 2$ is not less than $n + f_{k+1} - 2$, by Lemma 5, we have

$$\hat{\varepsilon}_n \hat{\varepsilon}_{n+1} \cdots \hat{\varepsilon}_{n+f_{k+1}-3} = \hat{\varepsilon}_{n+f_k} \hat{\varepsilon}_{n+f_k+1} \cdots \hat{\varepsilon}_{n+f_k+f_{k+1}-3} \ .$$

(2) Since the minimum $i \geq n$ such that either $i \equiv_{k+1} f_k - 1$ or $i \equiv_{k+1} f_k - 2$ is $n + f_{k+1} - 2$, by Lemma 5, we have

$$\hat{\varepsilon}_n \hat{\varepsilon}_{n+1} \cdots \hat{\varepsilon}_{n+f_{k+1}-3} = \hat{\varepsilon}_{n+f_{k-1}} \hat{\varepsilon}_{n+f_{k-1}+1} \cdots \hat{\varepsilon}_{n+f_{k-1}+f_{k+1}-3} \ .$$

■

Theorem 1 *The pair* (n, m) *of nonnegative integers satisfies* $n \in \mathcal{R}_m$ *if one of the following two conditions holds:*
(1) $f_k + 1 \leq m \leq f_{k+1} - 2$*,* $n - i \equiv_{k+1} 0$ *and* $i \leq n$ *for some* $k \geq 1$ *and* $i \in \mathbf{Z}$ *with* $f_k + 1 \leq m + i \leq f_{k+1} - 2$*.*
(2) $f_{k-1} + 1 \leq m \leq f_{k+1} - 2$*,* $i \leq n$ *and* $n - i \equiv_{k+1} f_k$ *for some* $k \geq 2$ *and* $i \in \mathbf{Z}$ *with* $f_{k-1} + 1 \leq m + i \leq f_{k+1} - 2$*.*

Remark 2 *The "if and only if" statement actually holds in Theorem 1 in place of "if" since we will prove later that* $H_{n,m} \neq 0$ *if none of the conditions (1) and (2) hold.*

Proof of Theorem 1. Assume (1) and $i \geq 0$. By (1) of Lemma 8, $n - i$ is an $(f_{k+1} - 2, f_k)$-shift invariant place. Then, n is an (m, f_k)-shift invariant place since $i + m \leq f_{k+1} - 2$. Thus, $n \in \mathcal{R}_m$ as $f_k < m$.

Assume (1) and $i < 0$. Then, since $n - i$ is an $(f_{k+1} - 2, f_k)$-shift invariant place and $m \leq f_{k+2} - 2$, it is an (m, f_k)-shift invariant place. Moreover, since $f_k - i < m$, n is a m-repetitive place.

Assume (2) and $i \geq 0$. Then, $n - i$ is an $(f_{k+1} - 2, f_{k-1})$-shift invariant place by (2) of Lemma 8. Then, n is an (m, f_{k-1})-shift invariance place since $i + m \leq f_{k+1} - 2$. Thus, n is an m-repetitive place as $f_{k-1} < m$.

Assume (2) and $i < 0$. Then, since $n - i$ is an $(f_{k+1} - 2, f_{k-1})$-shift invariant place and $m \leq f_{k+1} - 2$, it is an (m, f_{k-1})-shift invariant place. Then, n is an m-repetitive place, since $f_{k-1} - i < m$. Thus, $n \in \mathcal{R}_m$. ■

Corollary 1 *The place 0 is m-repetitive for an $m \geq 2$ if $m \notin \cup_{k=1}^{\infty}\{f_k - 1, f_k\}$.*

Remark 3 *The "if and only if" statement actually holds in Corollary 1 in place of "if" since we will prove later that $H_{0,m} \neq 0$ if $m \in \cup_{k=1}^{\infty}\{f_k - 1, f_k\}$.*

Proof of Corollary 1. Let $i = 0$ in (1) of Theorem 1. Then, 0 is m-repetitive if $f_k + 1 \leq m \leq f_{k+1} - 2$ for some $k \geq 1$. ■

Corollary 2 *Let $k \geq 2$. The place n is f_k-repetitive if*

$$W_k \prec \hat{\varepsilon}_{n+1}\hat{\varepsilon}_{n+2} \cdots \hat{\varepsilon}_{n+2f_k-3} .$$

Proof. By (2) of Theorem 1, for any $k \geq 2$, n is an f_k-repetitive place if $n - i \equiv_{k+1} f_k$ for some i with $i \leq n$ and $-f_{k-2} + 1 \leq i \leq f_{k-1} - 2$. Since the condition $n - i \equiv_{k+1} f_k$ is equivalent to $n - i \equiv_{k+2} f_k$ and there is no carry in addition of $-i$ to both sides of $n \equiv_{k+2} f_k + i$, the condition $n - i \equiv_{k+1} f_k$ is equivalent to $n \equiv_{k+2} f_k + i$. Hence, the place n is f_k-repetitive if $n \equiv_{k+2} j$ for some j with $f_{k-1} + 1 \leq j \leq f_{k+1} - 2$. By Lemma 6, this condition is equivalent to that W_k starts at one of the places in $\{n+1, n+2, \cdots, f_k - 2\}$, which completes the proof. ■

3 Hankel determinants

The aim of this section is to find the value of the Hankel determinants

$$\begin{aligned} H_{n,m} &:= H_{n,m}(\varepsilon) = \det(\varepsilon_{n+i+j})_{0 \leq i,j \leq m-1} \\ \overline{H}_{n,m} &:= H_{n,m}(\overline{\varepsilon}) = \det(\overline{\varepsilon}_{n+i+j})_{0 \leq i,j \leq m-1} \\ &(n = 0, 1, 2, \cdots;\ m = 1, 2, 3, \cdots) \end{aligned}$$

for the Fibonacci word $\varepsilon(a, b)$ at $(a, b) = (1, 0)$ and $(a, b) = (0, 1)$:

$$\begin{aligned} \varepsilon &:= \varepsilon(1, 0) = 10110101101101 \cdots, \\ \overline{\varepsilon} &:= \varepsilon(0, 1) = 01001010010010 \cdots. \end{aligned}$$

It is clear that $H_{n,m}(\varepsilon(a, b)) = 0$ if n is the m-repetitive place in $\varepsilon(a, b)$, where a, b are considered to be two independent variables, so that, in general, $H_{n,m}(\varepsilon(a, b))$ becomes a polynomial in a and b as is stated in Remark 1.

In the following lemmas, theorems and corollary, we give statements for ε and $\overline{\varepsilon}$ parallelly, while we give the proofs only for ε since the proofs for $\overline{\varepsilon}$ are similar to those for ε. The only difference between them is the starting point, Lemma 5, where $a-b$ in the right-hand side is 1 for ε and -1 for $\overline{\varepsilon}$.

We use the following notation: for any subset S of $\{0,1,2,3,4,5\}$, $\chi(k:S)$ is a function on $k \in \mathbf{Z}$ such that

$$\chi(k:S) = \begin{cases} -1 & (\text{if } k \equiv s \ (mod\ 6) \text{ for some } s \in S) \\ 1 & (\text{otherwise}). \end{cases}$$

The following corollary follows from Theorem 1.

Corollary 3 *$H_{n,m} = 0$ if one of the conditions (1), (2) in Theorem 1 is satisfied. The same statement holds for $\overline{H}_{n,m}$.*

Lemma 9 *For any $k \geq 2$, we have*

$$\begin{aligned} H_{0,f_k} &= \chi(k:2,3)\left(H_{0,f_{k-1}} - (-1)^{f_{k-1}} H_{f_k-1,f_{k-1}}\right) \\ \overline{H}_{0,f_k} &= \chi(k:1,3,4,5)\left(\overline{H}_{0,f_{k-1}} - (-1)^{f_{k-1}} \overline{H}_{f_k-1,f_{k-1}}\right). \end{aligned}$$

Proof. The matrix $(\varepsilon_{i+j})_{0\leq i,j<f_k}$ is decomposed into three parts:

$$(\varepsilon_{i+j})_{0\leq i,j<f_k} = \begin{pmatrix} A \\ A' \\ B \end{pmatrix},$$

where

$$\begin{aligned} A &= (\varepsilon_{i+j})_{0\leq i<f_{k-2}, 0\leq j<f_k} \\ A' &= (\varepsilon_{f_{k-2}+i+j})_{0\leq i<f_{k-3}, 0\leq j<f_k} \\ B &= (\varepsilon_{f_{k-1}+i+j})_{0\leq i<f_{k-2}, 0\leq j<f_k}. \end{aligned}$$

By Lemma 5, the following two subwords of ε:

$$\varepsilon_0\varepsilon_1\cdots\varepsilon_{f_{k-2}+f_k-2} \quad \text{and} \quad \varepsilon_{f_{k-1}}\varepsilon_{f_{k-1}+1}\cdots\varepsilon_{f_{k-1}+f_{k-2}+f_k-2}$$

differ only at two places, namely, $\varepsilon_{f_k-2} \neq \varepsilon_{f_{k-1}+f_k-2}$ and $\varepsilon_{f_k-1} \neq \varepsilon_{f_{k-1}+f_k-1}$. Thus, we get

$$B - A = \left(\begin{array}{cccccc} & & & & (-1)^k & (-1)^{k-1} \\ & & & (-1)^k & (-1)^{k-1} & \\ & 0 & & \cdots & & \\ & (-1)^k & (-1)^{k-1} & & 0 & \end{array}\right). \tag{15}$$

Let $A_0, A_1, \cdots, A_{f_k-1}$ be the columns of the matrix $\begin{pmatrix} A \\ A' \end{pmatrix}$ in order from the left. Since

$$\begin{array}{l}(A_0 A_1 \cdots A_{f_{k-2}-2}) = (\varepsilon_{i+j})_{0\leq i<f_{k-1}, 0\leq j<f_{k-2}-1} \\ (A_{f_{k-1}} A_{f_{k-1}+1} \cdots A_{f_k-2}) = (\varepsilon_{f_{k-1}+i+j})_{0\leq i<f_{k-1}, 0\leq j<f_{k-2}-1}\end{array}$$

and

$$\varepsilon_0\varepsilon_1 \cdots \varepsilon_{f_{k-2}+f_{k-1}-3} = \varepsilon_{f_{k-1}}\varepsilon_{f_{k-1}+1} \cdots \varepsilon_{f_{k-1}+f_{k-2}+f_{k-1}-3}$$

by Lemma 5, we get

$$(A_0 A_1 \cdots A_{f_{k-2}-2}) = (A_{f_{k-1}} A_{f_{k-1}+1} \cdots A_{f_k-2}). \tag{16}$$

Thus, from (15) and (16) we obtain

$$\begin{aligned} & H_{0,f_k} \\ = \; & \det \begin{pmatrix} A_0 & \cdots & A_{f_{k-1}-1} & A_{f_{k-1}} & \cdots & A_{f_k-2} & A_{f_k-1} \\ & & & & & (-1)^k & (-1)^{k-1} \\ & & & & (-1)^k & (-1)^{k-1} & \\ & \mathbf{0} & & \cdots & \cdots & & \\ & & (-1)^k & (-1)^{k-1} & & \mathbf{0} & \end{pmatrix} \\ = \; & \det \begin{pmatrix} A_0 & \cdots & A_{f_{k-1}-1} & 0 & \cdots & 0 & A_{f_k-1} \\ & & & & & (-1)^k & (-1)^{k-1} \\ & & & & (-1)^k & (-1)^{k-1} & \\ & \mathbf{0} & & \cdots & \cdots & & \\ & & (-1)^k & (-1)^{k-1} & & \mathbf{0} & \end{pmatrix} \qquad (17) \\ = \; & (-1)^{(k-1)f_{k-2}}(-1)^{\left[\frac{f_{k-2}}{2}\right]} \det(A_0 A_1 \cdots A_{f_{k-1}-1}) \\ & +(-1)^{kf_{k-2}}(-1)^{\left[\frac{f_{k-2}}{2}\right]+f_{k-1}} \det(A_{f_k-1} A_0 A_1 \cdots A_{f_{k-1}-2}). \end{aligned}$$

Since

$$\varepsilon_0\varepsilon_1 \cdots \varepsilon_{2f_{k-1}-3} = \varepsilon_{f_k}\varepsilon_{f_k+1} \cdots \varepsilon_{f_k+2f_{k-1}-3}$$

by Lemma 5, we get

$$\det(A_{f_k-1}A_0A_1\cdots A_{f_{k-1}-2}) = \det(\varepsilon_{f_k-1+i+j})_{0\le i,j<f_{k-1}} = H_{f_k-1,f_{k-1}}\,.$$

Thus we get

$$\begin{aligned} H_{0,f_k} &= (-1)^{(k-1)f_{k-2}}(-1)^{\left[\frac{f_{k-2}}{2}\right]}H_{0,f_{k-1}} \\ &\quad +(-1)^{kf_{k-2}}(-1)^{\left[\frac{f_{k-2}}{2}\right]+f_k-1}H_{f_k-1,f_{k-1}} \\ &= \chi(k:2,3)\left(H_{0,f_{k-1}}-(-1)^{f_{k-1}}H_{f_k-1,f_{k-1}}\right), \end{aligned}$$

where we have used the fact that

$$(-1)^{(k-1)f_{k-2}}(-1)^{\left[\frac{f_{k-2}}{2}\right]} = \chi(k:2,3).$$

■

Lemma 10 *For $k\ge 2$, we have*

$$\begin{aligned} H_{f_{k+1}-1,f_k} &= \chi(k:1,3,4,5)H_{f_{k+1}-1,f_{k-1}} \\ \overline{H}_{f_{k+1}-1,f_k} &= \chi(k:2,3)\overline{H}_{f_{k+1}-1,f_{k-1}} \end{aligned}$$

