Maximal pattern complexity as topological invariants

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Abstract: For an infinite word $\alpha = \alpha_0\alpha_1\alpha_2\cdots \in A^{\mathbb{N}}$ over a finite set A, the maximal pattern complexity was introduced as

$$p^*_\alpha(N) := \sup_{\Omega} \sharp\{\alpha_{i+\omega(0)}\alpha_{i+\omega(1)}\cdots\alpha_{i+\omega(N-1)};\ i \in \mathbb{N}\}$$

where the "sup" is taken over all subsets $\Omega := \{\omega(0) < \omega(1) < \cdots < \omega(N-1)\}$ of $\mathbb{N}$ of size N.

In this paper, we prove that if $\sharp A = \ell \geq 2$, then either

$$p^*_\alpha(N) = \ell^N \quad (N = 1, 2, \cdots)$$

or there exists $n = 1, 2, \cdots$ such that

$$p^*_\alpha(N) \leq \sum_{i=0}^{n-1} \binom{N}{i} (\ell - 1)^{N-i} \quad (N = 1, 2, \cdots).$$

Hence,

$$h^*(\alpha) := \limsup_{N\to\infty} \frac{1}{N} \log p^*_\alpha(N)$$

takes value either 0 or $\log 2$ if $\sharp A = 2$. We also define

$$d^*(\alpha) := \limsup_{N\to\infty} \frac{1}{\log N} \log p^*_\alpha(N).$$

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We prove that for any $\alpha \in A^{\mathbb{N}}$ and $\beta \in B^{\mathbb{N}}$, if there exists a continuous mapping $f : A^{\mathbb{N}} \to B^{\mathbb{N}}$ with $fT_A = T_B f$, where T_A and T_B are the shifts on $A^{\mathbb{N}}$ and $B^{\mathbb{N}}$, respectively, such that $\beta = f(\alpha)$, then we have $d^*(\beta) \leq d^*(\alpha)$. Thus, d^* is a monotone increasing topological invariant. In the same way, h^* is a monotone increasing topological invariant among infinite words over 2 letters.

1 Introduction

Let N be a positive integer. By a N-*window*, we mean a finite subset Ω of $\mathbb{N} := \{0, 1, 2, \cdots\}$ with $\sharp\Omega = N$, where $\sharp\Omega$ denotes the number of elements in Ω.

Let $\alpha = (\alpha_i)_{i\in\mathbb{N}} = \alpha_0\alpha_1\alpha_2 \cdots \in A^{\mathbb{N}}$ be a word over a finite alphabet A defined on $\mathbb{N}$ and Ω be a N-*window*. We denote

$$\alpha[i + \Omega] := (\alpha_{i+j})_{j\in\Omega} \in A^{\Omega},$$

and

$$F_\alpha(\Omega) := \{\alpha[i + \Omega];\ i \in \mathbb{N}\}.$$

The *maximal pattern complexity* $p^*_\alpha(N)$ as a function on $N \in \{1, 2, 3, \cdots\}$ was introduced by the author with Luca Zamboni [1] as

$$p^*_\alpha(N) = \sup_{\Omega} \sharp F_\alpha(\Omega) \tag{1}$$

where the "sup" is taken over all N-windows Ω. In [1], it is always assumed that $0 \in \Omega$ for a window Ω, while it is not assumed here. This change is irrelevant for the definition (1).

We also define the *maximal pattern entropy* $h^*(\alpha)$ and *maximal pattern dimension* $d^*(\alpha)$ of α by

$$\begin{aligned} h^*(\alpha) &:= \limsup_{N\to\infty} \frac{1}{N} \log p^*_\alpha(N) & (2)\\ d^*(\alpha) &:= \limsup_{N\to\infty} \frac{1}{\log N} \log p^*_\alpha(N). & (3) \end{aligned}$$

It is asked in [1] (Problem 2) whether the statement that for $\alpha \in A^{\mathbb{N}}$ with $\sharp A = 2$, if $p^*_\alpha(N)$ increases exponentially, then $p^*_\alpha(N) = 2^N$ ($N =$

$1, 2, \cdots)$ holds is true or not. Here, we solve this problem positively. Actually, we prove a stonger statement that either $p^*_\alpha(N)$ is full or of a polynomial order if $\sharp A = 2$. In fact, we prove that

Theorem 1. *Let* $\alpha \in A^{\mathbb{N}}$ *with* $\sharp A = \ell \geq 2$.
(i) *Either* $p^*_\alpha(N) = \ell^N \quad (N = 1, 2, \cdots)$ *holds or there exists* $n = 1, 2, \cdots$ *such that*

$$p^*_\alpha(N) \leq \sum_{i=0}^{n-1} \binom{N}{i} (\ell - 1)^{N-i} \quad (N = 1, 2, \cdots).$$

