Mixing properties of the numeration systems coming from weighted substitutions

(Ergodic Theory and Dynamical Systems 30-4 (2010), pp1111-1118

DOI 10.1017/S0143385709000510)

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Abstract

A weighted substitution is a substitution with weights for each occurrence of substituted symbols. It defines a tiling space admitting the translation and scaling operators. The translation is the additive $\mathbb{R}$-action and the scaling is the multiplicative G-action, where G is a closed multiplicative subgroup of $\mathbb{R}_+$. We obtained necessary and sufficient conditions for the additive action to be strongly mixing and weakly mixing.

1 Introduction

Let $\mathbb{A}$ be a nonempty finite set. Let $(\sigma, \tau) : \mathbb{A} \to \cup_{n \ge 2}(\mathbb{A} \times (0,1))^n$ be a *weighted substitution* on $\mathbb{A}$, that is, for any $a \in \mathbb{A}$, $(\sigma,\tau)(a)$ is a pair of elements $\sigma(a) = \sigma(a)_0\sigma(a)_1 \cdots \sigma(a)_{n-1} \in \mathbb{A}^n$ and $\tau(a) = \tau(a)_0\tau(a)_1 \cdots \tau(a)_{n-1} \in (0,1)^n$, where n is the *length* of $\sigma(a)$ and

$\tau(a)$, that is, $n = |\sigma(a)| = |\tau(a)| \geq 2$, which may depend on a. Moreover, $\sum_{0 \leq i < |\tau(a)|} \tau(a)_i = 1$ for any $a \in \mathbb{A}$. Here, $\sigma : \mathbb{A} \to \cup_{n \geq 2} \mathbb{A}^n$ is a substitution on $\mathbb{A}$ in the usual sense. We always assume that σ is *primitive*, that is, there exists $k > 0$ such that for any $a, b \in \mathbb{A}$, b appears in $\sigma^k(a)$ (the k times application of σ to a).

We define the repeated applications of (σ^k, τ^k) for $k = 2, 3, \cdots$, $(\sigma^k, \tau^k) : \mathbb{A} \to \cup_{n \geq 2} (\mathbb{A} \times (0,1))^n$, inductively as follows. For any $a \in \mathbb{A}$, $\sigma^k(a)$ is the k times application of σ to a as usual, while $\tau^k(a) = \tau^k(a)_0 \tau^k(a)_1 \cdots \tau^k(a)_{n-1}$ with $n = |\sigma^k(a)|$ is defined as

$$\begin{gathered} \tau^k(a)_i = \tau^{k-1}(a)_h \tau(\sigma^{k-1}(a)_h)_j \quad (0 \leq i < n), \\ \text{where } i = \textstyle\sum_{0 \leq h' < h} |\sigma(\sigma^{k-1}(a)_{h'})| + j \\ \text{with } 0 \leq h < |\sigma^{k-1}(a)| \text{ and } 0 \leq j < |\sigma(\sigma^{k-1}(a)_h)|. \end{gathered}$$

We define the *base set* $B(\sigma, \tau)$ of the weighted substitution (σ, τ) as the closed multiplicative subgroup of $\mathbb{R}_+$ generated by

$$\{\tau^k(a)_i \,;\; a \in \mathbb{A},\; k = 1, 2, \cdots,\; 0 \leq i < |\sigma^k(a)| \text{ such that } \sigma^k(a)_i = a\}.$$

There are 2 cases, either $B(\sigma, \tau) = \mathbb{R}_+$ or $B(\sigma, \tau) = \{\lambda^n, n \in \mathbb{Z}\}$ with $\lambda > 1$. Moreover, it is known [1] that in the latter case, λ is an algebraic number. For the latter case, we define $g : \mathbb{A} \to \mathbb{R}_+$, satisfying that

$$g(\sigma(a)_i) \in g(a)\tau(a)_i B(\sigma, \tau) \quad (\forall a \in \mathbb{A},\; 0 \leq \forall i < |\sigma(a)|). \tag{1}$$

Such a function g always exists. For example, take $a_0 \in \mathbb{A}$ and for any $a \in \mathbb{A}$, define $g(a) := \tau^k(a_0)_i$ for some fixed $k = 1, 2, \cdots$ and $0 \leq i < |\sigma^k(a_0)|$ such that $a = \sigma^k(a_0)_i$. For the former case that $B(\sigma, \tau) = \mathbb{R}_+$, we define $g(a) \equiv 1$.