Proof. Just like the proof of Lemma 9, we decompose the matrix $(\varepsilon_{f_{k+1}-1+i+j})_{0\le i,j<f_k}$ into three parts:

$$(\varepsilon_{f_{k+1}-1+i+j})_{0\le i,j<f_k} = \begin{pmatrix} A \\ A' \\ B \end{pmatrix},$$

where

$$\begin{aligned} A &= (\varepsilon_{f_{k+1}-1+i+j})_{0\le i<f_{k-2},0\le j<f_k} \\ A' &= (\varepsilon_{f_{k+1}-1+f_{k-2}+i+j})_{0\le i<f_{k-3},0\le j<f_k} \\ B &= (\varepsilon_{f_{k+1}-1+f_{k-1}+i+j})_{0\le i<f_{k-2},0\le j<f_k}\,. \end{aligned}$$

By Lemma 5, the following two subwords of ε:

$$\begin{aligned} &\varepsilon_{f_{k+1}-1}\varepsilon_{f_{k+1}}\cdots\varepsilon_{f_{k+1}+f_{k-2}+f_k-3} \quad \text{and} \\ &\varepsilon_{f_{k+1}-1+f_{k-1}}\varepsilon_{f_{k+1}+f_{k-1}}\cdots\varepsilon_{f_{k+1}+f_{k-1}+f_{k-2}+f_k-3} \end{aligned}$$

differ only at two places. Namely, $\varepsilon_{f_{k+1}+f_k-2} \neq \varepsilon_{f_{k+1}+f_{k-1}+f_k-2}$ and $\varepsilon_{f_{k+1}+f_k-1} \neq \varepsilon_{f_{k+1}+f_{k-1}+f_k-1}$. Therefore, we get

$$B - A = \begin{pmatrix} & & & & & (-1)^k \\ & & & & (-1)^k & (-1)^{k-1} \\ \mathbf{0} & & & \cdots & & \\ & & \cdots & & & \\ & (-1)^k & (-1)^{k-1} & & & \mathbf{0} \end{pmatrix}.$$

Thus, we have

$$\begin{aligned} & \det(\varepsilon_{f_{k+1}-1+i+j})_{0\le i,j<f_k} \\ = \ & \det \begin{pmatrix} A_0 & A_1 & \cdots & A_{f_{k-1}-1} & A_{f_{k-1}} & \cdots & A_{f_k-2} & A_{f_k-1} \\ & & & & & & & (-1)^k \\ & & & & & & (-1)^k & (-1)^{k-1} \\ & & \mathbf{0} & & & \cdots & \cdots & \\ & & & (-1)^k & (-1)^{k-1} & & & \mathbf{0} \end{pmatrix} \\ = \ & (-1)^{kf_{k-2}}(-1)^{\left[\frac{f_{k-2}}{2}\right]} \det(A_0 A_1 \cdots A_{f_{k-1}-1}) \\ = \ & \chi(k:1,3,4,5) H_{f_{k+1}-1,f_{k-1}} \ , \end{aligned} \quad (18)$$

which completes the proof for $H_{f_{k+1}-1,f_{k-1}}$. ■

Lemma 11 *For any $k \geq 2$, we have*

$$\begin{aligned} H_{f_{k+1}-1,f_{k-1}} &= \chi(k:2,5) H_{0,f_{k-1}} \\ \overline{H}_{f_{k+1}-1,f_{k-1}} &= \chi(k:2,5) \overline{H}_{0,f_{k-1}} \end{aligned}$$

Proof. Since by Lemma 5,

$$\begin{aligned} & \varepsilon_{f_{k+1}-1}\varepsilon_{f_{k+1}} \cdots \varepsilon_{f_{k+1}+f_{k-1}-2} \\ = \ & \varepsilon_{f_{k+1}+f_{k-1}-1}\varepsilon_{f_{k+1}+f_{k-1}} \cdots \varepsilon_{f_{k+1}+2f_{k-1}-2} \ , \end{aligned}$$

we get

$$(\varepsilon_{f_{k+1}-1+i+j})_{0\le i,j<f_{k-1}} = \begin{pmatrix} 0 & & & 0 & 1 \\ 1 & \cdot & & & 0 \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ \mathbf{0} & & & 1 & 0 \end{pmatrix} (\varepsilon_{f_{k+1}+i+j})_{0\le i,j<f_{k-1}} .$$

Also, by Lemma 5,

$$(\varepsilon_{f_{k+1}+i+j})_{0\le i,j<f_k} = (\varepsilon_{i+j})_{0\le i,j<f_k} .$$

Thus we obtain

$$\begin{aligned} H_{f_{k+1}-1,f_{k-1}} &= \det(\varepsilon_{f_{k+1}-1+i+j})_{0\le i,j<f_{k-1}} \\ &= (-1)^{f_{k-1}-1} det(\varepsilon_{f_{k+1}+i+j})_{0\le i,j<f_{k-1}} \\ &= \chi(k:2,5) H_{0,f_{k-1}} , \end{aligned}$$

which completes the proof. ■

Lemma 12 *For any $k \ge 3$, we have*

$$\begin{aligned} H_{0,f_k} &= \chi(k:2,3)H_{0,f_{k-1}} + \chi(k:2,4)H_{0,f_{k-2}} \\ \overline{H}_{0,f_k} &= \chi(k:1,3,4,5)\overline{H}_{0,f_{k-1}} + \chi(k:0,1,2,3)\overline{H}_{0,f_{k-2}} \end{aligned}$$

Proof. Clear from Lemmas 9–11. ■

Lemma 13 *For any $k \ge 0$, we have*

$$\begin{aligned} H_{0,f_k} &= \chi(k:2)f_{k-1} \\ \overline{H}_{0,f_k} &= \chi(k:1,2,4)f_{k-2} \end{aligned}$$

Proof. It holds that

$$\begin{aligned} &H_{0,f_0} = 1, \quad H_{0,f_1} = 1, \quad H_{0,f_2} = -2 \\ &\overline{H}_{0,f_0} = 0, \quad \overline{H}_{0,f_1} = -1, \quad \overline{H}_{0,f_2} = -1. \end{aligned}$$

Thus, the lemma holds for $k = 0, 1, 2$. For ≥ 3, we can prove it by induction on k using Lemma 12. ■

Lemma 14 *For any $k \geq 1$, we have*

$$\begin{aligned} H_{0,f_k-1} &= \chi(k:0,4) f_{k-2} \\ \overline{H}_{0,f_k-1} &= \chi(k:2,3,4,5) f_{k-3}. \end{aligned}$$

Proof. Since the matrix $(\varepsilon_{i+j})_{0\leq i,j<f_k-1}$ is obtained from the matrix $(\varepsilon_{i+j})_{0\leq i,j<f_k}$ by removing the last row and the last column, for any $k \geq 2$ we have by (17),

$$\begin{aligned} & H_{0,f_k-1} \\ = \ & \det \begin{pmatrix} A_0 & A_1 & \cdots & A_{f_{k-1}-1} & 0 & \cdots & 0 & 0 \\ & & & & & & & (-1)^k \\ & & & & & & (-1)^k & (-1)^{k-1} \\ & & \mathbf{0} & & & \cdots & \cdots & \\ & & & & (-1)^k & (-1)^{k-1} & & \mathbf{0} \end{pmatrix} \\ = \ & (-1)^{k(f_{k-2}-1)}(-1)^{\left[\frac{f_{k-2}-1}{2}\right]} \det(A_0 A_1 \cdots A_{f_{k-1}-1}) \\ = \ & (-1)^{k(f_{k-2}-1)}(-1)^{\left[\frac{f_{k-2}-1}{2}\right]} H_{0,f_{k-1}} . \end{aligned} \tag{19}$$

Hence, in view of Lemma 13, we obtain the formula for H_{0,f_k-1}. ■

Theorem 2 *For any $m, k \geq 1$ with $f_{k-1} < m \leq f_k$ and $n \in \mathbf{N}$ with $n \equiv_{k+1} 0$, we have*

$$\begin{aligned} H_{n,m} &= \begin{cases} \chi(k:2) f_{k-1} & (\textit{if } m = f_k\ ) \\ \chi(k:0,4) f_{k-2} & (\textit{ if } m = f_k - 1\ ) \\ 0 & (\textit{ otherwise })\end{cases} \\ \overline{H}_{n,m} &= \begin{cases} \chi(k:1,2,4) f_{k-2} & (\textit{if } m = f_k\ ) \\ \chi(k:2,3,4,5) f_{k-3} & (\textit{ if } m = f_k - 1\ ) \\ 0 & (\textit{ otherwise }).\end{cases} \end{aligned}$$

Proof. By Lemma 3 and 7, the matrix for $H_{n,m}$ coincides with that for $H_{0,m}$ so that $H_{n,m} = H_{0,m}$. Then, the first two cases follow from Lemma 13 and 14. For the last case, by Corollary 1, there exist two identical rows in the matrix $(\varepsilon_{i+j})_{0\leq i,j<m}$, so that $H_{0,m}=0$. ■

Theorem 3 *For any $k,n,i\in\mathbf{N}$ with $n\equiv_{k+1} i$ and $0\leq i\leq f_{k+1}-1$, we have*

$$H_{n,f_k} = \begin{cases} \chi(k:2)\chi(k:1,4)^i f_{k-1} & \\ \quad \left(\begin{array}{c} \textit{if either } \tau_{k+1}(n)=0 \textit{ and } 0\leq i<f_{k-1} \\ \textit{or } \tau_{k+1}(n)=1 \textit{ and } 0\leq i<f_k \end{array}\right) & \\ \chi(k:1,2,4)f_{k-2} & \\ \quad \left(\begin{array}{c} \textit{if either } \tau_{k+1}(n)=0 \textit{ and } i=f_{k-1} \\ \textit{or } i=f_{k+1}-1 \end{array}\right) & \\ 0 \quad (\textit{otherwise}) & \end{cases}$$

$$\overline{H}_{n,f_k} = \begin{cases} \chi(k:1,2,4)\chi(k:1,4)^i f_{k-2} & \\ \quad \left(\begin{array}{c} \textit{if either } \tau_{k+1}(n)=0 \textit{ and } 0\leq i<f_{k-1} \\ \textit{or } \tau_{k+1}(n)=1 \textit{ and } 0\leq i<f_k \end{array}\right) & \\ \chi(k:2)f_{k-3} & \\ \quad \left(\begin{array}{c} \textit{if either } \tau_{k+1}(n)=0 \textit{ and } i=f_{k-1} \\ \textit{or } i=f_{k+1}-1 \end{array}\right) & \\ 0 \quad (\textit{otherwise}). & \end{cases}$$

Proof. The theorem holds for $k=0$. Let $k\geq 1$.

Assume that either $\tau_{k+1}(n)=0$ and $0\leq i<f_{k-1}$ or $\tau_{k+1}(n)=1$ and $0\leq i<f_k$. Then we have by Lemma 3 and 7

$$\begin{aligned} \varepsilon_{i+j} &= \varepsilon_{n+j} \qquad (j=0,1,\cdots,f_k-i-1) \\ \varepsilon_{i+j-f_k} &= \varepsilon_{n+j} \qquad (j=f_k-i,f_k,\cdots,2f_k-2) \\ \varepsilon_j &= \varepsilon_{j+f_k} \qquad (j=0,1,\cdots,f_k-1). \end{aligned}$$

Hence, the columns of the matrix $(\varepsilon_{n+h+j})_{0\leq h,j\leq f_k}$ coincide with those of the matrix $(\varepsilon_{h+j})_{0\leq h,j\leq f_k}$. The j-th column of the former is the $(i+j)(mod\ f_k)$-th column of the latter for $j=0,\cdots,f_k-1$. Therefore, we get $H_{n,f_k}=(-1)^{i(f_k-i)}H_{0,f_k}$, which leads to the first case of our theorem by Theorem 2.

Assume that $i = f_{k+1} - 1$. Then we have $H_{n,f_k} = H_{f_{k+1}-1,f_k}$ by Lemmas 3 and 7. Thus, by Lemmas 10–12 we get

$$H_{n,f_k} = \chi(k:1,2,4) f_{k-2} \ .$$

Assume that $\tau_{k+1}(n) = 0$ and $i = f_{k-1}$. Then, since $n \equiv_{k+2} i$, we have $H_{n,f_k} = H_{f_{k-1},f_k}$ by Lemmas 3 and 7. By Lemma 1,

$$\begin{aligned} \xi &:= \varepsilon_{f_{k-1}} \varepsilon_{f_{k-1}+1} \cdots \varepsilon_{f_{k-1}+2f_k-2} \prec_1 W_{k-2} W_{k-1} W_k W_{k-1} W_{k-2} \ , \\ \eta &:= \varepsilon_{f_{k+1}-1} \varepsilon_{f_{k+1}} \cdots \varepsilon_{f_{k+1}+2f_k-3} \prec_{f_k} W_{k-2} W_{k-1} W_k W_{k-1} W_{k-2} \end{aligned}$$

holds. Since the last letter of η comes one letter before the last letter of the palindrome word $W_{k-2}W_{k-1}W_kW_{k-1}W_{k-2}$. Hence, ξ is the mirror image of η, so that

$$\left(\varepsilon_{f_{k-1}+i+j}\right)_{0 \le i,j < f_k} =$$

$$\begin{pmatrix} & 0 & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ & 1 & & 0 & \\ 1 & & & & \end{pmatrix} \left(\varepsilon_{f_{k+1}-1+i+j}\right)_{0 \le i,j < f_k} \begin{pmatrix} & 0 & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ & 1 & & 0 & \\ 1 & & & & \end{pmatrix} .$$

Thus, we obtain $H_{f_{k-1},f_k} = H_{f_{k+1}-1,f_k}$ and

$$H_{n,f_k} = \chi(k:1,2,4) f_{k-2} \, .$$

Assume that n does not belong to the above two cases. Then, since $\tau_{k+1}(n) = 1$ implies $i < f_k$, we have the following condition:

$$\tau_{k+1}(n) = 0 \text{ and } f_{k-1} + 1 \le i \le f_{k+1} - 2.$$

This condition is nonempty only if $k \ge 2$, which we assume. Then, the condition (2) of Theorem 1 is satisfied with f_k (resp. $i - f_k$) in place of m (resp. i). Thus, by Corollary 3, $H_{n,f_k} = 0$. ■

