Numeration systems as dynamical systems

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Abstract: A numeration system originally implies a digitization of real numbers, but in this paper it rather implies a compactification of real numbers as a result of the digitization.

By definition, a numeration system with G, where G is a nontrivial closed multiplicative subgroup of $\mathbb{R}_+$, is a nontrivial compact metrizable space Ω admitting a continuous $(\lambda\omega + t)$-action of $(\lambda, t) \in G \times \mathbb{R}$ to $\omega \in \Omega$, such that the $(\omega + t)$-action is strictly ergodic with the unique invariant probability measure μ_Ω, which is the unique G-invariant probability measure attaining the topological entropy $|\log \lambda|$ of the transformation $\omega \mapsto \lambda\omega$ for any $\lambda \neq 1$.

We construct a class of numeration systems coming from weighted substitutions, which contains those coming from substitutions or β-expansions with algebraic β. It also contains those with $G = \mathbb{R}_+$.

We obtained the ζ-function ζ_Ω of the numeration systems Ω coming from weighted substitutions. For $\alpha \in \mathbb{C}$ with $0 < \mathcal{R}e(\alpha) < 1$, there exists a nontrivial adapted α-homogeneous cocycle on Ω if and only if ζ_Ω has a pole at α. These cocycles are generalizations of the fractal functions of Peano type. Moreover, in the case $G = \mathbb{R}_+$, these cocycles define self-similar processes of order α under μ_Ω. We discuss one of them with $\alpha = 1/2$. For a pole α of ζ_Ω with $\mathcal{R}e(\alpha) < 0$, there exsists a nontrivial adapted α-homogeneous cocycle on the set of integer points $\mathcal{I}(\Omega)$ in Ω, which is proved to be a coboundary. The image of $\mathcal{I}(\Omega)$ under this coboundary function becomes a fractal set

of Rauzy type.

This paper is based on [10] changing the way of presentation.

1 Numeration systems

By a *numeration system*, we mean a compact metrizable space Ω with at least 2 elements as follows:

($\sharp$1) There exists a nontrivial closed multiplicative subgroup G of $\mathbb{R}_+$ and a continuous action $\lambda\omega + t$ of $(\lambda, t) \in G \times \mathbb{R}$ to $\omega \in \Omega$ such that $\lambda'(\lambda\omega + t) + t' = \lambda'\lambda\omega + \lambda' t + t'$.

($\sharp$2) The $(\omega + t)$-action of $t \in \mathbb{R}$ to $\omega \in \Omega$ is strictly ergodic with the unique invariant probability measure μ_Ω called the *equilibrium measure* on Ω. Consequently, it is invariant under the $(\lambda\omega + t)$-action of $(\lambda, t) \in G \times \mathbb{R}$ to $\omega \in \Omega$ as well.

($\sharp$3) For any fixed $\lambda_0 \in G$, the transformation $\omega \mapsto \lambda_0\omega$ on Ω has the $|\log \lambda_0|$-topological entropy. For any probability measure ν on Ω other than μ_Ω which is invariant under the $\lambda\omega$-action of $\lambda \in G$ to ω, and $1 \neq \lambda_0 \in G$, it holds that

$$h_\nu(\lambda_0) < h_{\mu_\Omega}(\lambda_0) = |\log \lambda_0|.$$

The $(\omega + t)$-action of $t \in \mathbb{R}$ to $\omega \in \Omega$ is called the *additive* action or $\mathbb{R}$-action, while the $\lambda\omega$-action of $\lambda \in G$ to $\omega \in \Omega$ is called the *multiplicative* action or G-action.

Note that if Ω is a numeration system, then Ω is a connected space with the continuum cardinality. Also, note that the multiplicative group G as above is either $\mathbb{R}_+$ or $\{\lambda^n;\ n \in \mathbb{Z}\}$ for some $\lambda > 1$. Moreover, the additive action is faithful, that is, $\omega + t = \omega$ implies $t = 0$ for any $\omega \in \Omega$ and $t \in \mathbb{R}$.