Given a weighted substitution (σ, τ) and the function g as above. We define a numeration system $\Omega(\sigma, \tau, g)$ as in [1]. Here, we repeat the construction for the convenience of the readers.

Let $\mathbb{H} = \{x + iy;\; y > 0\}$ be the upper half complex plane. An open rectangle $(x_1, x_2) \times (y_1, y_2)$ in $\mathbb{H}$ is called an *admissible tile* if $0 < x_2 - x_1 = y_1 < y_2$. Let $\mathcal{R}$ be the set of admissible tiles. A subset $\omega \subset \mathcal{R} \times \mathbb{A}$ is called a *colored tiling* with colors in $\mathbb{A}$ if

$$S \cap S' = \emptyset \text{ for any } (S, a) \neq (S', a') \text{ in } \omega \tag{2}$$

$$\text{and } \cup_{(S,a) \in \omega} \overline{S} = \mathbb{H} \tag{3}$$

Let $\Omega(\mathbb{A})$ be the set of colored tilings with colors in $\mathbb{A}$. A topology is introduced on $\Omega(\mathbb{A})$ so that a net $\{\omega_n\}_{n\in I} \subset \Omega(\mathbb{A})$ converges to $\omega \in \Omega(\mathbb{A})$ if for every $(R, a) \in \omega$, there exists $(R_n, a_n) \in \omega_n$ such that

$$a_n = a \text{ for any sufficiently large } n \in I \text{ and } \lim_{n\to\infty} \rho(R, R_n) = 0,$$

where ρ is the Hausdorff metric between rectangles.

For $(x_1, x_2) \times (y_1, y_2) \in \mathcal{R},\ t \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+$, we denote

$$\begin{aligned} R + t &:= (x_1 + t, x_2 + t) \times (y_1, y_2) \\ \lambda R &:= (\lambda x_1, \lambda x_2) \times (\lambda y_1, \lambda y_2). \end{aligned}$$

Note that they are also admissible tiles.

For $\omega \in \Omega(\mathbb{A})$, $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+$, we define $\omega + t \in \Omega(\mathbb{A})$ and $\lambda\omega \in \Omega(\mathbb{A})$ as follows:

$$\begin{aligned} \omega + t &= \{(R - t, a);\ (R, a) \in \omega\} \\ \lambda\omega &= \{(\lambda R, a);\ (R, a) \in \omega\}. \end{aligned}$$

Thus, we define a continuous group action $\lambda\omega + t$ of $(\lambda, t) \in \mathbb{R}_+ \times \mathbb{R}$ to $\omega \in \Omega(\mathbb{A})$.

Let (σ, τ) be a weighted substitution together with g as (1). Let $\Omega(\sigma, \tau, g)'$ be the set of all elements ω in $\Omega(\mathbb{A})$ such that for any $((x_1, x_2) \times (y_1, y_2),\ a) \in \omega$,

(I) $y_1 \in g(a)B(\sigma, \tau)$, and

(II) $(R^i, \sigma(a)_i) \in \omega$ holds for $i = 0, 1, \cdots, |\sigma(a)| - 1$, where

$$\begin{aligned} R^i \ := \ & (x_1 + (x_2 - x_1)\sum_{j=0}^{i-1} \tau(a)_j,\ x_1 + (x_2 - x_1)\sum_{j=0}^{i} \tau(a)_j) \\ & \times (\tau(a)_i y_1,\ y_1). \end{aligned}$$

In this case, $(x_1, x_2) \times (y_1, y_2)$ is called the *mother tile* of R^i's in ω.

A vertical line $\gamma := \{x\} \times (-\infty, \infty)$ is called a *separating line* of $\omega \in \Omega(\sigma, \tau, g)'$ if for any $(R, a) \in \omega$, $R \cap \gamma = \emptyset$. Let $\Omega(\sigma, \tau, g)''$ be the set of all $\omega \in \Omega(\sigma, \tau, g)'$ which do not have a separating line and $\Omega(\sigma, \tau, g)$ be the closure of $\Omega(\sigma, \tau, g)''$.

Definition: By a *numeration system* with a nontrivial closed multiplicative subgroup G of $\mathbb{R}_+$, we mean a compact metrizable space Ω having at least 2 elements as follows:

(#1) There exists a continuous action $\lambda\omega + t$ of $(\lambda, t) \in G \times \mathbb{R}$ to $\omega \in \Omega$ such that $\lambda'(\lambda\omega + t) + t' = \lambda'\lambda\omega + \lambda' t + t'$.