(ii) *Either* $p^*_\alpha(N) \geq 2^N \quad (N = 1, 2, \cdots)$ *holds or there exists* $n = 1, 2, \cdots$ *such that*

$$p^*_\alpha(N) \leq \left(\sum_{i=0}^{n-1} \binom{N}{i} \right)^{\ell - 1} \quad (N = 1, 2, \cdots).$$

(iii) *The maximal pattern entropy* $h^*(\alpha)$ *does not take value in* $(0, \log 2) \cup (\log(\ell - 1), \log \ell)$. *Moreover, if* $h^*(\alpha) = 0$*, then* $p^*_\alpha(N)$ *is a polynomial order in* N *and if if* $h^*(\alpha) = \ell$*, then* $p^*_\alpha(N)$ *is full, that is* $p^*_\alpha(N) = \ell^N$ $(N = 1, 2, \cdots)$.

For $\alpha \in A^{\mathbb{N}}$ and $\beta \in B^{\mathbb{N}}$, we say that β is a *factor* of α if there exists a continuous mapping $f : A^{\mathbb{N}} \to B^{\mathbb{N}}$ such that $fT_A = T_B f$ and $\beta = f(\alpha)$, where T_A and T_B are the shifts on $A^{\mathbb{N}}$ and $B^{\mathbb{N}}$, respectively. If in the above, f is a homeomorphism, then we say that α and β are *conjugate*. A function P defined on the set of infinite words over finite sets is called a *monotone increasing (decreasing) topological invariant* if $P(\beta) \leq P(\alpha)$ $(P(\beta) \leq P(\alpha)$, respectively) holds for any infinite words α, β such that β is a factor of α. Here, if $P(\beta) \leq P(\alpha)$ $(P(\beta) \leq P(\alpha))$ holds only for a special kind S of infinite words α, β such that β is a factor of α, we say that P is a monotone increasing (decreasing, respectively) topological invariant *among the class* S. A property P on the set of infinite words over finite sets is consisered as $\{0, 1\}$-valued function in the usual way, so that the above definitions will be applied for proreties as well.

Theorem 2. (i) *The maximal pattern dimension* d^* *is an increasing topological invariant.*
(ii) *The maximal pattern entropy* h^* *is an increasing topological invariant among the infinite words over* 2 *letters.*
(iii) *The property on* α *that* $h^*(\alpha) = 0$ *is a decreasing topological invariant.*

It is suggested by Xiangdon Ye [3] that for any $\alpha \in A^{\mathbb{N}}$, $h^*(\alpha) = 0$ if and only if the dynamical system $(\overline{O}(\alpha), T_A)$ has 0 topological sequence entropy defined by Wen Huang and others [4], where $\overline{O}(\alpha)$ is the closure of $\{{T_A}^i\alpha;\ i \in \mathbb{N}\}$.

2 Combinatorial Lemma

Let A and Ω be nonempty finite sets. Denote by $\mathcal{P}(\Omega)$ the set of all subsets of Ω.

The set $\cup_{S\in\mathcal{P}(\Omega)} A^S$ consists of all words over A defined on some $S \in \mathcal{P}(\Omega)$. Let $\xi = (\xi_i)_{i\in S} \in A^S$. We denote $S = dom(\xi)$. For $\xi, \eta \in \cup_{S\in\mathcal{P}(\Omega)} A^S$, η is called a *restriction* of ξ if $dom(\eta) \subset dom(\xi)$ and $\eta_i = \xi_i$ for any $i \in dom(\eta)$. In this case, we call η a *restriction* of ξ and denote $\eta \subset \xi$ or $\eta = \xi|_S$ with $S = dom(\eta)$.

For $F \subset \cup_{S\in\mathcal{P}(\Omega)} A^S$, we denote by $A^\Omega\langle F\rangle$ the set of words $\xi \in A^\Omega$ such that $\eta \subset \xi$ does not holds for any $\eta \in F$. In this setting, we call $\eta \in F$ a *forbidden* word and $\xi \in A^\Omega\langle F\rangle$ an *admissible* word.

Let n be a positive integer. We denote by $\mathcal{P}_n(\Omega)$ the set of $S \in \mathcal{P}(\Omega)$ with $\sharp S = n$. We call $F \subset \cup_{S\in\mathcal{P}(\Omega)} A^S$ a *simple complete list of forbidden words of size* n *on* Ω if $F \subset \cup_{S\in\mathcal{P}_n(\Omega)} A^S$ and $\sharp(F \cap A^S) = 1$ for any $S \in \mathcal{P}_n(\Omega)$.