This is because if there exist $\omega_1 \in \Omega$ and $t_1 \neq 0$ such that $\omega_1 + t_1 = \omega_1$, then take a sequence λ_n in G such that $\lambda_n \to 0$ and $\lambda_n\omega_1$ converges as $n \to \infty$. Let $\omega_\infty := \lim_{n\to\infty} \lambda_n\omega_1$. For any $t \in \mathbb{R}$, let a_n be a sequence of integers such that $a_n\lambda_n t_1 \to t$ as $n \to \infty$. Then we have

$$\begin{aligned} \omega_\infty + t &= \lim_{n\to\infty} (\lambda_n\omega_1 + \lambda_n a_n t_1) \\ = \quad \lim_{n\to\infty} \lambda_n (\omega_1 + a_n t_1) &= \lim_{n\to\infty} \lambda_n\omega_1 = \omega_\infty. \end{aligned}$$

Thus, ω_∞ becomes a fixed point of the $(\omega+t)$-action of $t \in \mathbb{R}$ to $\omega \in \Omega$. S ince this action is minimal, we have $\Omega = \{\omega_\infty\}$, cotradicting with that Ω has at least 2 elements.

An example of a numeration system is the set $\{0,1\}^{\mathbb{Z}}$ with the product topology divided by the closed equivalence relation $\sim$ such that

$$(\cdots \alpha_{-2}, \alpha_{-1}; \alpha_0, \alpha_1, \alpha_2 \cdots) \sim (\cdots \beta_{-2}, \beta_{-1}; \beta_0, \beta_1, \beta_2 \cdots)$$

if and only if there exists $N \in \mathbb{Z} \cup \{\pm\infty\}$ satisfying that $\alpha_n = \beta_n$ $(\forall n > N)$, $\alpha_N = \beta_N + 1$ and $\alpha_n = 0,\ \beta_n = 1$ $(\forall n < N)$ or the same statement with α and β exchanged. Let $\Omega(2) := \{0,1\}^{\mathbb{Z}}/\sim$ and the equivalence class containing $(\cdots \alpha_{-2}, \alpha_{-1}; \alpha_0, \alpha_1 \alpha_2 \cdots) \in \{0,1\}^{\mathbb{Z}}$ is denoted by $\sum_{n=-\infty}^{\infty} \alpha_n 2^n \in \Omega(2)$. Then, $\Omega(2)$ is an additive topological group with the addition as follows:

$$\sum_{n=-\infty}^{\infty} \alpha_n 2^n + \sum_{n=-\infty}^{\infty} \beta_n 2^n = \sum_{n=-\infty}^{\infty} \gamma_n 2^n$$

if and only if there exists $(\cdots \eta_{-2}, \eta_{-1}; \eta_0, \eta_1, \eta_2 \cdots) \in \{0,1\}^{\mathbb{Z}}$ satisfying that

$$2\eta_{n+1} + \gamma_n = \alpha_n + \beta_n + \eta_n \quad (\forall n \in \mathbb{Z}).$$

This is isomorphic to the 2-adic *solenoidal group* which is by definition the projective limit of the projective system $\theta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ with $\theta(\alpha) = 2\alpha$ $(\alpha \in \mathbb{R}/\mathbb{Z})$.

Moreover, $\mathbb{R}$ is imbedded in $\Omega(2)$ continuously as a dense additive subgroup in the way that a nonnegative real number α is identified with $\sum_{n=-\infty}^{\infty} \alpha_n 2^n$ such that $\alpha = \sum_{n=-\infty}^{N} \alpha_n 2^n$ and $\alpha_n = 0$ $(\forall n > N)$ for some $N \in \mathbb{Z}$, while a negative real number $-\alpha$ with α as above is identified with $\sum_{n=-\infty}^{\infty} (1-\alpha_n) 2^n$. Then, $\mathbb{R}$ acts additively to $\Omega(2)$ by this addition. Furthermore, $G := \{2^k;\ k \in \mathbb{Z}\}$ acts multiplicatively to $\Omega(2)$ by

$$2^k \sum_{n=-\infty}^{\infty} \alpha_n 2^n = \sum_{n=-\infty}^{\infty} \alpha_{n-k} 2^n.$$

Thus, we have a group of actions on $\Omega(2)$ satisfying ($\sharp$1), ($\sharp$2)and ($\sharp$3) with $G := \{2^k;\ k \in \mathbb{Z}\}$ and the equilibrium measure $(1/2, 1/2)^{\mathbb{Z}}$.