(#2) The $(\omega + t)$-action of $t \in \mathbb{R}$ to $\omega \in \Omega$ is strictly ergodic with the unique invariant probability measure μ_Ω called the *equilibrium measure* on Ω. Consequently, it is invariant under the $(\lambda\omega + t)$-action of $(\lambda, t) \in G \times \mathbb{R}$ to $\omega \in \Omega$ as well.

(#3) For any fixed $\lambda_0 \in G$, the transformation $\omega \mapsto \lambda_0\omega$ on Ω has the $|\log \lambda_0|$-topological entropy. For any probability measure ν on Ω other than μ_Ω which is invariant under the $\lambda\omega$-action of $\lambda \in G$ to ω, and $1 \neq \lambda_0 \in G$, it holds that

$$h_\nu(\lambda_0) < h_{\mu_\Omega}(\lambda_0) = |\log \lambda_0|.$$

Theorem 1. [1] *The space $\Omega(\sigma, \tau, g)$ is a numeration system with $G = B(\sigma, \tau)$.*

In this paper, we study the spectral property of the additive action $\omega \to \omega + t$ on the probability space (Ω, μ_Ω) for $\Omega = \Omega(\sigma, \tau, g)$. For to prove or disprove the weak mixing property, we use the same technic as B. Solomyak [4], and for to disprove strong mixing property, we use the same technic as F.M. Dekking and M. Keane [3]. In fact, we prove

Theorem 2. *The additive action on the space $\Omega(\sigma, \tau, g)$ with $B(\sigma, \tau) = \{\lambda^n;\ n \in \mathbb{Z}\}$ for some $\lambda > 1$ is not strongly mixing. Moreover, it is weakly mixing if and only if λ is not a Pisot number.*

Since it is known [1] that the additive action on $\Omega(\sigma, \tau, g)$ with $B(\sigma, \tau) = \mathbb{R}_+$ has a pure Lebesgue spectrum and is strongly mixing. Therefore, we have the following classification.

Corollary 1. *The additive action on the space $\Omega(\sigma, \tau, g)$ is strongly mixing if and only if $B(\sigma, \tau) = \mathbb{R}_+$. It is not strongly mixing but weakly mixing if and only if $B(\sigma, \tau) = \{\lambda^n;\ n \in \mathbb{Z}\}$ with a non-Pisot number λ. It is not weakly mixing if and only if $B(\sigma, \tau) = \{\lambda^n;\ n \in \mathbb{Z}\}$ with a Pisot number λ.*

As for the pure discrete spectrum, we have a conjecture:

Conjecture The additive action on the space $\Omega(\sigma, \tau, g)$ has a pure disctrete spectrum if and only if $B(\sigma, \tau) = \{\lambda^n;\ n \in \mathbb{Z}\}$ with a Pisot number λ and that there is only one element in $\Omega(\sigma, \tau, g)$ having the seperating line equal to $\{0\} \times (-\infty, \infty)$. (The "if" part can be proved.)

Our basic notations and terminology follow from [1]. See also [2], where the same contents are presented in a slightly different way.

2 Pisot case

Let $\Omega = \Omega(\sigma, \tau, g)$ with $G := B(\sigma, \tau) = \{\lambda^n;\ n \in \mathbb{Z}\}$ for some $\lambda > 1$. We know that λ is an algebraic number ([1]).

Lemma 1. *For any $a \in \mathbb{A}$ and $0 \leq i < |\sigma(a)|$, $\tau(a)_i \in \mathbb{Q}(\lambda)$. Moreover, for the function g in (1), there exists $g_0 > 0$ such that $g(a)/g_0 \in \mathbb{Q}(\lambda)$ for any $a \in \mathbb{A}$.*