Lemma 15 *For any $k,n,i \in \mathbf{N}$ with $k \geq 1$ and $n \equiv_{k+1} i$, assume that either $\tau_{k+1}(n) = 0$ and $0 \leq i < f_{k-1}$ or $\tau_{k+1}(n) = 1$ and $0 \leq i < f_k$. Then we have*

$$H_{n,f_k-1} = \begin{cases} \chi(k:0,4)f_{k-2} & (i=0) \\ \chi(k:2,3)\chi(k:1,2,4,5)^i H_{i+f_k,f_{k-1}-1} \\ \quad +\chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-2} & (0 < i \leq f_{k-2}) \\ \chi(k:2,3)\chi(k:1,2,4,5)^i H_{i+f_k,f_{k-1}-1} & (f_{k-2} < i \leq f_{k-1}) \\ \chi(k:0,4)\chi(k:1,4)^i f_{k-2} & (f_{k-1} < i < f_k) \end{cases}$$

$$\overline{H}_{n,f_k-1} = \begin{cases} \chi(k:2,3,4,5)f_{k-3} & (i=0) \\ \chi(k:1,3,4,5)\chi(k:1,2,4,5)^i \overline{H}_{i+f_k,f_{k-1}-1} \\ \quad +\chi(k:0,1)\chi(k:1,4)^i f_{k-3} & (0 < i \leq f_{k-2}) \\ \chi(k:1,3,4,5)\chi(k:1,2,4,5)^i \overline{H}_{i+f_k,f_{k-1}-1} & (f_{k-2} < i \leq f_{k-1}) \\ \chi(k:2,3,4,5)\chi(k:1,4)^i f_{k-3} & (f_{k-1} < i < f_k). \end{cases}$$

Proof. If $i = 0$, then the statement follows from Theorem 2. Let

$$\begin{aligned} A_j &= {}^t(\varepsilon_j, \varepsilon_{j+1}, \cdots, \varepsilon_{j+f_{k-1}-1}) \\ A'_j &= {}^t(\varepsilon_j, \varepsilon_{j+1}, \cdots, \varepsilon_{j+f_{k-1}-2}) \\ B'_j &= {}^t(\varepsilon_{j+f_{k-1}}, \varepsilon_{j+f_{k-1}+1} \cdots, \varepsilon_{j+f_k-1}) \\ & \quad\; (j = 0,1,2,\cdots). \end{aligned} \tag{20}$$

Then, by the same argument as in the proof of Theorem 3, we obtain

$$\begin{aligned} H_{n,f_k-1} &= \det \begin{pmatrix} A_i \cdots A_{f_k-1} A_0 \cdots A_{i-2} \\ B'_i \cdots B'_{f_k-1} B'_0 \cdots B'_{i-2} \end{pmatrix} \\ &= (-1)^{(i-1)(f_k-i)} \det \begin{pmatrix} A_0 \cdots A_{i-2} A_i \cdots A_{f_k-1} \\ B'_0 \cdots B'_{i-2} B'_i \cdots B'_{f_k-1} \end{pmatrix}. \end{aligned}$$

Therefore, if $f_{k-2} < i \leq f_{k-1}$, then by the same argument to get (17), we obtain

$$(-1)^{(i-1)(f_k-i)} H_{n,f_k-1} =$$

$$\det\begin{pmatrix} A_0\cdots A_{i-2}A_i\cdots A_{f_{k-1}-1} & 0 & \cdots & 0 & A_{f_k-1} \\ & & & (-1)^k & (-1)^{k-1} \\ & & & (-1)^{k-1} & \\ \mathbf{0} & & \cdots & & \mathbf{0} \\ & (-1)^k & (-1)^{k-1} & & \end{pmatrix}.$$

Since by Lemma 5

$$A_{f_k-1} - A_{f_{k-2}-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^k \end{pmatrix},$$

we get

$$\begin{aligned}
&(-1)^{(i-1)(f_k-i)}H_{n,f_k-1} = \\
&\det\begin{pmatrix} A'_0\cdots A'_{i-2}A'_i\cdots A'_{f_{k-1}-1} & 0 & \cdots & 0 & 0 \\ * \ \cdots \ * \ \ * \ \cdots \ * & 0 & \cdots & 0 & (-1)^k \\ & & & (-1)^k & (-1)^{k-1} \\ & & & (-1)^{k-1} & \\ \mathbf{0} & & \cdots & & \mathbf{0} \\ & (-1)^k & (-1)^{k-1} & & \end{pmatrix} \\
&= (-1)^{kf_{k-2}}(-1)^{\left[\frac{f_{k-2}}{2}\right]}\det(A'_0\cdots A'_{i-2}A'_i\cdots A'_{f_{k-1}-1}) \\
&= \chi(k:1,3,4,5)(-1)^{(i-1)(f_{k-1}-i)}H_{i+f_k,f_{k-1}-1}.
\end{aligned}$$

Thus we obtain

$$H_{n,f_k-1} = \chi(k:2,3)\chi(k:1,2,4,5)^i H_{i+f_k,f_{k-1}-1}.$$

Assume that $f_{k-1} < i < f_k$. Then as above we have

$$(-1)^{(i-1)(f_k-i)}H_{n,f_k-1} =$$

$$\det\left(\begin{array}{cccccc} A_0\cdots A_{f_{k-1}-1} & 0 & \cdots 0 & 0\cdots & 0 & A_{f_k-1} \\ & & & & (-1)^k & (-1)^{k-1} \\ & & & & \cdots & \\ & & 0 & (-1)^k & \cdots & \\ \mathbf{0} & & 0 & (-1)^{k-1} & & \\ & & (-1)^k & 0 & & \mathbf{0} \\ & & \cdots & & & \\ & (-1)^k & \cdots & & & \end{array}\right)$$

$$= (-1)^{k(i-f_{k-1}-1)+(k-1)(f_k-i)+\left[\frac{f_{k-2}-1}{2}\right]}\det(A_0\cdots A_{f_{k-1}-1}).$$

Hence, by Lemma 13

$$\begin{aligned} H_{n,f_k-1} &= \chi(k:0,3,4)\chi(k:1,4)^i H_{0,f_{k-1}} \\ &= \chi(k:0,4)\chi(k:1,4)^i f_{k-2}. \end{aligned}$$

Assume that $0 < i < f_{k-2}$. Then, since $A_{i-1+f_{k-1}} = A_{i-1}$, by the same arguments as above we get

$$(-1)^{(i-1)(f_k-i)}H_{n,f_k-1} =$$

$$\det\left(\begin{array}{ccccc} A'_0\cdots A'_{i-2}A'_i\cdots A'_{f_{k-1}-1} & 0 & \cdots A'_{i-1}\cdots & & 0 \\ * \ \cdots \ * \ \ * \ \cdots \ * & 0 & \cdots * \cdots & & (-1)^k \\ & & & (-1)^k & (-1)^{k-1} \\ & & & (-1)^{k-1} & \\ \mathbf{0} & & \cdots & & \mathbf{0} \\ & (-1)^k & & & \end{array}\right)$$

$$= (-1)^{kf_{k-2}}(-1)^{\left[\frac{f_{k-2}}{2}\right]}\det(A'_0\cdots A'_{i-2}A'_i\cdots A'_{f_{k-1}-1})$$

$$+(-1)^{k(i-1)+(k-1)(f_{k-2}-i)}(-1)^{i-1+\left[\frac{f_{k-2}-1}{2}\right]}$$

$$\det(A_0\cdots A_{i-2}A_i\cdots A_{f_{k-1}-1}A_{i-1})).$$

Since

$$\det(A_0\cdots A_{i-2}A_i\cdots A_{f_{k-1}-1}A_{i-1})) = (-1)^{f_{k-1}-i}H_{0,f_{k-1}}\ ,$$

we obtain by Lemma 13

$$H_{n,f_k-1} = \chi(k:2,3)\chi(k:1,2,4,5)^i H_{i+f_k,f_{k-1}-1} + \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-2}\ . \tag{21}$$

Note that (21) holds also for $i = f_{k-2}$ since in this case,

$$H_{n,f_k-1} = (-1)^{k(f_{k-2}-1)}(-1)^{f_{k-2}-1+\left[\frac{f_{k-2}-1}{2}\right]} \det(A_0 \cdots A_{f_{k-2}-2}A_{f_{k-2}} \cdots A_{f_{k-1}-2}A_{f_k-1})$$

and

$$A_{f_k-1} = A_{f_{k-1}-1} +^t (0,\cdots,0,(-1)^k),$$

which completes the proof for H_{n,f_k-1}. ∎

Lemma 16 *For any $k,n,i \in \mathbf{N}$ with $k \geq 1$ and $n \equiv_{k+1} i$, assume that either $\tau_{k+1}(n) = 0$ and $0 \leq i < f_{k-1}$ or $\tau_{k+1}(n) = 1$ and $0 \leq i < f_k$. Then we have*

$$H_{n,f_k-1} = \begin{cases} \chi(k:0,4)f_{k-2} & (i=0) \\ \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-3} & (0 < i \leq f_{k-1}) \\ \chi(k:0,4)\chi(k:1,4)^i f_{k-2} & (f_{k-1} < i < f_k) \end{cases}$$

$$\overline{H}_{n,f_k-1} = \begin{cases} \chi(k:2,3,4,5)f_{k-3} & (i=0) \\ \chi(k:0,1)\chi(k:1,4)^i f_{k-4} & (0 < i \leq f_{k-1}) \\ \chi(k:2,3,4,5)\chi(k:1,4)^i f_{k-3} & (f_{k-1} < i < f_k). \end{cases}$$

Proof. The first and the third cases have been already proved in Lemma 15. Let us consider the second case where $0 < i \leq f_{k-1}$. We divide it into two subcases, and use induction on k.

Case 1. $i = 1$:
If $k = 1$, then

$$H_{n,f_{k-1}} = H_{n,1} = \varepsilon_n = 0$$

since $n \equiv_2 1$ and $\tau_0(n) = 1$. On the other hand, $f_{k-3} = f_{-2} = 0$, and hence, we get the statement. Assume that $k \geq 2$ and the assertion holds for $k-1$.

Then, by Lemma 15 and the induction hypothesis, we get

$$\begin{aligned}
& H_{n,f_k-1} \\
= \ & \chi(k:2,3)\chi(k:1,2,4,5)^i H_{i+f_k,f_{k-1}-1} + \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-2} \\
= \ & \chi(k:1,3,4,5) H_{1+f_k,f_{k-1}-1} + \chi(k:2,3,4,5) f_{k-2} \\
= \ & \chi(k:1,3,4,5)\chi(k-1:2,3,4,5) f_{k-4} + \chi(k:2,3,4,5) f_{k-2} \\
= \ & \chi(k:0,1) f_{k-4} + \chi(k:2,3,4,5) f_{k-2} \\
= \ & \chi(k:2,3,4,5) f_{k-3} \ ,
\end{aligned}$$

which is the desired statement.

Case 2. $i \geq 2$:
If $f_{k-2} < i \leq f_{k-1}$, then it follows from the third case and then the fourth case of Lemma 15 that

$$\begin{aligned}
& H_{n,f_k-1} \\
= \ & \chi(k:2,3)\chi(k:1,2,4,5)^i H_{i+f_k,f_{k-1}-1} \\
= \ & \chi(k:2,3)\chi(k:1,2,4,5)^i \chi(k-1:0,4)\chi(k-1:1,4)^i f_{k-3} \\
= \ & \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-3} \ .
\end{aligned}$$

Assume that $i \leq f_{k-2}$ and the statement holds for $k-1$. Then by Lemma 15, we get

$$\begin{aligned}
& H_{n,f_k-1} \\
= \ & \chi(k:2,3)\chi(k:1,2,4,5)^i H_{i+f_k,f_{k-1}-1} + \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-2} \\
= \ & \chi(k:2,3)\chi(k:1,2,4,5)^i \chi(k-1:1,2,3,5)\chi(k-1:1,4)^i f_{k-4} \\
& + \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-2} \\
= \ & \chi(k:0,4)\chi(k:1,4)^i f_{k-4} + \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-2} \\
= \ & \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-3} \ .
\end{aligned}$$

This completes the proof for H_{n,f_k-1}. ■

Lemma 17 *For any $k, n \in \mathbf{N}$ with $k \geq 2$ and $\tau_{k+1}(n) = 0$, we have*

$$H_{n,f_k-1} = \begin{cases} \chi(k:2,3,4,5) f_{k-3} & (n \equiv_{k+1} f_{k-1}) \\ \chi(k:0,4) f_{k-2} & (n \equiv_{k+1} f_{k-1}+1) \end{cases}$$

$$\overline{H}_{n,f_k-1} = \begin{cases} \chi(k:0,4)f_{k-4} & (n \equiv_{k+1} f_{k-1}) \\ \chi(k:2,3,4,5)f_{k-3} & (n \equiv_{k+1} f_{k-1}+1). \end{cases}$$