Theorem 1. *$\Omega(2)$ is a numeration system with $G = \{2^n;\ n \in \mathbb{Z}\}$.*

We can express $\Omega(2)$ in the following different way. By a *partition* of the upper half plane $\mathbb{H} := \{z = x + iy;\ y > 0\}$, we mean a disjoint family of open sets such that the union of their closures coincides with $\mathbb{H}$. Let us consider the space $\Omega(2)'$ of partitions ω of $\mathbb{H}$ by open squares of the form $(x_1, x_2) \times (y_1, y_2)$ with $x_2 - x_1 = y_2 - y_1 = y_1$ and $y_1 \in G$ such that $(x_1, x_2) \times (y_1, y_2) \in \omega$ implies

$$\begin{aligned} &(x_1, (x_1 + x_2)/2) \times (y_1/2, y_1) \in \omega \quad \text{(type 0)} \\ \text{and}& \\ &((x_1 + x_2)/2, x_2) \times (y_1/2, y_1) \in \omega \quad \text{(type 1)}. \end{aligned} \tag{1}$$

An example of $\omega \in \Omega(2)'$ is shown in Figure 1. For $\omega \in \Omega(2)'$, let $(\alpha_0, \alpha_1, \cdots)$ be the sequence of the types defined in (1) of the squares in ω intersecting with the half vertical line from $+0 + i$ to $+0 + i\infty$ and let $(\alpha_{-1}, \alpha_{-2}, \cdots)$ be the sequence of the types of the squares in ω intersecting with the line segment from $+0 + i$ to $+0$. Then, ω is identified with $\sum_{n=-\infty}^{\infty} \alpha_n 2^n$. Note that replacing $+0$ by -0, we get $\sum_{n=-\infty}^{\infty} \beta_n 2^n$ such that $(\cdots \alpha_{-2}, \alpha_{-1}; \alpha_0, \alpha_1, \cdots) \sim (\cdots \beta_{-2}, \beta_{-1}; \beta_0, \beta_1, \cdots)$.

The topology on $\Omega(2)'$ is defined so that $\omega_n \in \Omega(2)'$ converges to $\omega \in \Omega(2)'$ as $n \to \infty$ if for every $R \in \omega$, there exist $R_n \in \omega_n$ such that $\lim_{n\to\infty} \rho(R, R_n) = 0$, where ρ is the Hausdorff metric between sets $R,\ R' \subset \mathbb{H}$

$$\rho(R, R') := \max\{\sup_{z\in R} \inf_{z'\in R'} |z - z'|,\ \sup_{z'\in R'} \inf_{z\in R} |z - z'|\}. \tag{2}$$

For $\omega \in \Omega(2)'$, $t \in \mathbb{R}$ and $\lambda \in \{2^n;\ n \in \mathbb{R}\}$, $\omega + t \in \Omega(2)'$ and $\lambda\omega \in \Omega(2)'$ are defined as the partitions

$$\omega + t := \{(x_1 - t, x_2 - t) \times (y_1, y_2);\ (x_1, x_2) \times (y_1, y_2) \in \omega\}$$

and

$$\lambda\omega := \{(\lambda x_1, \lambda x_2) \times (\lambda y_1, \lambda y_2);\ (x_1, x_2) \times (y_1, y_2) \in \omega\}.$$

Let $\kappa : \Omega(2)' \to \Omega(2)$ be the identification mapping defined above. Then, κ is a homeomorphism between $\Omega(2)'$ and $\Omega(2)$ such that $\kappa(\omega+$