Proof. By (1), for any $a \in \mathbb{A}$ and i with $0 \leq i < |\sigma(a)|$, there exists an integer $n(a, i)$ such that $\tau(a)_i = (g(\sigma(a)_i)/g(a))\lambda^{n(a,i)}$. Let $P_{ab} := \sum_{i;\ \sigma(a)_i = b} \lambda^{n(a,i)}$ and $M := (P_{ab})_{a,b\in\mathbb{A}}$ be a square matrix with the index set $\mathbb{A}$. Then for any $a \in \mathbb{A}$, we have

$$\begin{aligned}
1 &= \sum_{0\leq i<|\sigma(a)|} \tau(a)_i \\
&= \sum_{0\leq i<|\sigma(a)|} (g(\sigma(a)_i)/g(a))\lambda^{n(a,i)} \\
&= \sum_{b\in\mathbb{A}} g(b)/g(a) \sum_{i;\ \sigma(a)_i=b} \lambda^{n(a,i)} \\
&= \sum_{b\in\mathbb{A}} (g(b)/g(a))P_{ab}.
\end{aligned}$$

Hence, $g(a) = \sum_{b \in \mathbb{A}} P_{ab} g(b)$. That is,

$$\mathbf{g} = M\mathbf{g} \ \text{ with } \ \mathbf{g} = \begin{pmatrix} \cdot \\ \cdot \\ g(a) \\ \cdot \\ \cdot \end{pmatrix}. \tag{4}$$

Since some power of M is a positive matrix and $\mathbf{g}$ is a positive vector, $\mathbf{g}$ is an eigen vector of M corresponding to the simple eigenvalue 1. Take $a_0 \in \mathbb{A}$ and put $g_0 = g(a_0)$. Let $\overline{g}(a) = g(a)/g_0$ for $a \in \mathbb{A} \setminus \{a_0\}$ and $\overline{M}$ be the restriction of M to the index set $\overline{\mathbb{A}} := \mathbb{A} \setminus \{a_0\}$. Then, it follows from (4) that

$$(I - \overline{M}) \begin{pmatrix} \cdot \\ \cdot \\ \overline{g}(a) \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \\ M_{aa_0} \\ \cdot \\ \cdot \end{pmatrix}.$$

Since $I - \overline{M}$ is a regular matrix with entries in $\mathbb{Q}(\lambda)$ by an appropriate choice of $a_0 \in \mathbb{A}$ as 1 is the simple eigenvalue of of M, and the vector in the right hand also has entries in $\mathbb{Q}(\lambda)$, every $\overline{g}(a)$ is in $\mathbb{Q}(\lambda)$ for any $a \in \mathbb{A} \setminus \{a_0\}$. Thus, $g(a)/g_0 \in \mathbb{Q}(\lambda)$ for any $a \in \mathbb{A}$. From this, $\tau(a)_i \in \mathbb{Q}(\lambda)$ follows since $\tau(a)_i = (g(\sigma(a)_i)/g(a))\lambda^{n(a,i)}$. □

Let $\mathcal{I}(\Omega)$ be the set of integer points in Ω. That is,

$$\begin{aligned} \mathcal{I}(\Omega) = \{\omega \in \Omega; \ \ &\text{there exists } ((x_1, x_2) \times (y_1, y_2),\ a) \in \omega \\ &\text{such that } x_1 = 0 \text{ and } 1 \in [y_1, y_2)\}. \end{aligned} \tag{5}$$

Lemma 2. *If λ is a Pisot number, then the additive action on Ω has a nonconstant continuous eigen function.*

Proof. For $\omega \in \mathcal{I}(\otimes)$, let $\{((\xi_j, \xi'_j) \times (\zeta_j, \zeta_{j+1}),\ a_j);\ j = 0, 1, 2, \cdots\}$ with $0 = \xi_0 \geq \xi_1 \geq \xi_2 \geq \cdots$ be the collection of tiles in ω such that

$\xi_0 = 0$, $\zeta_0 \leq 1 < \zeta_1$ and for any $j = 1, 2, \cdots$, $(\xi_j, \xi'_j) \times (\zeta_j, \zeta_{j+1})$ is the mother tile of $(\xi_{j-1}, \xi'_{j-1}) \times (\zeta_{j-1}, \zeta_j)$. Then, we have

$$\xi_{j-1} - \xi_j = \zeta_j \sum_{0 \leq i < h} \tau(a_j)_i$$

for some h with $0 \leq h < |\sigma(a_j)| - 1$. On the other hand, $\zeta_j = g(a_j)\lambda^n$ for some $n \in \mathbb{Z}$ which tends to ∞ linearly fast as $j \to \infty$.