Proof. Assume that $n \equiv_{k+1} f_{k-1}$. Then since $\tau_{k+1}(n) = 0$, we have $n \equiv_{k+2} f_{k-1}$. Therefore, by Lemma 3 and 7, we get

$$H_{n,f_k-1} = \det \begin{pmatrix} A_{f_{k-1}} \cdots A_{f_k-1} A_{f_k} \cdots A_{f_{k+1}-2} \\ B'_{f_{k-1}} \cdots B'_{f_k-1} B'_{f_k} \cdots B'_{f_{k+1}-2} \end{pmatrix},$$

where we use the notation (20). By Lemma 5, the following two subwords of ε:

$$\varepsilon_n \varepsilon_{n+1} \cdots \varepsilon_{n+f_{k-2}+f_k-3} \quad \text{and} \quad \varepsilon_{n+f_{k-1}} \varepsilon_{n+f_{k-1}+1} \cdots \varepsilon_{n+f_{k-1}+f_{k-2}+f_k-3}$$

differ only at two places, namely, at the $(f_k - 2 - f_{k-1})$-th and the $(f_k - 1 - f_{k-1})$-th places. Hence, we have

$$H_{n,f_k-1} = \det \begin{pmatrix} A_{f_{k-1}} \cdots A_{f_k-1} A_{f_k} \cdots A_{f_{k+1}-2} \\ B'_{f_{k-1}} \cdots B'_{f_k-1} B'_{f_k} \cdots B'_{f_{k+1}-2} \end{pmatrix} =$$

$$\det \begin{pmatrix} A_{f_{k-1}} & \cdots & \cdots & A_{f_k-1} & A_{f_k} \cdots A_{f_{k+1}-2} \\ & & (-1)^k & (-1)^{k-1} & \\ & & (-1)^{k-1} & & \\ \mathbf{0} & & \cdots & & \\ & \cdots & & & \mathbf{0} \\ (-1)^k & (-1)^{k-1} & & & \end{pmatrix}.$$

By adding the first $f_{k-2} - 1$ columns and subtracting the last $f_{k-2} - 1$ columns to and from the column beginning by A_{f_k-1}, we get the column

$${}^t(A_{f_k-1} 0 \cdots 0) + {}^t((-1)^{k-1} 0 \cdots 0 (-1)^k 0 \cdots 0),$$

where $(-1)^k$ is at the $(f_{k-2} - 1)$-th place. Since, by Lemma 5

$$(A_{f_{k-1}} \cdots A_{f_k-2}) - (A_{2f_{k-1}} \cdots A_{f_{k+1}-2}) =$$

$$
\begin{pmatrix}
 & & & & (-1)^{k-1} \\
 & 0 & & (-1)^{k-1} & (-1)^{k} \\
 & & \cdots & \cdots & \\
 & \cdots & \cdots & & \\
(-1)^{k-1} & (-1)^{k} & & & 0 \\
(-1)^{k} & & & & \\
 & & 0 & &
\end{pmatrix},
$$

hence, we get

$$
\begin{aligned}
& H_{n,f_k-1} \\
= \; & (-1)^{k(f_{k-2}-1)}(-1)^{f_{k-1}(f_{k-2}-1)+\left[\frac{f_{k-2}-1}{2}\right]} \\
& \Big\{\det(A_{f_k-1}A_{f_k}\cdots A_{f_{k+1}-2}) + (-1)^{k-1}\det(A''_{f_k}\cdots A''_{f_{k+1}-2}) \\
& \quad + (-1)^{k+f_{k-2}-1}\det(A'''_{f_k}\cdots A'''_{f_{k+1}-2})\Big\}, \qquad (22)
\end{aligned}
$$

where

$$
\begin{aligned}
A''_j &:= {}^t(\varepsilon_{j+1}\cdots\varepsilon_{j+f_{k-1}-1}) \\
A'''_j &= {}^t(\varepsilon_j\cdots\varepsilon_{j+f_{k-2}-2}\varepsilon_{j+f_{k-2}}\cdots\varepsilon_{j+f_{k-1}-1}).
\end{aligned}
$$

Here, we have

$$
\begin{aligned}
\det(A_{f_k-1}A_{f_k}\cdots A_{f_{k+1}-2}) &= H_{f_k-1,f_{k-1}} \\
\det(A''_{f_k}\cdots A''_{f_{k+1}-2}) &= H_{f_k+1,f_{k-1}-1}\ , \qquad (23)
\end{aligned}
$$

and by Lemma 5

$$
\det(A'''_{f_k}\cdots A'''_{f_{k+1}-2}) =
$$

$$\begin{pmatrix} & & & 0 & & \\ & & & & & (-1)^{k-1} \\ & & & & (-1)^{k-1} & (-1)^k \\ A'''_{f_k}\cdots A'''_{f_k+f_{k-2}-1} & & & \cdots & \cdots & \\ & & & \cdots & \cdots & \\ & & (-1)^{k-1} & \cdots & & \\ & & (-1)^k & & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} C_{f_k} & & & & & \\ \vdots & & 0 & & & \\ C_{f_k+f_{k-2}-2} & & & & & \\ C_{f_k+f_{k-2}} & & & & & (-1)^{k-1} \\ & & & & (-1)^{k-1} & (-1)^k \\ \vdots & & & \cdots & \cdots & \\ & & & \cdots & \cdots & \\ & & (-1)^{k-1} & \cdots & & \\ C_{f_{k+1}-1} & & (-1)^k & & & 0 \end{pmatrix},$$

where we put

$$C_j = (\varepsilon_j \varepsilon_{j+1} \cdots \varepsilon_{j+f_{k-2}-1}).$$

Since $C_{f_k+f_{k-2}+j} = C_{f_k+j}$ $(j = 0, 1, \cdots, f_{k-3} - 2)$ by Lemma 5, it holds that

$$\begin{aligned} &\det(A'''_{f_k}\cdots A'''_{f_{k+1}-2}) \\ = \ &(-1)^{(k-1)(f_{k-3}-1)+f_{k-3}-1+\left[\frac{f_{k-3}-1}{2}\right]} \det \begin{pmatrix} C_{f_k} \\ \vdots \\ C_{f_k+f_{k-2}-2} \\ C_{f_{k+1}-1} \end{pmatrix}. \end{aligned}$$

Moreover it follows from Lemma 5 that

$$\det \begin{pmatrix} C_{f_k} \\ \vdots \\ C_{f_k+f_{k-2}-2} \\ C_{f_{k+1}-1} \end{pmatrix} = \det \begin{pmatrix} C_{f_{k+1}} \\ \vdots \\ C_{f_{k+1}+f_{k-2}-2} \\ C_{f_{k+1}-1} \end{pmatrix} = (-1)^{f_{k-2}-1} H_{f_{k+1}-1, f_{k-2}} ,$$

which implies

$$\det(A'''_{f_k} \cdots A'''_{f_{k+1}-2}) = \chi(k:0,3,5) H_{f_{k+1}-1, f_{k-2}} \ .$$

Thus by (22), (23), Theorem 3 and Lemma 16, we obtain

$$\begin{array}{rl} & H_{n,f_k-1} \\ = & \chi(k:4) H_{f_k-1, f_{k-1}} + \chi(k:0,2) H_{f_k+1, f_{k-1}-1} + \chi(k:1,3,4) H_{f_{k+1}-1, f_{k-2}} \\ = & \chi(k:2,3,4,5) f_{k-3} + \chi(k:2,3,4,5) f_{k-4} + \chi(k:0,1) f_{k-4} \\ = & \chi(k:2,3,4,5) f_{k-3}, \end{array}$$

which is the first case of our lemma.

To prove the second case, assume that $n \equiv_{k+1} f_{k-1}+1$. Then, as above we get

$$H_{n,f_k-1} = \det \begin{pmatrix} A_{f_{k-1}+1} \cdots A_{f_k-1} A_{f_k} \cdots A_{f_{k+1}-1} \\ B'_{f_{k-1}+1} \cdots B'_{f_k-1} B'_{f_k} \cdots B'_{f_{k+1}-1} \end{pmatrix} =$$

$$\det \begin{pmatrix} A_{f_{k-1}+1} & \cdots & \cdots & A_{f_k-1} & A_{f_k} \cdots A_{f_{k+1}-1} \\ & & (-1)^k & (-1)^{k-1} & \\ & & (-1)^{k-1} & & \\ \mathbf{0} & & \cdots & & \\ & \cdots & & & \mathbf{0} \\ (-1)^k & (-1)^{k-1} & & & \\ (-1)^{k-1} & & & & \end{pmatrix}$$

$$= (-1)^{(k-1)(f_{k-2}-1)} (-1)^{(f_{k-2}-1)f_{k-1} + \left[\frac{f_{k-2}-1}{2}\right]} \det(A_{f_k} \cdots A_{f_{k+1}-1}) \ .$$

Therefore, we get by Theorem 3

$$\begin{array}{rcl} H_{n,f_k-1} & = & \chi(k:0,3,4) \chi(k-1:2) f_{k-2} \\ & = & \chi(k:0,4) f_{k-2} \ , \end{array}$$

which completes the proof for H_{n,f_k-1}. ∎

Theorem 4 *For any $k, n, i \in \mathbf{N}$ with $k \geq 1$, $n \equiv_{k+1} i$ and $0 \leq i < f_{k+1}$, we have*

$$H_{n,f_k-1} = \begin{cases} \chi(k:0,4)f_{k-2} & (i=0) \\ \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-3} & (0 < i \leq f_{k-1}) \\ \chi(k:0,4)\chi(k:1,4)^i f_{k-2} & \left(\begin{array}{l} f_{k-1} < i < f_k \\ \textit{and } \tau_{k+1}(n) = 1 \end{array}\right) \\ \chi(k:0,4)f_{k-2} & \left(\begin{array}{l} i = f_{k-1}+1 \\ \textit{and } \tau_{k+1}(n) = 0 \end{array}\right) \\ 0 & (\textit{ otherwise }) \end{cases}$$

$$\overline{H}_{n,f_k-1} = \begin{cases} \chi(k:2,3,4,5)f_{k-3} & (i=0) \\ \chi(k:0,1)\chi(k:1,4)^i f_{k-4} & (0 < i \leq f_{k-1}) \\ \chi(k:2,3,4,5)\chi(k:1,4)^i f_{k-3} & \left(\begin{array}{l} f_{k-1} < i < f_k \\ \textit{and } \tau_{k+1}(n) = 1 \end{array}\right) \\ \chi(k:2,3,4,5)f_{k-3} & \left(\begin{array}{l} i = f_{k-1}+1 \\ \textit{and } \tau_{k+1}(n) = 0 \end{array}\right) \\ 0 & (\textit{ otherwise }). \end{cases}$$

Proof. The first four cases follow from Lemma 16 and 17. Note that for $i = f_{k-1}$, the assertion in these lemmas coincide, so that H_{n,f_k-1} is independent of $\tau_{k+1}(n)$. Let us consider the last case, where $\tau_{k+1}(n) = 0$ and $f_{k-1} + 2 \leq i \leq f_{k+1} - 1$. We may assume that $k \geq 2$. Then, with $m = f_k - 1$ and $i - f_k$ in place of i there, the condition (2) of Theorem 1 is satisfied. Therefore by Theorem 1, $n \in \mathcal{R}_m$ which implies that $H_{n,f_k-1} = 0$. ■

Lemma 18 *For any $n, m \in \mathbf{N}$ such that $f_{k-2} + 1 \leq m \leq f_k - 2$, $i \leq n$ and $n - i \equiv_{k+1} 0$ for some $i, k \in \mathbf{Z}$ with $k \geq 2$ and $m + i = f_k$. Then, we have*

$$H_{n,m} = \chi(k:2)\chi(k:3,4,5)^i(-1)^{[i/2]}f_{k-3}$$

$$\overline{H}_{n,m} \quad = \quad \chi(k:1,4)\chi(k:0,1,2)^i(-1)^{[i/2]}f_{k-3} \ .$$

Proof. At first, we consder the case $i < f_{k-2}$. By arguments similar to those used in the proof of Lemma 15, we get with the notation (20)

$$H_{n,m} =$$
$$\det\begin{pmatrix} A_iA_{i+1} & \cdots & A_{f_{k-1}+i-1} & 0 & & \cdots & 0 & A_{f_k-1} \\ & & & & & & (-1)^k & (-1)^{k-1} \\ & & & & & & \cdots & \\ & 0 & & & & \cdots & & 0 \\ & & (-1)^k & & (-1)^{k-1} & & & \end{pmatrix}.$$

Therefore, by Theorem 3 and 4,

$$\begin{aligned} & H_{n.m} \\ = \ & (-1)^{k(f_{k-2}-i+1)+\left[\frac{f_{k-2}-i+1}{2}\right]}H_{i,f_{k-1}-1} + (-1)^{(k-1)(f_{k-2}-i)+\left[\frac{f_{k-2}-i}{2}\right]}H_{i,f_{k-1}} \\ = \ & \chi(k:2)\chi(k:3,4,5)^i(-1)^{[i/2]}(-f_{k-4}+f_{k-2}) \\ = \ & \chi(k:2)\chi(k:3,4,5)^i(-1)^{[i/2]}f_{k-3} \ . \end{aligned}$$

If $i = f_{k-2}$, then the lemma follows from Theorem 3.