Lemma 1. *Let* A *be a set with* $\sharp A = 2$. *Let* Ω *be a finite set with* $N = \sharp\Omega$. *Let* n *be a positive integer such that* $n \le N$. *Let* F *be a simple complete list of forbidden words of size* n *on* Ω. *Then, we have*

$$\sharp A^\Omega\langle F\rangle \le \sum_{i=0}^{n-1} \binom{N}{i}. \tag{4}$$

Proof. We use the induction on $n = 1, 2, \cdots$.

Let $n = 1$ and F be a simple complete list of forbidden words of size 1 on Ω. Then for any $i \in \Omega$, there exists a unique $\eta_i \in A$ with $(\eta_i)_{i\in\{i\}} \in F$, so that for any $\xi \in A^\Omega\langle F\rangle$, $\xi_i \neq \eta_i$ for any $i \in \Omega$. Hence, $\xi \in A^\Omega\langle F\rangle$ is uniquely determined since $\sharp A = 2$. Thus, $\sharp A^\Omega\langle F\rangle = 1$ and we have (3) for $n = 1$.

Let $n \geq 2$ and (3) hold for $n-1$ in place of n. We further use the induction on N. Let $N = n$ and F be a simple complete list of forbidden words of size n on Ω. Then, F consists of one element belonging to A^Ω. Hence, $\sharp A^\Omega\langle F\rangle = 2^N - 1$, which implies (3) for $N = n$.

Let $N > n$ and (3) holds for $N-1$ in place of N. Let F be a simple complete list of forbidden words of size n on Ω. Take $\omega \in \Omega$ and $\Omega' := \Omega \setminus \{\omega\}$. Let $F' := \{\eta \in F;\ \omega \notin dom(\eta)\}$ and $F'' := \{\eta|_{dom(\eta)\cap\Omega'};\ \eta \in F \setminus F'\}$. Then, F' is a simple complete list of forbidden words of size n on Ω', while F'' is a simple complete list of forbidden words of size $n-1$ on Ω'. Since $\sharp\Omega' = N-1$, we have by the induction hypothesis that

$$\sharp A^{\Omega'}\langle F'\rangle \leq \sum_{i=0}^{n-1} \binom{N-1}{i}. \tag{5}$$

Moreover, since $\sharp F'' = n-1$, we have by the induction hypothesis that

$$\sharp A^{\Omega'}\langle F''\rangle \leq \sum_{i=0}^{n-2} \binom{N-1}{i}. \tag{6}$$

For any $\xi \in A^\Omega\langle F\rangle$, we have $\xi|_{\Omega'} \in A^{\Omega'}\langle F'\rangle$ since $F' \subset F$. Define the mapping $\pi : A^\Omega\langle F\rangle \to A^{\Omega'}\langle F'\rangle$ by $\pi(\xi) = \xi|_{\Omega'}$. Then, for any $\xi' \in A^{\Omega'}\langle F'\rangle$, $\sharp\pi^{-1}(\xi') \leq 2$ since $\sharp A = 2$. Moreover, if $\xi' \in A^{\Omega'} \setminus A^{\Omega'}\langle F''\rangle$, then there exists $\eta' \in F''$ such that $\eta' \subset \xi'$. Since there exists $\eta \in F$ such that $\eta' \subset \eta$, there exists $\xi \in A^\Omega$ such that $\xi' \subset \xi$ and $\eta \subset \xi$, and hence, $\xi \notin A^\Omega\langle F\rangle$. Therefore, in this case, we have $\sharp\pi^{-1}(\xi') \leq 1$.

Hence, $\sharp\pi^{-1}(\xi') = 2$ holds only if $\xi' \in A^{\Omega'}\langle F''\rangle$. Thus, we have

$$\begin{aligned}
\sharp A^{\Omega}\langle F\rangle &\leq \sharp A^{\Omega'}\langle F'\rangle + \sharp A^{\Omega'}\langle F''\rangle \\
&\leq \sum_{i=0}^{n-1}\binom{N-1}{i} + \sum_{i=0}^{n-2}\binom{N-1}{i} \\
&= \sum_{i=0}^{n-1}\left(\binom{N-1}{i} + \binom{N-1}{i-1}\right) \\
&= \sum_{i=0}^{n-1}\binom{N}{i}
\end{aligned}$$