By Lemma 1, there exists a positive integer q such that all of $qg(a)/g_0$ for $a \in \mathbb{A}$, and all of $q\tau(a)_j$ for $a \in \mathbb{A}$ and $0 \leq j < |\sigma(a)|$ are algegraic integers belonging to $\mathbb{Q}(\lambda)$. Let

$$D := \{q\tau(a)_j \ ; \ a \in \mathbb{A}, \ 0 \leq j < |\sigma(a)|\} \cup \{qg(a)/g_0 \ ; \ a \in \mathbb{A}\} \cup \{0\}.$$

Then, D is a finite set of algebraic integers belonging to $\mathbb{Q}(\lambda)$ such that $q^2(\xi_{j-1} - \xi_j)/g_0 \in D\lambda^n$ for some $n \in \mathbb{Z}$ which tends to ∞ linearly fast as $j \to \infty$. Since λ is a Pisot number, this implies that the distance of $q^2(\xi_{j-1} - \xi_j)/g_0$ to the nearest integer tends to 0 exponentially fast. Hence,

$$\begin{aligned} \psi(\omega) &:= \lim_{N \to \infty} \exp[2\pi i \sum_{j=1}^{N} q^2(\xi_{j-1} - \xi_j)/g_0] \\ &= \lim_{N \to \infty} \exp[-2\pi i \eta \xi_N] \quad (\text{with } \eta = q^2/g_0) \end{aligned}$$

exists as a continuous function of $\omega \in \mathcal{I}(\otimes)$. It is continuous since ξ_N is determined by the local information of the tiling ω.

For $\omega \in \Omega$, let $s(\omega) \geq 0$ be the smallest number such that $\omega + s(\omega) \in \mathcal{I}(\otimes)$. Define a function f on Ω by

$$f(\omega) = \psi(\omega + s(\omega)) \exp[-2\pi i \eta s(\omega)].$$

Then f is a nonconstant continous function such that

$$f(\omega + t) = \exp[2\pi i \eta t] f(\omega) \tag{6}$$

for any $t \in \mathbb{R}$.

This is because if $\omega + s(\omega) = \omega + t + s(\omega + t)$, then (6) follows as $s(\omega + t) = s(\omega) - t$. If $\omega, \omega + t \in \mathcal{I}(\otimes)$ with $t > 0$, then $s(\omega) = s(\omega + t) = 0$ and

$$\begin{aligned} f(\omega) = \psi(\omega) &= \lim_{N\to\infty} \exp[-2\pi i\eta\xi_N] \\ f(\omega + t) = \psi(\omega + t) &= \lim_{N\to\infty} \exp[-2\pi i\eta\theta_N], \end{aligned}$$

where both of $\xi_N - t$ and θ_N are left coordinates of tiles in $\omega + t$ with horizontal sizes at least CN for some $C > 0$. On the other hand, there is $D > 0$ such that $-DN < \xi_N - t \leq \theta_N \leq 0$. This implies that $\theta_N - (\xi_N - t)$ is a bounded number of sums of horizontal sizes of tiles with horizontal sizes at least CN, and hence, a bounded number of sums of numbers of the form $g(a)\lambda^n$ with $a \in \mathbb{A}$ and $n \geq CN$. Therefore, $\exp[2\pi i\eta(\theta_N - (\xi_N - t))] \to 1$ as $N \to \infty$. Thus, we have $f(\omega + t) = \exp[2\pi i\eta t]f(\omega)$. Combining these 2 cases, we complete the proof. $\square$

Remark 1. *If ω has a seperating line $\{t\} \times (-\infty, \infty)$, then we have $f(\omega) = \exp[-2\pi i\eta t]$. Moreover, if there is only one element, say ω_0, in Ω which has the seperating line $\{0\} \times (-\infty, \infty)$, then the mapping $\omega_0 + t \mapsto (\lambda^{-n}\eta t;\ n \in \mathbb{N}) \in \Theta_\lambda$ is extended to an homeomorphism between Ω and Θ_λ, where*

$$\Theta_\lambda = \{(x_0, x_1, \cdots) \in (\mathbb{R}/\mathbb{Z})^{\mathbb{N}};\ x_n + a_1x_{n+1} + \cdots + a_mx_{n+m} = 0\ (\forall n \in \mathbb{N})\}$$

and $z^m + a_1z^{m-1} + \cdots + a_m$ is the minimal polynomial of λ. This proves the "if" part of Conjecture in Section 1.