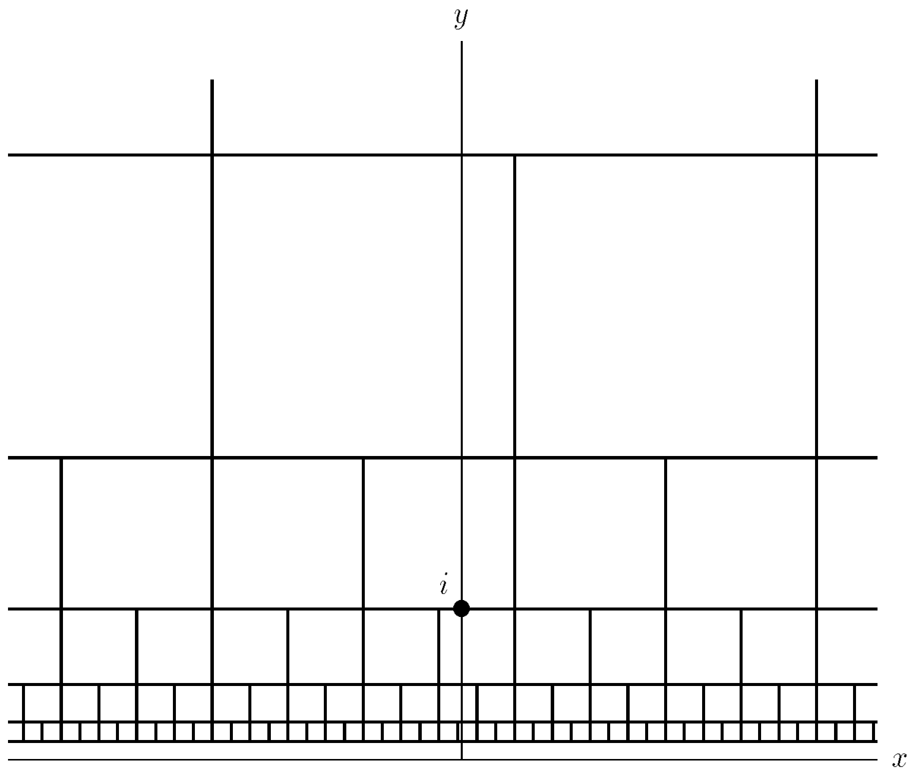

Figure 1: the tiling corresponding to $\cdots 01.101 \cdots$

$t) = \kappa(\omega) + t$ and $\kappa(\lambda\omega) = \lambda\kappa(\omega)$ for any $\omega \in \Omega(2)'$, $t \in \mathbb{R}$ and $\lambda \in \{2^n;\ n \in \mathbb{Z}\}$. Thus, $\Omega(2)'$ is isomorphic to $\Omega(2)$ as a numeration system and will be identified with $\Omega(2)$.

We generalize this construction. Let $\mathbb{A}$ be a nonempty finite set. An element in $\mathbb{A}$ is called a *color*. An open rectangle $(x_1, x_2) \times (y_1, y_2)$ in $\mathbb{H}$ is called an *admissible tile* if

$$x_2 - x_1 = y_1 \tag{3}$$

is satisfied (see Figure 2). In another word, an admissible tile is a

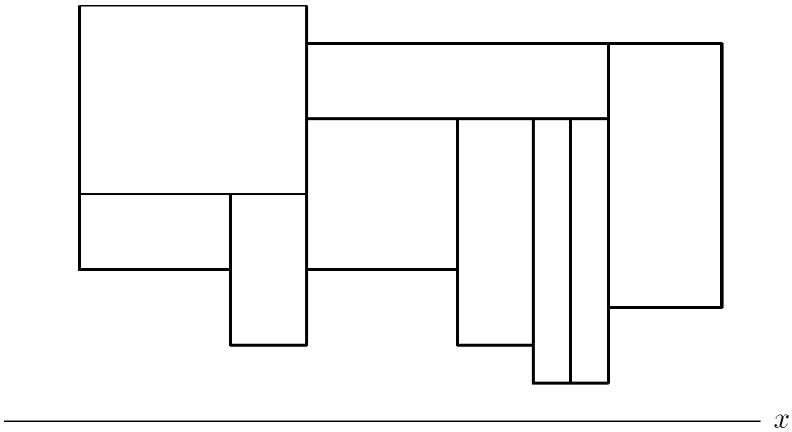

Figure 2: admissible tiles

rectangle $(x_1, x_2) \times (y_1, y_2)$ in $\mathbb{H}$ such that the lower side has the hyperbolic length 1. Let $\mathcal{R}$ be the set of admissible tiles in $\mathbb{H}$.

A *colored tiling* ω is a subset of $\mathcal{R} \times \mathbb{A}$ such that
(1) $R \cap R' = \emptyset$ for any (R, a) and (R', a') in ω with $(R, a) \neq (R', a')$, and
(2) $\cup_{(R,A)\in\omega} \overline{R} = \mathbb{H}$.

An element in $\mathcal{R} \times \mathbb{A}$ is called a *colored tile*. We denote

$$dom(\omega) := \{R;\ (R, a) \in \omega \text{ for some } a \in \mathbb{A}\}.$$

For $R \in dom(\omega)$, there exists a unique $a \in \mathbb{A}$ such that $(R, a) \in \omega$, which is denoted by $\omega(R)$ and is called the *color* of the tile R (in ω). Let $R = (x_1, x_2) \times (y_1, y_2)$. We call y_2/y_1 the *vertical size* of the tile R which is denoted by $S(R)$.