Finally, we consider the case $f_{k-2} < i < f_{k-1}$. Then, denoting

$$A_j^r =^t (\varepsilon_j\varepsilon_{j+1}\cdots\varepsilon_{i+r-1}), \tag{24}$$

we obtain by Theorem 3

$$H_{n,m} = \det(A_i^{f_k-i}A_{i+1}^{f_k-i}\cdots A_{f_k-1}^{f_k-i}) =$$

$$\det\begin{pmatrix} A_i^{f_{k-2}} & A_{i+1}^{f_{k-2}} & \cdots & A_{f_{k-1}-2}^{f_{k-2}} & A_{f_{k-1}-1}^{f_{k-2}} & A_{f_{k-1}}^{f_{k-2}} & \cdots & A_{f_k-1}^{f_{k-2}} \\ & & & (-1)^{k-1} & (-1)^k & & & \\ & 0 & & (-1)^k & & & & \\ & & \cdots & \cdots & & & & \\ & & \cdots & \cdots & & & 0 & \\ (-1)^{k-1} & (-1)^k & & & & & & \\ (-1)^k & & & & & & & \end{pmatrix}$$

$$\begin{aligned} &= (-1)^{k(f_{k-1}-i)}(-1)^{(f_{k-1}-i)f_{k-2}+\left[\frac{f_{k-1}-i}{2}\right]}H_{f_{k-1},f_{k-2}} \\ &= \chi(k:2)\chi(k:3,4,5)^i(-1)^{[i/2]}f_{k-3} \ , \end{aligned}$$

which completes the proof for $H_{n,m}$. ■

Lemma 19 *For any* $n, m \in \mathbf{N}$ *such that* $f_{k-1}+1 \le m \le f_k - 2$, $i \le n$, $n - i \equiv_k f_{k-1}$ *for some* $i, k \in \mathbf{Z}$ *with* $k \ge 2$ *and* $m + i = f_k$, *we have*

$$\begin{aligned} H_{n,m} &= \chi(k:1,2,4)\chi(k:0,1,2)^i(-1)^{[i/2]} f_{k-2} \\ \overline{H}_{n,m} &= \chi(k:2)\chi(k:3,4,5)^i(-1)^{[i/2]} f_{k-3} \ . \end{aligned}$$

Proof. By the same arguments and in the same notations as in the second part of the proof of Lemma 18, we obtain

$$H_{n,m} = \det(A^{f_k-i}_{f_{k-1}+i} \cdots A^{f_k-i}_{f_k-1} A^{f_k-i}_{f_k} \cdots A^{f_k-i}_{f_{k+1}-1}) =$$

$$\det \begin{pmatrix} A_i^{f_{k-1}} & A_{i+1}^{f_{k-1}} & \cdots & A^{f_{k-1}}_{f_{k-2}-2} & A^{f_{k-1}}_{f_{k-2}-1} & A^{f_{k-1}}_{f_k} & \cdots & A^{f_{k-1}}_{f_{k+1}-1} \\ & & & (-1)^k & (-1)^{k-1} & & & \\ & 0 & & (-1)^{k-1} & & & & \\ & & \cdots & \cdots & & & & \\ & & \cdots & \cdots & & & 0 & \\ (-1)^k & (-1)^{k-1} & & & & & & \\ (-1)^{k-1} & & & & & & & \end{pmatrix}$$

$$\begin{aligned} &= (-1)^{(k-1)(f_{k-2}-i)}(-1)^{(f_{k-2}-i)f_{k-1}+\left[\frac{f_{k-2}-i}{2}\right]} H_{f_k, f_{k-1}} \\ &= \chi(k:1,2,4)\chi(k:0,1,2)^i(-1)^{[i/2]} f_{k-2} \ , \end{aligned}$$

which completes the proof for $H_{n,m}$. ■

Lemma 20 *For any* $n, m \in \mathbf{N}$ *such that* $f_{k-1}+1 \le m \le f_k - 2$, $i \le n$ *and* $n - i \equiv_{k+1} 0$ *for some* $i, k \in \mathbf{Z}$ *with* $k \ge 2$ *and* $m + i = f_k - 1$, *we have*

$$\begin{aligned} H_{n,m} &= \chi(k:0,4)\chi(k:3,4,5)^i(-1)^{[i/2]} f_{k-2} \\ \overline{H}_{n,m} &= \chi(k:2,3,4,5)\chi(k:0,1,2)^i(-1)^{[i/2]} f_{k-3} \ . \end{aligned}$$

Proof. The proof is similar to the first part of the proof of Lemma 18. With the notation in (20), we get

$$H_{n,m} =$$

$$\det\begin{pmatrix} A_iA_{i+1} & \cdots & A_{f_{k-1}+i-1} & 0 & 0 & \cdots & 0 \\ & & & & & & (-1)^k \\ & & & & & \cdots & \\ & 0 & & & \cdots & & \\ & & & (-1)^k & (-1)^{k-1} & & 0 \end{pmatrix}$$

$$= (-1)^{k(f_{k-2}-1-i)}(-1)^{\left[\frac{f_{k-2}-1-i}{2}\right]}\det(A_iA_{i+1}\cdots A_{f_{k-1}+i-1}).$$

Hence, by Theorem 3

$$H_{n,m} = \chi(k:0,4)\chi(k:3,4,5)^i(-1)^{[i/2]}f_{k-2}\ ,$$

which completes the proof for $H_{n,m}$.. ■

Lemma 21 *For any $n, m \in \mathbf{N}$ such that $f_{k-2}+1 \le m \le f_k - 2$, $i \le n$ and $n - i \equiv_k f_{k-1}$ for some $i, k \in \mathbf{Z}$ with $k \ge 2$ and $m + i = f_k - 1$, we have*

$$\begin{aligned} H_{n,m} &= \chi(k:2,3,4,5)\chi(k:0,1,2)^i(-1)^{[i/2]}f_{k-3} \\ \overline{H}_{n,m} &= \chi(k:0,4)\chi(k:3,4,5)^i(-1)^{[i/2]}f_{k-4}\ . \end{aligned}$$

Proof. Since $i = f_k - 1 - m$, we get $1 \le i \le f_{k-1} - 2$.

If $i = f_{k-2} - 1$, then $m = f_{k-1}$ and $n \equiv_k f_k - 1$. Therefore, by Theorem 3, we get

$$H_{n,m} = \chi(k-1:1,2,4)f_{k-3},$$

which coincides with the required identity since

$$\chi(k:0,1,2)^{f_{k-2}-1} = \chi(k:\{0,1,2\}\cap\{0,3\}) = \chi(k:0),$$
$$(-1)^{\left[\frac{f_{k-2}-1}{2}\right]} = \chi(k:0,4).$$

If $i = f_{k-2}$, then $m = f_{k-1} - 1$ and $n \equiv_k 0$. Therefore, by Theorem 4, we get

$$H_{n,m} = \chi(k-1:0,4)f_{k-3},$$

which coincides with the required statement since

$$\chi(k:0,1,2)^{f_{k-2}} = \chi(k:\{0,1,2\}\cap\{1,2,4,5\}) = \chi(k:1,2),$$
$$(-1)^{\left[\frac{f_{k-2}}{2}\right]} = \chi(k:3,4).$$

If $f_{k-2}+1 \leq i \leq f_{k-1}-2$, then $n-i' \equiv_k 0$ with $i' := i - f_{k-2}$. Then, since $m+i' = f_{k-1}-1$ and $f_{k-2}+1 \leq m \leq f_{k-1}-2$, applying Lemma 20, we obtain

$$\begin{aligned} & H_{n,m} = \chi(k-1:0,4)\chi(k-1:3,4,5)^{i'}(-1)^{[i'/2]}f_{k-3} \\ = \ & \chi(k:1,5)\chi(k:0,4,5)^{i}\chi(k:\{0,4,5\}\cap\{1,2.4.5\})(-1)^{[i'/2]}f_{k-3} \\ = \ & \chi(k:1,4)\chi(k:0,4,5)^{i}(-1)^{[i/2]}(-1)^{\left[\frac{f_{k-2}+1}{2}\right]}(-1)^{if_{k-2}}f_{k-3} \\ = \ & \chi(k:2,3,4,5)\chi(0,1,2)^{i}(-1)^{[i/2]}f_{k-3}\ . \end{aligned}$$

Now, we consider the case $1 \leq i \leq f_{k-2}-2$. Then, with the notations in (24) and in (20), we get

$$H_{n,m} = \det(A^{f_k-i}_{f_{k-1}+i}\cdots A^{f_k-i}_{f_k-1}A^{f_k-i}_{f_k}\cdots A^{f_k-i}_{f_{k+1}-2}) =$$

$$\det\begin{pmatrix} A_{f_{k-1}+i} & A_{f_{k-1}+i+1} & \cdots & A_{f_k-2} & A_{f_k-1} & A_{f_k} & \cdots & A_{f_{k+1}-2} \\ & & & (-1)^k & (-1)^{k-1} & & & \\ & 0 & & (-1)^{k-1} & & & & \\ & & \cdots & \cdots & & & & \\ & & \cdots & \cdots & & & 0 & \\ & (-1)^k & & & & & & \\ (-1)^k & (-1)^{k-1} & & & & & & \end{pmatrix}.$$

Therefore, by arguments similar to those used in the first part of the proof of Lemma 17, we get

$$\begin{aligned} H_{n,m} \ = \ & (-1)^{k(f_{k-2}-1-i)}(-1)^{f_{k-1}(f_{k-2}-1-i)+\left[\frac{f_{k-2}-1-i}{2}\right]} \\ & \Big\{\det(A_{f_k-1}A_{f_k}\cdots A_{f_{k+1}-2}) + (-1)^{k-1}\det(A''_{f_k}\cdots A''_{f_{k+1}-2}) \\ & \quad + (-1)^{k+f_{k-2}-1-i}\det(A'''_{f_k}\cdots A'''_{f_{k+1}-2})\Big\}, \end{aligned}$$

where we use the same notations as in the proof of Lemma 17 except for A'''_j's which are defined by

$$A'''_j = {}^t\,(\varepsilon_j\cdots\varepsilon_{j+f_{k-2}-i-2}\ \varepsilon_{j+f_{k-2}-i}\cdots\varepsilon_{j+f_{k-1}-1}).$$

Then, following the arguments there, we get

$$\begin{aligned} H_{n,m} &= \chi(k:4)\chi(k:0,1,2)^i(-1)^{[i/2]}\Big\{H_{f_k-1,f_{k-1}} \\ &\quad + (-1)^{k-1}H_{f_k+1,f_{k-1}-1} + (-1)^{k+f_{k-2}-1-i}E\Big\} \end{aligned}$$

with

$$\begin{aligned} E &:= \det(A'''_{f_k}\cdots A'''_{f_{k+1}-2}) \\ &= \det(A'_{f_k}\cdots A'_{f_k+f_{k-2}-i-2}A'_{f_k+f_{k-2}-i}\cdots A'_{f_{k+1}-1}) \\ &= \det(A'_{f_{k+1}}\cdots A'_{f_{k+1}+f_{k-2}-i-2}A'_{f_k+f_{k-2}-i}\cdots A'_{f_{k+1}-1}) \\ &= (-1)^{(f_{k-2}-i-1)(f_{k-3}+i)}\det(A'_{f_k+f_{k-2}-i}\cdots A'_{f_{k+1}+f_{k-2}-i-2}) \\ &= (-1)^{(f_{k-2}-i-1)(f_{k-3}+i)}H_{f_{k-2}-i,f_{k-1}-1}\ , \end{aligned}$$

where we have used Lemma 5. Therefore, by Theorem 3 and 4, we have

$$\begin{aligned} &H_{n,m} = \chi(k:4)\chi(k:0,1,2)^i(-1)^{[i/2]}\{\chi(k-1:1,2,4)f_{k-3} \\ &+(-1)^{k-1}\chi(k-1:2,3,4,5)f_{k-4} + (-1)^{k+f_{k-2}-1-i}(-1)^{(f_{k-2}-i-1)(f_{k-3}+i)} \\ &\chi(k-1:1,2,3,5)\chi(k-1:1,4)^{f_{k-2}-i}f_{k-4}\Big\} \\ &= \chi(k:2,3,4,5)\chi(k:0,1,2)^i(-1)^{[i/2]}f_{k-3}, \end{aligned}$$

which completes the proof for $H_{n,m}$. ■

4 Tiling for $H_{n,m}$ and $\overline{H}_{n.m}$

In this section, we collect the values of $H_{n,m}$ and $\overline{H}_{n.m}$ obtained in the last section and arrange them in the quarter plane $\Omega := \{0,1,2,\cdots\}\times\{1,2,3,\cdots\}$. We will tile Ω by the following tiles on which the values $H_{n,m}$ are written in. That is,

$$\begin{aligned} U_1 &:= V_1 := \{(1,-1)\} \\ U_k &:= \{(i,j)\in \mathbf{Z}^2;\ 0\le i+j\le f_{k-1}-1,\ -f_{k-1}\le j\le -1\} \\ V_k &:= \{(i,j)\in \mathbf{Z}^2;\ 0\le i+j\le f_{k-2}-1,\ -f_{k-2}\le j\le -1\} \\ &\qquad (\ k=2,3,4,\cdots\ ) \end{aligned}$$

with the written-in values $u_k : U_k \to \mathbf{Z}$, $v_k : V_k \to \mathbf{Z}$:

$$u_1(1,-1) := 0 \ , \quad v_1(1,-1) := 1$$

$$u_k(i,j) \ := \ \begin{cases} \chi(k:2)\chi(k:3,4,5)^i(-1)^{[i/2]}f_{k-3} & (i+j=0) \\ \chi(k:0,3,4)\chi(k:0,3)^i f_{k-3} & (j=-f_{k-1}) \\ \chi(k:3,5)\chi(k:2,3,4)^i(-1)^{[i/2]}f_{k-3} & (i+j=f_{k-1}-1) \\ \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-3} & (j=-1) \\ 0 & (\text{ otherwise }) \end{cases}$$

$$v_k(i,j) \ := \ \begin{cases} \chi(k:1,2,4)\chi(k:0,1,2)^i(-1)^{[i/2]}f_{k-2} & (i+j=0) \\ \chi(k:2,3,5)\chi(k:2,5)^i f_{k-2} & (j=-f_{k-2}) \\ \chi(k:0,1,2,3)\chi(k:1,2,3)^i(-1)^{[i/2]}f_{k-2} & (i+j=f_{k-2}-1) \\ \chi(k:0,1)\chi(k:1,4)^i f_{k-2} & (j=-1) \\ 0 & (\text{ otherwise }) \end{cases}$$

$$(\ k = 2,3,4,\cdots \ ) \ ,$$

$\overline{u}_k : U_k \to \mathbf{Z}$ and $\overline{v}_k : V_k \to \mathbf{Z}$:

$$\overline{u}_1(1,-1) := 1 \ , \quad \overline{v}_1(1,-1) := 0$$

$$\overline{u}_k(i,j) \ := \ \begin{cases} \chi(k:1,4)\chi(k:0,1,2)^i(-1)^{[i/2]}f_{k-4} & (i+j=0) \\ \chi(k:4)\chi(k:0,3)^i f_{k-4} & (j=-f_{k-1}) \\ \chi(k:1,2,3,4)\chi(k:0,1,5)^i(-1)^{[i/2]}f_{k-4} & (i+j=f_{k-1}-1) \\ \chi(k:0,1)\chi(k:1,4)^i f_{k-4} & (j=-1) \\ 0 & (\text{ otherwise }) \end{cases}$$

$$\overline{v}_k(i,j) \ := \ \begin{cases} \chi(k:2)\chi(k:3,4,5)^i(-1)^{[i/2]}f_{k-3} & (i+j=0) \\ \chi(k:3)\chi(k:2,5)^i f_{k-3} & (j=-f_{k-2}) \\ \chi(k:2,4)\chi(k:0,4,5)^i(-1)^{[i/2]}f_{k-3} & (i+j=f_{k-2}-1) \\ \chi(k:1,2,3,5)\chi(k:1,4)^i f_{k-3} & (j=-1) \\ 0 & (\text{ otherwise }) \end{cases}$$

$$(\ k = 2,3,4,\cdots \ ).$$

Let

$$\mathcal{U}_k \ := \ \{(n, f_k);\ n \in \mathbf{N} \text{ and } n \equiv_{k+1} 0\}$$

$$\begin{aligned}\mathcal{V}_k &:= \{(n, f_k); n \in \mathbf{N} \text{ and } n \equiv_{k+2} f_{k+1} + f_{k-1}\} \\ T_k &:= (V_k + (-f_{k-2}, f_k)) \cap \Omega \\ &\qquad (k = 1, 2, 3, \cdots),\end{aligned}$$

where $V + (x, y) := \{v + x, w + y); (v, w) \in V\}$ for $V \subset \mathbf{Z}^2$, $(x.y) \in \mathbf{Z}^2$.