3 non-Pisot case

Denote

$$C_0 := \min\{\tau(a)_i;\ a \in \mathbb{A},\ 0 \leq i < |\sigma(a)|\} \tag{7}$$

Lemma 3. *If λ is not a Pisot number, then the additive action on Ω is weakly mixing.*

Proof. Suppose that the additive action on Ω has a nonconstant measurable eigenfunction, say f, such that

$$f(\omega + t) = \exp[2\pi i\eta t]f(\omega) \tag{8}$$

for any $\omega \in \Omega$ and $t \in \mathbb{R}$ with some $\eta \in \mathbb{R} \setminus \{0\}$. Since the additive action on Ω is ergodic, we may assume that $|f(\omega)| = 1$ for all $\omega \in \Omega$.

For $\omega \in \Omega$, there exists a unique tile $((x_1^0, x_2^0) \times (y_1^0, y_2^0),\ a^0) \in \omega$ with $x_1^0 \leq 0 < x_2^0$ and $y_1^0 \leq 1 < y_2^0$. This tile is called the *central tile* of ω and is denoted by $T^0(\omega)$. Then, $\omega \in \mathcal{I}(\Omega)$ if and only if $x_1^0 = 0$. Let $\Omega_0 := \Omega \setminus \mathcal{I}(\Omega)$.

It is clear that $\mu_\Omega(\Omega_0) = 1$. For $\omega \in \Omega$, the values $x_1^0, x_2^0, y_1^0, y_2^0, a^0$ of $T^0(\omega)$ are determine by ω, so that they may be written as functions of ω like $x_1^0(\omega),\ a^0(\omega)$, etc. There are only finitely many different values of (y_1^0, y_2^0, a^0)'s, since $a^0 \in \mathbb{A},\ y_1^0 \in g(a^0)G,\ y_1^0 \leq 1 < y_2^0$ and $y_1^0/y_2^0 \in \{\tau(a)_i;\ a \in \mathbb{A},\ 0 \leq i < |\sigma(a)|\}$.

Take the minimum $\theta = \theta(\omega) > 0$ for $\omega \in \Omega$ such that

$$T^0(\omega) = T^0(\omega + \theta).$$

Since Ω is minimal with respect to the additive action and G is discrete, such θ always exists and $\theta(\omega) \geq C_0 y_2^0 > C_0$ ($\forall \omega \in \Omega_0$) (see (7)). Moreover, θ is a continuous and locally constant function if the domain is restricted to Ω_0.

Take an arbitrary $\omega \in \Omega_0$. Since ω and $\omega + \theta$ share the central tile $T^0(\omega)$, $\lambda^n\omega$ and $\lambda^n(\omega + \theta)$ share a colored tile $\lambda^n T^0(\omega)$ for any $n \in \mathbb{Z}$. This implies the tilings of $\lambda^n\omega$ and $\lambda^n(\omega + \theta)$ restricted to the region $(\lambda^n x_1^0, \lambda^n x_2^0) \times (0, \lambda^n y_2^0)$ are the same. Note that

$$\lambda^n x_1^0 \to -\infty,\ \lambda^n x_2^0 \to \infty,\ \lambda^n y_2^0 \to \infty$$

as $n \to \infty$. Therefore, for any metric Ξ on Ω which is consistent with the topology, it holds that

$$\Xi(\lambda^n\omega, \lambda^n(\omega + \theta)) \to 0$$

as $n \to \infty$.

This implies that

$$||(f(\lambda^n\omega) - f(\lambda^n(\omega + \theta)))||_{L^1} \to 0$$

as $n \to \infty$, where L^1 is with respect to the equilibrium measure μ_Ω. Since $|f(\omega)| \equiv 1$, we have by (8),

$$||1 - \exp[2\pi i \eta \lambda^n \theta]||_{L^1} \to 0$$

as $n \to \infty$.

Since θ is a locally constant function on Ω_0, there exists $\theta_0 > 0$ such that

$$|1 - \exp[2\pi i \eta \lambda^n \theta_0]| \to 0$$

as $n \to \infty$, so that the distance of $\eta \lambda^n \theta_0$ to the nearest integer converges to 0. Since λ is an algeraic number and $\eta\theta_0 \neq 0$, it is known that this is possible only when λ is a Pisot number.