using (4) and (5). This completes the proof. □

3 Proof of Theorem 1

Let $\alpha \in A^{\mathbb{N}}$ with $\sharp A = 2$. Assume that there exists $n = 1, 2, \cdots$ such that $p^*_\alpha(n) < 2^n$. Take any $N \geq n$ and any N-window Ω. Take any subset S of Ω with $\sharp S = n$. Since $\sharp F_\alpha(S) \leq p^*_\alpha(n) < 2^n$, there exists $\eta^{(S)} \in A^S \setminus F_\alpha(S)$. Since $\eta^{(S)} \in A^S$ and $S \in \mathcal{P}_n(\Omega)$, $F := \{\eta^{(S)};\ S \in \mathcal{P}_n(\Omega)\}$ is a simple complete list of forbidden words of size n on Ω.

Let $\xi \in F_\alpha(\Omega)$. Note that $\xi \in A^\Omega$ and there exists $i \in \mathbb{N}$ such that $\xi = \alpha[i + \Omega]$. Hence, $\xi|_S = \alpha[i + S] \in F_\alpha(S)$ for any $S \in \mathcal{P}_n(\Omega)$. Therefore, $\xi|_S \neq \eta^{(S)}$ since $\eta^{(S)} \notin F_\alpha(S)$, so that $\eta^{(S)} \subset \xi$ does not holds for any $S \in \mathcal{P}_n(\Omega)$. Thus, $\xi \in A^\Omega\langle F\rangle$ holds for any $\xi \in F_\alpha(\Omega)$. Since it follows that $F_\alpha(\Omega) \subset A^\Omega\langle F\rangle$, we have

$$\sharp F_\alpha(\Omega) \leq \sum_{i=0}^{n-1}\binom{N}{i}$$

for any N-window Ω by Lemma 1. Thus, $p^*_\alpha(N) \leq \sum_{i=0}^{n-1}\binom{N}{i}$ holds for any $N = n, n+1, \cdots$. If $N < n$, then $\sum_{i=0}^{n-1}\binom{N}{i} = 2^N$, so that $p^*_\alpha(N) \leq \sum_{i=0}^{n-1}\binom{N}{i}$ holds trivially. Thus, the proof is completed. □

4 Proof of Theorem 2

Since (i) follows from Theorem 1 and (iii) follows from (ii), it is sufficient to prove (ii). By (i), it is sufficient to prove that if β is a factor of α and if $h^*(\alpha) = 0$, then $h^*(\beta) = 0$.

Assume that β is a factor of α and $h^*(\alpha) = 0$. Let $f : A^{\mathbb{N}} \to B^{\mathbb{N}}$ be a continuous mapping such that $fT_A = T_B f$ and $\beta = f(\alpha)$. Then, there exists k such that $f(\gamma)_0$ is determined by $\gamma_0, \ \gamma_1, \ \cdots , \ \gamma_{k-1}$. Let Ω be any N-window. Let

$$\Omega' := \{\omega + j; \ \omega \in \Omega, \ j = 0, 1, \cdots , k-1\}.$$

Then, since $\beta[i + \Omega]$ is determined by $\alpha[i + \Omega']$, we have $\sharp F_\beta(\Omega) \leq \sharp F_\alpha(\Omega')$. Since $\sharp\Omega' \leq Nk$, $\sharp F_\beta(\Omega) \leq p^*_\alpha(Nk)$ holds for any N-window Ω. Hence, we have $p^*_\beta(N) \leq p^*_\alpha(Nk)$. Thus,

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{N} \log p^*_\beta(N) &\leq \limsup_{N\to\infty} \frac{1}{N} \log p^*_\alpha(Nk) \\ &\leq k \limsup_{N\to\infty} \frac{1}{N} \log p^*_\alpha(N) = 0,\end{aligned}$$

which completes the proof.

5 Remarks

For $\alpha \in A^{\mathbb{N}}$ with $\sharp A = \ell$, we can prove just in the same way as Theorem 1 that either $p^*_\alpha(N) = \ell^N$ $(N = 1, 2, \cdots)$ or there exists $n = 1, 2, \cdots$ such that

$$p^*_\alpha(N) \leq \sum_{i=0}^{n-1} \binom{N}{i} (\ell - 1)^{N-i} \quad (N = 1, 2, \cdots).$$

Hence, $h^*(\alpha)$ does not take value in $(\log(\ell - 1), \ \log \ell)$.

There is a conjecture by Xiangdon Ye [3] and the author that for $\alpha \in A^{\mathbb{N}}$ with $\sharp A = \ell$, $h^*(\alpha)$ takes values in $\{0, \log 2, \log 3 \cdots , \log \ell\}$.

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