Theorem 5 *It holds that*

$$\Omega = \bigcup_{k=1}^{\infty} \left(\bigcup_{(i,j)\in\mathcal{U}_k} (U_k + (i, j)) \cup \bigcup_{(i,j)\in\mathcal{V}_k} (V_k + (i, j)) \cup T_k \right),$$

where the right hand side is a disjoint union, so that Ω *is tiled by the tiles* U_k*'s,* V_k*'s and* T_k*'s. Moreover, for any* $(n, m) \in \Omega$*, if* $(n, m) = (i, j) + (i', j')$ *with* $(i, j) \in U_k$ *and* $(i', j') \in \mathcal{U}_k$ *, then we have* $H_{n,m} = u_k(i, j)$ *and* $\overline{H}_{n,m} = \overline{u}_k(i, j)$*. Also, if* $(n, m) = (i, j) + (i', j')$ *with* $(i, j) \in V_k$ *and either* $(i', j') \in \mathcal{V}_k$ *or* $(i', j') = (-f_{k-2}, f_k)$*, then we have* $H_{n,m} = v_k(i, j)$ *and* $\overline{H}_{n,m} = \overline{v}_k(i, j)$*. Furthermore, in this tiling, the tiles* U_k*,* V_k *and* T_k *with* $k \geq 2$ *are followed by the sequences of smaller tiles* $U_{k-1}V_{k-1}U_{k-1}$*,* U_{k-1} *and* U_{k-1}*, respectively, as shown in Figure 1.*

Proof. Take an arbitrary point $(n, m) \in \Omega$. Let $f_{k-1} \leq m < f_k$. If $n + m - f_k \geq 0$, define $0 \leq i < f_{k+2}$ by $i \equiv_{k+2} n$.

Case 1 $n + m - f_k < 0$: We get $(n, m) \in T_k$.

Case 2 $0 \leq i < f_{k-1}$: We get $(n, m) \in U_k + (n + m - i - f_k, f_k)$.

Case 3 $f_{k-1} \leq i < f_{k+1}$: We get $(n, m) \in U_{k+1} + (n+m-i-f_{k+1}, f_{k+1})$.

Case 4 $f_{k+1} \leq i < f_{k+1}+f_{k-1}$: We get $(n, m) \in U_k+(n+m-i+f_{k-1}, f_k)$.

Case 5 $f_{k+1}+f_{k-1} \leq i < f_{k+2}$: We get $(n, m) \in V_k+(n+m-i+2f_{k-1}, f_k)$.

The fact that the written-in values coincide with $H_{n,m}$ and $\overline{H}_{n,m}$ follows from Lemma 18 (first case in u_k and $\overline{u}_k$), Theorem 3 (second case), Lemma 21 (third case), Theorem 4 (fourth case), Corollary 3 (fifth case), Lemma 19 (first case in v_k and $\overline{v}_k$), Theorem 3 (second case), Lemma 20 (third case), Lemma 20 (fourth case) and Corollary 3 (fifth case). The m in the preceding lemmas and theorems coincides with $f_k + j$ in Theorem 5 while the meanings of the symbols k, i, n are not necessarily the same between them. ∎

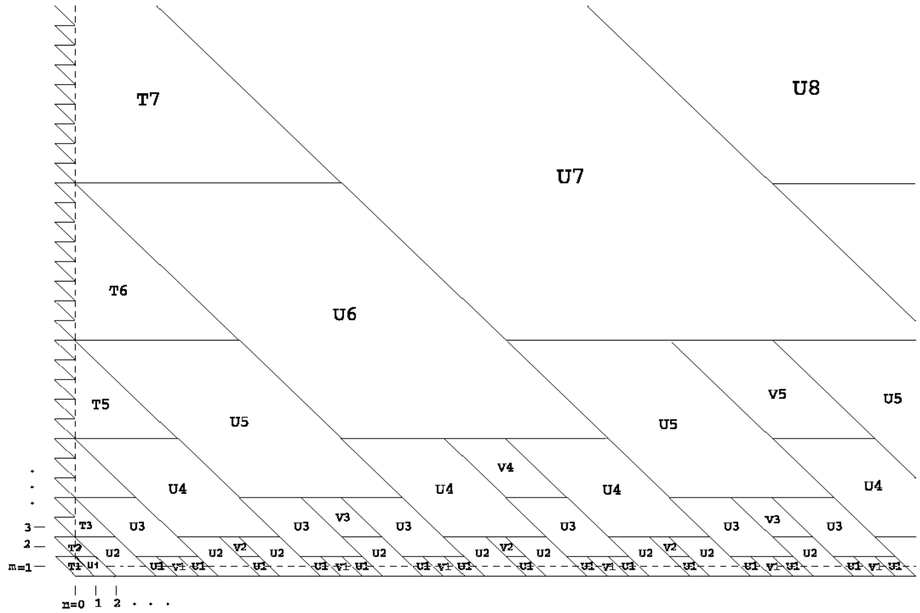

Figure 1: Tiling for $H_{m,n}$

5 Padé approximation

Let $\varphi = \varphi_0\varphi_1\varphi_2\cdots$ be an infinite sequence over a field $\mathbf{K}$, $\hat{H}_{n,m} := H_{n,m}(\varphi)$ be the Hankel determinant (3), and $\varphi(z)$ the formal Laurent series (4) with $h = -1$. We also denote the **Hankel matrices** by

$$\hat{M}_{n,m} := (\varphi_{n+i+j})_{i,j=0,1,\cdots m-1} \qquad (25)$$
$$(n = 0, 1, 2, \ldots;\ m = 1, 2, 3, \cdots),$$

so that $\hat{H}_{n,m} = \det \hat{M}_{n,m}$.

The following proposition is well known ([1], for example). But we give a proof for self-containedness.

Proposition 1 *(1) For any $m = 1, 2, \cdots$, a Padé pair (P, Q) of order m for φ exists. Moreover, for each m, the rational function $P/Q \in \mathbf{K}(z)$ is determined uniquely for such Padé pairs (P, Q).*
(2) For any $m = 1, 2, \cdots$, m is a normal index for φ if and only if $\hat{H}_{0,m}(\varphi) \neq 0$.

Proof. Let

$$\begin{aligned} P &= p_0 + p_1 z + p_2 z^2 + \cdots + p_m z^m \\ Q &= q_0 + q_1 z + q_2 z^2 + \cdots + q_m z^m. \end{aligned}$$

Then, the condition $\| Q\varphi - P \| < \exp(-m)$ is equivalent to

$$\begin{array}{llllll} & & & & -p_m & = 0 \\ & & & q_m\varphi_0 & -p_{m-1} & = 0 \\ & & & \cdots & \cdots & \\ & q_1\varphi_0 + & \cdots + & q_m\varphi_{m-1} & -p_0 & = 0 \\ q_0\varphi_0 + & q_1\varphi_1 + & \cdots + & q_m\varphi_m & & = 0 \\ & \cdots & \cdots & & & \\ & \cdots & \cdots & & & \\ q_0\varphi_{m-1} + & q_1\varphi_{m-2} + & \cdots + & q_m\varphi_{2m-1} & & = 0. \end{array} \tag{26}$$

Furthermore, Equation (26) for $(q_0 q_1 \cdots q_m)$ is equivalent to

$$(q_0 q_1 \cdots q_{m-1})\hat{M}_{0,m} + q_m(\varphi_m \varphi_{m+1} \cdots \varphi_{2m-1}) = (00 \cdots 0), \tag{27}$$

where $(p_0 p_1 \cdots p_m)$ is determined by $(q_0 q_1 \cdots q_m)$ by the upper half of Equation (26). There are two cases.

Case 1: $\hat{H}_{0,m} = 0$. In this case, since $\det \hat{M}_{0,m} = \hat{H}_{0,m} = 0$, there exists a nonzero vector $(q_0 q_1 \cdots q_{m-1})$ such that $(q_0 q_1 \cdots q_{m-1})\hat{M}_{0,m} = O$. Then, Equation (27) is satisfied with this $(q_0 q_1 \cdots q_{m-1})$ and $q_n = 0$.

Case 2: $\hat{H}_{0,m} \neq 0$. In this case, since $\det \hat{M}_{0,m} = \hat{H}_{0,m} \neq 0$, there exists a unique vector $(q_0 q_1 \cdots q_{m-1})$ such that

$$(q_0 q_1 \cdots q_{m-1})\hat{M}_{0,m} = -(\varphi_m \varphi_{m+1} \cdots \varphi_{2m-1}). \tag{28}$$

Then, (27) is satisfied with this $(q_0 q_1 \cdots q_{m-1})$ and $q_m = 1$.

Thus, a Padé pair of order m exists. Moreover, by the above arguments, a Padé pair (P, Q) of order m with $\deg Q < m$ exists if and only if $\hat{H}_{0,m} = 0$, since if $\hat{H}_{0,m} \neq 0$, then by (27), $q_m = 0$ implies $(q_0 q_1 \cdots q_{m-1}) = (00 \cdots 0)$ and hence, $Q = 0$.

Now we prove that for any Padé pairs (P, Q) and (P', Q') of order m, it holds $P/Q = P'/Q'$. By (5), we have

$$\| \varphi - P/Q \| < \exp(-n - \deg Q)$$

and

$$\| \varphi - P'/Q' \| < \exp(-m - \deg Q').$$

Hence, we have

$$\| P/Q - P'/Q' \| < \exp(-m - \deg Q \wedge \deg Q').$$

Therefore,

$$\| PQ' - P'Q \| < \exp(-m + \deg Q \vee \deg Q') \leq 1.$$

Since $PQ' - P'Q$ is a polynomial of z, $\| PQ' - P'Q \|$ is either 0 or not less than 1. Hence, the above inequality implies $PQ' - P'Q = 0$, which completes the proof. ∎

In view of (26), without loss of generality, we can put

$$\begin{aligned} P &= p_0 + p_1 z + p_2 z^2 + \cdots + p_{m-1} z^{m-1} \\ Q &= q_0 + q_1 z + q_2 z^2 + \cdots + q_m z^m. \end{aligned} \tag{29}$$

Theorem 6 *Let (P, Q) be the normalized Padé pair for φ with $\deg Q$ as its normal index m with P, Q given by (29). Then, we have*
(1) $Q(z) = \hat{H}_{0,m}^{-1} \det(z\hat{M}_{0,m} - \hat{M}_{1,m})$.
(2) $\det(zI - \hat{M}_{0,m}) =$

$$\begin{vmatrix}
z & & & & & & & & & & & & & 1 \\
 & z & & & & & & & & & & & 1 & \\
 & & \ddots & & & & & & & & & \iddots & & \\
 & & & z & & & & & & & 1 & & & \\
 & & & & z & & & & & 1 & & & & \\
p_{m-1} & & & & & 1 & & & & & & & & \\
p_{m-2} & p_{m-1} & & & & q_{m-1} & 1 & & & & & & & \\
\vdots & \ddots & \ddots & & & \vdots & \ddots & \ddots & & & & & & \\
p_1 & \cdots & & p_{m-1} & & q_2 & \cdots & & 1 & & & & & \\
p_0 & \cdots & & p_{m-2} & p_{m-1} & q_1 & \cdots & & q_{m-1} & 1 & & & & \\
 & p_0 & \cdots & \vdots & p_{m-2} & q_0 & \cdots & \cdots & \vdots & q_{m-1} & 1 & & & \\
 & & \ddots & p_1 & \vdots & & q_0 & \cdots & \vdots & \vdots & & \ddots & & \\
 & & & p_0 & p_1 & & & \ddots & q_1 & q_2 & \cdots & \cdots & 1 & \\
 & & & & p_0 & & & & q_0 & q_1 & \cdots & \cdots & q_{m-1} & 1
\end{vmatrix}$$

where I is the unit matrix of size m.
(3)

$$\hat{H}_{0,m} = (-1)^{[m/2]} \prod_{z;Q(z)=0} P(z) = (-1)^{[m/2]} p_k^m \prod_{z;P(z)=0} Q(z),$$

where $\prod_{z;R(z)=0}$ *denotes the product over all the roots of the polynomial $R(z)$ with their multiplicity and p_k is the leading coefficient of $P(z)$, that is, $p_{m-1} = \cdots = p_{k+1} = 0$, $p_k \neq 0$ if $P(z)$ is not the zero polynomial, otherwise, $p_k = 0$.*