Thus, we have a contradiction, which completes the proof. □

4 Strong mixing

Let $\Omega = \Omega(\sigma, \tau, g)$ with $G := B(\sigma, \tau) = \{\lambda^n;\ n \in \mathbb{Z}\}$ for some $\lambda > 1$. We prove that the additive action on Ω is not strongly mixing.

Take an arbitrary $\omega_0 \in \Omega_0$. Let $x_1^0 := x_1^0(\omega_0)$, $x_2^0 := x_2^0(\omega^0)$, $T_0 := T^0(\omega_0) + x_1^0$ and $\theta_0 := \theta(\omega_0)$. Define

$$A := \{\omega \in \Omega_0;\ T^0(\omega) + x_1^0(\omega) = T_0 \text{ and } \theta(\omega) = \theta_0\}$$

and

$$A_\epsilon := \{\omega \in A;\ x_1^0(\omega) \in (-\epsilon,\ 0)\}$$

for $\epsilon \in (0,\ x_2^0 - x_1^0)$.

Since A is a nonempty open set in Ω such that the boundaries have measure 0 with respect to μ_Ω and the additive action on Ω is uniquely ergodic, we have

$$0 < \mu_\Omega(A) = \lim_{T \to \infty} (1/T) \int_0^T 1_A(\omega + t)dt$$

for any $\omega \in \Omega$. Moreover, this convergence is uniform in ω. In the same way,

$$\begin{aligned} \mu_\Omega(A_\epsilon) &= \lim_{T \to \infty} (1/T) \int_0^T 1_{A_\epsilon}(\omega + t)dt \\ &= \mu_\Omega(A)\epsilon/(x_2^0 - x_1^0). \end{aligned}$$

For a sufficiently large integer n, $t \in \mathbb{R}$ and $\omega \in \Omega$, consider the set $I := \{t \in \mathbb{R};\ \omega + \lambda^{-n}t \in A\}$. It is a union of sufficiently long intervals, say I_i $(i \in \mathbb{Z})$. Let $t \in I$. Then, we have

$$T^0(\omega + \lambda^{-n}t) = T^0(\omega + \lambda^{-n}t + \theta_0).$$

Therefore, we have

$$T^0(\lambda^n\omega + t) = T^0(\lambda^n\omega + t + \lambda^n\theta_0),$$

since they are the descendants of the identical tile

$$\lambda^n T^0(\omega + \lambda^{-n}t) = \lambda^n T^0(\omega + \lambda^{-n}t + \theta_0)$$

at the same position. Then for any $\epsilon > 0$, $\lambda^n\omega + t \in A_\epsilon$ implies that $\lambda^n\omega + t + \lambda^n\theta_0 \in A_\epsilon$. That is, if $t \in I$, $\lambda^n\omega + t \in A_\epsilon$ implies $\lambda^n\omega + t \in A_\epsilon \cap (A_\epsilon - \lambda^n\theta_0)$. Hence, we have

$$\begin{aligned}
& \mu_\Omega(A_\epsilon \cap (A_\epsilon - \lambda^n\theta_0)) \\
= \ & \lim_{T\to\infty} (1/T) \int_0^T 1_{A_\epsilon \cap (A_\epsilon - \lambda^n\theta_0)}(\lambda^n\omega + t)dt \\
\geq \ & \lim_{T\to\infty} (1/T) \int_{[0,T]\cap I} 1_{A_\epsilon}(\lambda^n\omega + t)dt \\
\geq \ & \lim_{T\to\infty} (1/T) \int_0^T 1_I(t)dt \cdot \frac{1}{|I \cap [0,T]|} \int_{I\cap[0,T]} 1_{A_\epsilon}(\lambda^n\omega + t)dt \\
= \ & \mu_\Omega(A)\epsilon/(x_2^0 - x_1^0) \geq \mu_\Omega(A)\mu_\Omega(A_\epsilon)
\end{aligned}$$

where $\delta > 0$ tends to 0 as $n \to \infty$. Thus, taking $\epsilon > 0$ so that $\mu_\Omega(A_\epsilon) < (1/2)\mu_\Omega(A)$, we have

$$\liminf_{n\to\infty} \mu_\Omega(A_\epsilon \cap (A_\epsilon - \lambda^n\theta_0)) \geq 2\mu_\Omega(A_\epsilon)^2,$$

which implies that the additive action on (Ω, μ_Ω) is not strongly mixing. □

Acknowledgement: The author thanks Prof. Boris Solomyak (Washington Univ.) for his useful suggestions to the author.

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