Proof. (1) Note that $q_m = 1$ by the assumption that (P, Q) is the normalized Padé pair. By (28), we have

$$\begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ -q_0 & -q_1 & & \cdots & -q_{m-2} & -q_{m-1} \end{pmatrix} \hat{M}_{0,m} = \hat{M}_{1,m} \ .$$

Since $\hat{H}_{0,m} = \det \hat{M}_{0,m} \neq 0$ by the normality of the index m, it follows that

$$\begin{aligned} Q(z) &= \det \left(zI - \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ -q_0 & -q_1 & & \cdots & -q_{m-2} & -q_{m-1} \end{pmatrix} \right) \\ &= \det(zI - \hat{M}_{1,m} \hat{M}_{0,m}^{-1}) \\ &= \hat{H}_{0,m}^{-1} \det(z\hat{M}_{0,m} - \hat{M}_{1,m}). \end{aligned}$$

(2) We define the matrices:

$$
P_m := \begin{pmatrix} p_{m-1} & p_{m-2} & \cdots & p_1 & p_0 \\ p_{m-2} & \cdots & \cdots & p_0 & \\ \vdots & & \iddots & & \\ p_1 & \iddots & & \mathbf{0} & \\ p_0 & & & & \end{pmatrix}
$$

$$
P'_{m-1} := \begin{pmatrix} & & & & p_{m-1} \\ & \mathbf{0} & & p_{m-1} & p_{m-2} \\ & & \iddots & \vdots & \vdots \\ & \iddots & & \vdots & p_2 \\ p_{m-1} & p_{m-2} & \cdots & p_2 & p_1 \end{pmatrix}
$$

$$
Q_m := \begin{pmatrix} 1 & & & & \\ q_{m-1} & 1 & & \mathbf{0} & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ q_1 & q_2 & \cdots & q_{m-1} & 1 \end{pmatrix}
$$

$$
Q'_m := \begin{pmatrix} & & & & 1 \\ & \mathbf{0} & & 1 & q_{m-1} \\ & & \iddots & & \vdots \\ & \iddots & & & \vdots \\ 1 & q_{m-1} & \cdots & q_2 & q_1 \end{pmatrix}
$$

$$
Q''_{m-1} := \begin{pmatrix} 1 & & & & \\ q_{m-1} & 1 & & \mathbf{0} & \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ q_2 & q_3 & \cdots & q_{m-1} & 1 \end{pmatrix}
$$

$$Q_{m,m-1} := \begin{pmatrix} q_1 & q_2 & \cdots & q_{m-2} & q_{m-1} \\ q_0 & q_1 & \cdots & q_{m-3} & q_{m-2} \\ & q_0 & q_1 & \cdots & q_{m-3} \\ & & \ddots & \ddots & \vdots \\ & \mathbf{0} & & \ddots & q_1 \\ & & & & q_0 \end{pmatrix}$$

$$\Phi_{m-1} := \begin{pmatrix} & & & & \varphi_0 \\ & \mathbf{0} & & \varphi_0 & \varphi_1 \\ & & \iddots & \vdots & \vdots \\ & \iddots & & \vdots & \varphi_{m-3} \\ \varphi_0 & \varphi_1 & \cdots & \varphi_{m-3} & \varphi_{m-2} \end{pmatrix}.$$

We denote by O the zero matrices of various sizes. We also denote by I_n the unit matrix of size n. By (26), we have

$$\begin{aligned} & \det(zI - \hat{M}_{0,m}) \\ = & \det\left(z \begin{pmatrix} O & O \\ O & I_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \\ Q_m^{-1} Q_{m,m-1} & \hat{M}_{0,m} \end{pmatrix} \right) \\ = & \det\left(\begin{pmatrix} I_{m-1} & O \\ O & Q_m \end{pmatrix} \left(z \begin{pmatrix} O & O \\ O & I_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \\ Q_m^{-1} Q_{m,m-1} & \hat{M}_{0,m} \end{pmatrix} \right) \right) \\ = & \det\left(z \begin{pmatrix} O & O \\ O & Q_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \\ Q_{m,m-1} & Q_m \hat{M}_{0,m} \end{pmatrix} \right) \\ = & \det\left(\left(z \begin{pmatrix} O & O \\ O & Q_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \\ Q_{m,m-1} & Q_m \hat{M}_{0,m} \end{pmatrix} \right) \begin{pmatrix} I_{m-1} & O \ \Phi_{m-1} \\ O & I_m \end{pmatrix} \right) \\ = & \det\left(z \begin{pmatrix} O & O \\ O & Q_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \ -\Phi_{m-1} \\ Q_{m,m-1} & P_m \end{pmatrix} \right), \end{aligned}$$

where we use (26) to get the last equality. Hence

$$\begin{aligned} & \det(zI - \hat{M}_{0,m}) \\ = & \det\left(z \begin{pmatrix} O & O \\ O & Q_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \ -\Phi_{m-1} \\ Q_{m,m-1} & P_m \end{pmatrix} \right) \\ = & \det\left(\begin{pmatrix} Q''_{m-1} & O \\ O & I_m \end{pmatrix} \left(z \begin{pmatrix} O & O \\ O & Q_m \end{pmatrix} - \begin{pmatrix} -I_{m-1} & O \ -\Phi_{m-1} \\ Q_{m,m-1} & P_m \end{pmatrix} \right) \right) \end{aligned}$$

$$
\begin{aligned}
&= \det\left(z\begin{pmatrix} O & O \\ O & Q_m \end{pmatrix} - \begin{pmatrix} -Q''_{m-1} & O \; -P'_{m-1} \\ Q_{m,m-1} & P_m \end{pmatrix}\right) \\
&= (-1)^m \det\left(\begin{pmatrix} Q''_{m-1} & O \; P'_{m-1} \\ Q_{m,m-1} & P_m \end{pmatrix} - z\begin{pmatrix} O & O \\ O & Q_m \end{pmatrix}\right) \\
&= (-1)^m \det\begin{pmatrix} I_m & O & zI_m \\ O & Q''_{m-1} & OP'_{m-1} \\ Q'_m & Q_{m,m-1} & P_m \end{pmatrix},
\end{aligned}
$$

which implies (2).

(3) By (2), we have

$$
\hat{H}_{0,m} = (-1)^m \det(0I - \hat{M}_{0,m})
$$

$$
= (-1)^{[m/2]} \begin{vmatrix}
p_{m-1} & & & & & 1 & & & \\
p_{m-2} & p_{m-1} & & & & q_{m-1} & 1 & & \\
\vdots & & \ddots & & & \vdots & \ddots & \ddots & \\
p_1 & \ddots & & p_{m-1} & & q_2 & \ddots & & 1 \\
p_0 & \ddots & & p_{m-2} & p_{m-1} & q_1 & \ddots & & q_{m-1} \\
 & p_0 & & & p_{m-2} & q_0 & \ddots & & q_{m-2} \\
 & & \ddots & \vdots & \vdots & & \ddots & & \vdots \\
 & & & p_0 & p_1 & & & \ddots & q_1 \\
 & & & & p_0 & & & & q_0
\end{vmatrix},
$$

which completes the proof since the determinant in the last-side in the above equality is Sylvester's determinant for $P(z)$ and $Q(z)$. ■

For a finite or infinite sequence $a_0(z), a_1(z), a_2(z), \cdots$ of elements in $\mathbf{K}((z^{-1}))$, we use the notation

$$
[a_0(z); a_1(z), a_2(z), \cdots, a_n(z)\ ] := a_0(z) + \cfrac{1}{a_1(z) + \cfrac{1}{a_2(z) + \ddots + \cfrac{1}{a_n(z)}}}
$$

and

$$
[a_0(z); a_1(z), a_2(z), \cdots\ ] := \lim_{n\to\infty} [a_0(z); a_1(z), a_2(z), \cdots, a_n(z)\ ] \qquad (30)
$$

provided that the limit exists, where the limit is taken with respect to the metric induced by the non-Archimedean norm in $\mathbf{K}((z^{-1}))$.

We define

$$\begin{aligned} p_0(z) &= a_0(z) \ , \quad p_{-1}(z) = 1 \ , \quad q_0(z) = 1 \ , \quad q_{-1}(z) = 0 \\ p_n(z) &= a_n(z)p_{n-1}(z) + p_{n-2}(z) \\ q_n(z) &= a_n(z)q_{n-1}(z) + q_{n-2}(z) \\ & (n = 1, 2, 3, \cdots) \end{aligned} \tag{31}$$

for any given sequence $a_1(z), a_2(z), \cdots \in \mathbf{K}((z^{-1}))$. Then $p_n(z),\ q_n(z) \in \mathbf{K}((z^{-1})),\ p_n(z) \neq 0$ if $q_n(z) = 0$, and

$$\frac{p_n(z)}{q_n(z)} = [a_0(z); a_1(z), a_2(z), \cdots, a_n(z)\ ] \in \mathbf{K}((z^{-1})) \cup \{\infty\} \quad (n \geq 0)$$

holds, where we mean $\psi/0 := \infty$ for $\psi \in \mathbf{K}((z^{-1})) \setminus \{0\}$, and $\psi + \infty := \infty$, $\psi/\infty := 0$ for $\psi \in \mathbf{K}((z^{-1}))$. By using (31), it can be shown that the limit (30) always exists in the set $\mathbf{K}((z^{-1}))$ as far as

$$a_n(z) \in \mathbf{K}[z] \quad (n \geq 0), \quad \deg a_n(z) \geq 1 \quad (n \geq 1). \tag{32}$$

For $\varphi(z) \in \mathbf{K}((z^{-1}))$ given by (4), we denote by $\lfloor \varphi(z) \rfloor$ the polynomial part of $\varphi(z)$, which is defined as follows:

$$\lfloor \varphi(z) \rfloor := \sum_{k=0}^{h} \varphi_h z^{-k+h} \in \mathbf{K}[z].$$

By T, we denote the mapping $T : \mathbf{K}((z^{-1})) \setminus \{0\} \to \mathbf{K}((z^{-1}))$ defined by

$$T(\psi(z)) := \frac{1}{\psi(z)} - \lfloor \frac{1}{\psi(z)} \rfloor \quad \left(\psi(z) \in \mathbf{K}((z^{-1})) \setminus \{0\}\right).$$

Then, for any given $\varphi(z) \in \mathbf{K}((z^{-1}))$, we can define the continued fraction expansion of $\varphi(z)$:

$$\varphi(z) = \begin{cases} [\ a_0(z);\ a_1(z), a_2(z), \cdots, a_{N-1}(z)\ ] & \text{if } \varphi(z) \in \mathbf{K}(z) \\ [\ a_0(z);\ a_1(z), a_2(z), a_3(z), \cdots\ ] & \text{otherwise} \end{cases} \tag{33}$$

with $a_n(z)$ satisfying (32) according to the following algorithm.

Continued Fraction Algorithm :

$$a_0(z) = \lfloor \varphi(z) \rfloor \ , \quad a_n(z) = \lfloor \frac{1}{T^{n-1}(\varphi(z) - a_0(z))} \rfloor$$
$$N = N(\varphi(z)) := \inf\{m;\ T^{m-1}(\varphi(z)) = 0\} \quad (\inf \emptyset := \infty).$$

We note that if $\varphi(z) \in \mathbf{K}(z)$, then $N < \infty$; if $\varphi(z) \in \mathbf{K}((z^{-1})) \setminus \mathbf{K}(z)$, then $N = \infty$ and the continued fraction (33) converges to the given $\varphi(z) \in \mathbf{K}(z)$. We say a continued fraction is **admissible** if it is obtained by the algorithm. We remark that a continued fraction (33) is admissible if and only if (32) holds.

The following proposition is known [2], but we give a proof for completeness.

Proposition 2 *The set of all $P/Q \in \mathbf{K}(z)$ for Padé pairs (P, Q) for $\varphi(z) \in \mathbf{K}((z))$ coincides with the set of convergents $p_n(z)/q_n(z)$ $(0 \leq n < N)$ of the continued fraction expansion of $\varphi(z)$. Moreover, m is a normal index if and only if m is a degree of $q_n(z)$ for some $n = 0, 1, 2, \cdots$ (with $n < N$ if $\varphi(z) \in \mathbf{K}(z)$).*

Proof. Note that

$$\begin{aligned} \varphi(z) &= \frac{(a_n(z) + T^n(\varphi(z) - a_0))p_{n-1}(z) + p_{n-2}(z)}{(a_n(z) + T^n(\varphi(z) - a_0))q_{n-1}(z) + q_{n-2}(z)} \\ (-1)^n &= p_{n-1}(z)q_{n-2}(z) - p_{n-2}(z)q_{n-1}(z). \end{aligned}$$

Hence, we have

$$\begin{aligned} & \| \ q_n(z)\varphi(z) - p_n(z) \ \| \\ = & \ \| \ \frac{(-1)^n T^n(\varphi(z) - a_0(z))}{q_n(z) + T^n(\varphi(z) - a_0(z))q_{n-1}(z)} \ \| \\ = & \ \exp(-\deg a_{n+1}(n) - \deg q_n(z)), \end{aligned}$$

so that

$$\| \ q_n(z)\varphi(z) - p_n(z) \ \| < \exp(-\deg q_n(z)) \quad (n < N). \tag{34}$$

In the case $N < \infty$, the left-hand side of (34) turns out to be 0 for $n = N - 1$. Therefore, $(p_n(z), q_n(z))$ is a Padé pair of order $m = \deg q_n(z)$ for all $m \in \{\deg q_n(z);\ 0 \leq n < N\}$.

Conversely, for any $k = 1, 2, \cdots$, let (P, Q) be a Padé pair of order k. Let $\deg q_n(z) \le k < \deg q_{n+1}$ for some $n = 0, 1, 2, \cdots$ with $n < N$ ($\deg q_N(z) := \infty$). Then, since $\deg Q \le k < \deg q_{n+1}$, it follows from (34) that

$$\begin{aligned} \| \varphi(z) - \frac{p_n(z)}{q_n(z)} \| &= \exp(-\deg q_n(z) - \deg q_{n+1}(z)) \\ &< \exp(-\deg q_n(z) - \deg Q). \end{aligned}$$

Since (P, Q) be a Padé pair of order k, we have

$$\begin{aligned} \| \varphi(z) - \frac{P}{Q} \| &< \exp(-k - \deg Q) \\ &\le \exp(-\deg q_n(z) - \deg Q). \end{aligned}$$

Therefore, we have

$$\| \frac{P}{Q} - \frac{p_n(z)}{q_n(z)} \| < \exp(-\deg q_n(z) - \deg Q).$$

On the other hand, if $P/Q \ne p_n(z)/q_n(z)$, then

$$\begin{aligned} \| \frac{P}{Q} - \frac{p_n(z)}{q_n(z)} \| &= \| \frac{Pq_n(z) - Qp_n(z)}{Qq_n(z)} \| \\ &\ge \exp(-\deg q_n(z) - \deg Q), \end{aligned}$$

which is a contradiction. Thus we have $P/Q = p_n(z)/q_n(z)$.

Note that $p_n(z)/q_n(z)$ is irreducible for any $n = 1, 2, \cdots$ with $n < N$, since $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$. Let $m = \deg q_n(z)$ for some $n = 1, 2, \cdots$ with $n < N$. Take any Padé pair (P, Q) of order m. Then $\deg Q \le m$. On the other hand, by the above argument, we have $P/Q = p_n(z)/q_n(z)$. Since $p_n(z)/q_n(z)$ is irredusible, this implies that $\deg Q \ge \deg q_n(z) = m$. Thus, m is a normal index.

Conversely, let $m \ge 0$ be any normal index. Take any Padé pair (P, Q) of order m. Then, by the above argument, there exists $n = 0, 1, 2, \cdots$ with $n < N$ such that $P/Q = p_n(z)/q_n(z)$. Hence the irreducibility of $p_n(z)/q_n(z)$ implies $\deg q_n(z) \le \deg Q (\le m)$. Hence, $(p_n(z),\ q_n(z))$ is a Padé pair of order m. Since m is a normal index, $\deg q_n(z) = m$. ■

Let us obtain the continued fraction expansions for

$$\varphi_{\hat{\varepsilon}}(z) = \hat{\varepsilon}_0 z^{-1} + \hat{\varepsilon}_1 z^{-2} + \hat{\varepsilon}_2 z^{-3} + \cdots \in \mathbf{Q}((z^{-1}))$$

corresponding to the Fibonacci words $\hat{\varepsilon} = \varepsilon(a,b)$ with $(a,b) = (1,0)$ and $(a,b) = (0,1)$. As in §3, we use the notations ε and $\overline{\varepsilon}$ for them. The proofs in the following theorems are given only for ε, since the proof is similar for $\overline{\varepsilon}$. In [3], J. Tamura gave the Jacobi-Perron-Parusnikov expansion for a vector consisting of Laurent series with coefficients given by certain substitutions, which contains the following as its special case, cf. the footnote, p. 301 [3]:

Proposition 3 *It holds that*

$$(z-1)\varphi_\varepsilon(z) = [\ 0; z^{f_{-2}}, z^{f_{-1}}, z^{f_0}, z^{f_1}, z^{f_2}, \cdots\ ].$$

Theorem 7 *We have the following admissible continued fraction for* $\varphi_\varepsilon(z)$ *and* $\varphi_{\overline{\varepsilon}}(z)$*:*

$$\begin{aligned} \varphi_\varepsilon(z) &= [\ 0; a_1, a_2, a_3, \cdots\ ] \\ \varphi_{\overline{\varepsilon}}(z) &= [\ 0; \overline{a}_1, \overline{a}_2, \overline{a}_3, \cdots\ ] \end{aligned}$$

with

$$\begin{aligned} & a_1 = z,\ \ a_2 = -z+1,\ \ a_3 = -\frac{1}{2}(z+1) \\ & a_{2n+2} = (-1)^{n-1} f_n^2 (z^{f_n - 1} + z^{f_n - 2} + \cdots + 1) \\ & a_{2n+3} = (-1)^{n-1} \frac{1}{f_n f_{n+1}} (z-1) \\ & \qquad\qquad\qquad\qquad (n = 1, 2, \cdots\ ), \end{aligned}$$

and

$$\begin{aligned} & \overline{a}_1 = z^2\ ,\ \ \ \overline{a}_2 = -z, \\ & \overline{a}_{2n+1} = (-1)^{n-1} f_{n-1}^2 (z^{f_n - 1} + z^{f_n - 2} + \cdots + 1) \\ & \overline{a}_{2n+2} = (-1)^{n-1} \frac{1}{f_{n-1} f_n} (z-1) \\ & \qquad\qquad\qquad\qquad (n = 1, 2, \cdots\ ). \end{aligned}$$

Proof. We put

$$\begin{aligned}
\theta_n &:= [0\,; z^{f_n}, z^{f_{n+1}}, z^{f_{n+2}}, \cdots\,] \quad (n \geq -2) \\
\xi_n &:= (-1)^{n-1}\frac{f_n^2 z^{f_n} + f_{n-1}f_n + f_n^2\theta_{n+1}}{z-1} \quad (n \geq 1) \\
\eta_n &:= (-1)^{n-1}\frac{z-1}{f_n f_{n+1} + f_n^2\theta_{n+1}} \quad (n \geq 1) \\
c_n &:= (-1)^{n-1} f_n^2 (z^{f_n - 1} + z^{f_n - 2} + \cdots + 1) \quad (n \geq 1) \\
d_n &:= (-1)^{n-1}\frac{1}{f_n f_{n+1}}(z-1) \quad (n \geq 1).
\end{aligned}$$

Then we have

$$\xi_n = [c_n\ ;\ \eta_n](= c_n + \frac{1}{\eta_n})\ , \quad \eta_n = [d_n\ ;\ \xi_n] \tag{35}$$

Using

$$\theta_n^{-1} = z^{f_n} + \theta_{n+1}$$

and Proposition 3, we get

$$\begin{aligned}
\varphi_\varepsilon(z) &= \frac{\theta_{-2}}{z-1} \qquad (\parallel \theta_{-2}/(z-1) \parallel < 1) \\
&= [0\ ;\ (z-1)\theta_{-2}^{-1}] \\
&= [0\ ;\ z - 1 + (z-1)\theta_{-1}] \quad (\parallel -1 + (z-1)\theta_{-1} \parallel < 1) \\
&= [0\ ;\ z,\ \frac{\theta_{-1}^{-1}}{-\theta_{-1}^{-1} + z - 1}\ ] \\
&= [0\ ;\ z,\ \frac{z + \theta_0}{-1 - \theta_0}\ ] \\
&= [0\ ;\ z,\ \ -z + 1 + \frac{1 + (-z+2)\theta_0}{-1 - \theta_0}\ ] \\
&\qquad\qquad\qquad\qquad (\parallel \frac{1 + (-z+2)\theta_0}{-1 - \theta_0} \parallel < 1) \\
&= [0\ ;\ z,\ \ -z + 1,\ \frac{-1 - \theta_0^{-1}}{-z + 2 + \theta_0^{-1}}\ ] \\
&= [0\ ;\ z,\ \ -z + 1,\ \frac{-z - 1 - \theta_1}{2 + \theta_1}\ ]
\end{aligned}$$

$$
\begin{aligned}
&= [0\ ;\ z,\ \ -z+1,\ \ -\frac{1}{2}(z+1),\ \frac{4\theta_1^{-1}+2}{z-1}\ ] \\
&= [0\ ;\ z,\ \ -z+1,\ \ -\frac{1}{2}(z+1),\ \frac{4z+2+4\theta_2}{z-1}\ ].
\end{aligned}
$$

Hence, we have

$$
f(z) = [0\ ;\ z,\ \ -z+1,\ \ -\frac{1}{2}(z+1),\ \xi_1\ ] \quad (\| \xi_1^{-1} \| < 1). \tag{36}
$$

From (35) and (36), it follows that

$$
\begin{aligned}
f(z) &= [0\ ;\ z,\ \ -z+1,\ \ -\frac{1}{2}(z+1)\ c_1,\ d_1,\ \cdots,\ c_n,\ d_n,\ \xi_{n+1}\ ] \\
&= [0\ ;\ z,\ \ -z+1,\ \ -\frac{1}{2}(z+1)\ c_1,\ d_1,\ c_2,\ d_2,\ \cdots\ ]
\end{aligned}
$$

which completes the proof for $\varphi_\varepsilon(z)$.

Starting from the identity $\varphi_{\overline{\varepsilon}}(z) = \frac{1-\theta_{-2}}{z-1}$ instead of $\varphi_\varepsilon(z) = \frac{\theta_{-2}}{z-1}$, we can get the admissible continued fraction for $\varphi_{\overline{\varepsilon}}(z)$ by the similar fashion as above. ■

Theorem 8 *The numerator $p_n := p_n(z)$ ($\overline{p}_n := \overline{p}_n(z)$, resp.) and the denominator $q_n := q_n(z)$ ($\overline{q}_n := \overline{q}_n(z)$, resp.) of the n-th convergent of the continued fraction expansion for $\varphi_\varepsilon(z)$ (and $\varphi_{\overline{\varepsilon}}(z)$, resp.) are given as follows:*

$$
\begin{aligned}
&p_0 = 0\ ,\quad p_1 = 1\ ,\quad p_2 = -z+1 \\
&q_0 = 1\ ,\quad q_1 = z\ ,\quad q_2 = -z^2+z+1 \\
&p_{2n-1} = \frac{1}{f_{n-1}}(\varepsilon_0 z^{f_n-1} + \varepsilon_1 z^{f_n-2} + \cdots + \varepsilon_{f_n-1}) \\
&p_{2n} = (-1)^n\{f_{n-1} z^{f_n}(\varepsilon_0 z^{f_{n-1}-1} + \varepsilon_1 z^{f_{n-1}-2} + \cdots + \varepsilon_{f_{n-1}-1}) \\
&\qquad - f_{n-2}(\varepsilon_0 z^{f_n-1} + \varepsilon_1 z^{f_n-2} + \cdots + \varepsilon_{f_n-1})\}/(z-1) \\
&q_{2n-1} = \frac{1}{f_{n-1}}(z^{f_n}-1) \\
&q_{2n} = (-1)^n\{f_{n-1} z^{f_n}(z^{f_{n-1}-1} + z^{f_{n-1}-2} + \cdots + 1) \\
&\qquad - f_{n-2}(z^{f_n-1} + z^{f_n-2} + \cdots + 1)\} \\
&\qquad (\ n = 2, 3, \cdots\ )\ ,
\end{aligned}
$$

and

$$
\begin{aligned}
&\overline{p}_0 = 0\ , \quad \overline{p}_1 = 1\\
&\overline{q}_0 = 1\ , \quad \overline{q}_1 = z^2\\
&\overline{p}_{2n-2} = -\frac{1}{f_{n-2}}(\overline{\varepsilon}_0 z^{f_n-1} + \overline{\varepsilon}_1 z^{f_n-2} + \cdots + \overline{\varepsilon}_{f_n-1})\\
&\overline{p}_{2n-1} = (-1)^{n-1}\{f_{n-2}z^{f_n}(\overline{\varepsilon}_0 z^{f_{n-1}-1} + \overline{\varepsilon}_1 z^{f_{n-1}-2} + \cdots + \overline{\varepsilon}_{f_{n-1}-1})\\
&\qquad - f_{n-3}(\overline{\varepsilon}_0 z^{f_n-1} + \overline{\varepsilon}_1 z^{f_n-2} + \cdots + \overline{\varepsilon}_{f_n-1})\}/(z-1) + f_{n-2}\\
&\overline{q}_{2n-2} = -\frac{1}{f_{n-2}}(z^{f_n} - 1)\\
&\overline{q}_{2n-1} = (-1)^{n-1}\{f_{n-2}z^{f_n}(z^{f_{n-1}-1} + z^{f_{n-1}-2} + \cdots + 1)\\
&\qquad - f_{n-3}(z^{f_n-1} + z^{f_n-2} + \cdots + 1)\}\\
&\qquad\quad (\ n = 2, 3, \cdots\ )\ ,
\end{aligned}
$$

where p_{2n} *and* $\overline{p}_{2n-1}$ *in the above are polynomials since the numerators are divisible by* $z-1$*.*

Proof. The values for $p_0, p_1, p_2, q_0, q_1, q_2$ are obtained from Theorem 7 by direct calculations. For a general n, we can prove the formula for p_n, q_n by induction on n using (31) and Theorem 7 without difficulty. ■

Remark 4 *From Proposition 2 and Theorem 8, it follows that the set of normal indices for* $\varphi_\varepsilon(z)$ *(and* $\varphi_{\overline{\varepsilon}}(z)$*, resp.) is* $\{0,\ f_0 = f_1 - 1,\ f_1 = f_2 - 1,\ f_2,\ f_3 - 1,\ \cdots\ \}$ *(*$\{0,\ f_1 = f_2 - 1,\ f_2,\ f_3 - 1,\ \cdots\ \}$*, resp.) which together with Proposition 1 give another proof of the third cases of Theorem 2 with* $n = 0$*.*

Remark 5 *In [4], the continued fraction expansion for Laurent series corresponding to infinite words over* $\{a, b\}$ *generated by substitutions of "Fibonacci type" are considered, where* a*,* b *will be considered as independent variables.*

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