Let $\Omega(\mathbb{A})$ be the set of colored tilings with colors in $\mathbb{A}$. A topology is introduced on $\Omega(\mathbb{A})$ so that a net $\{\omega_n\}_{n\in I} \subset \Omega(\mathbb{A})$ converges to $\omega \in \Omega(\mathbb{A})$ if for every $(R, a) \in \omega$, there exists $(R_n, a_n) \in \omega_n$ such that

$$a_n = a \text{ for any sufficiently large } n \in I \text{ and } \lim_{n\to\infty} \rho(R, R_n) = 0,$$

where ρ is the Hausdorff metric defined in (2).

For an admissible tile $R := (x_1, x_2) \times (y_1, y_2)$, $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+$, we denote

$$\begin{aligned} R + t &:= (x_1 + t, x_2 + t) \times (y_1, y_2) \\ \lambda R &:= (\lambda x_1, \lambda x_2) \times (\lambda y_1, \lambda y_2). \end{aligned}$$

Note that they are also admissible tiles.

For $\omega \in \Omega(\mathbb{A})$, $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+$, we define $\omega + t \in \Omega(\mathbb{A})$ and $\lambda\omega \in \Omega(\mathbb{A})$ as follows:

$$\begin{aligned} \omega + t &= \{(R - t, a);\ (R, a) \in \omega\} \\ \lambda\omega &= \{(\lambda R, a);\ (R, a) \in \omega\}. \end{aligned}$$

Thus, we define a continuous group action $\lambda\omega + t$ of $(\lambda, t) \in \mathbb{R}_+ \times \mathbb{R}$ to $\omega \in \Omega(\mathbb{A})$. We construct compact metrizable subspaces of $\Omega(\mathbb{A})$ corresponding to weighted substitutions which are numeration systems. Though $\sharp\mathbb{A} \geq 2$ is assumed in [8], we consider the case $\sharp\mathbb{A} = 1$ as well.

2 Remarks on the notations

In this paper, the notations are changed in a large scale from the previous papers [8], [9] and [10] of the author. The main changes are as follows:

(1) Here, the colored tilings are defined on the upper half plane $\mathbb{H}$, not on $\mathbb{R}^2$ as in the previous papers. The multiplicative action here agree with the multiplication on $\mathbb{H}$, while it agree with the logarithmic version of the multiplication at one coodinate in the previous papers. Here, the tiles are open rectangles, not half open rectangles as in the previous papers.

(2) Here, we simplified the proof in [9] for the space of colored tilings coming from weighted substitutions to be numeration systems by omitting the arguments on the topological entropy.

(3) The roles of x-axis and y-axis for colored tilings are exchanged here and in [10] from those in [8] and [9].

(4) Here and in [10], the set of colors is denoted by $\mathbb{A}$ instead of Σ.

Colors are denoted by $a,\ a',\ a_i$ (etc.) instead of $\sigma,\ \sigma',\ \sigma_i$ (etc.).
(5) Here and in [10], the weighted substitution is denoted by (σ,τ) instead of (φ,η).
(6) Here and in [10], admissible tiles are denoted by $R,\ R',\ R_i,\ R^i$ (etc.) instead of $S,\ S',\ S_i,\ S^i$ (etc.).
(7) Here and in [10], the terminology "primitive" for substitutions is used instead of "mixing" in [8] and [9].

3 Weighted substitutions

A *substitution* σ on a set $\mathbb{A}$ is a mapping $\mathbb{A}\to\mathbb{A}^+$, where $\mathbb{A}^+ = \bigcup_{\ell=1}^{\infty}\mathbb{A}^\ell$. For $\xi\in\mathbb{A}^+$, we denote $|\xi| := \ell$ if $\xi\in\mathbb{A}^\ell$, and ξ with $|\xi|=\ell$ is usually denoted by $\xi_0\xi_1\cdots\xi_{\ell-1}$ with $\xi_i\in\mathbb{A}$. We can extend σ to be a homomorphism $\mathbb{A}^+\to\mathbb{A}^+$ as follows:

$$\sigma(\xi) := \sigma(\xi_0)\sigma(\xi_1)\cdots\sigma(\xi_{\ell-1}),$$

where $\xi\in\mathbb{A}^\ell$ and the right-hand side is the concatenations of $\sigma(\xi_i)$'s. We can define $\sigma^2,\sigma^3,\cdots$ as the compositions of $\sigma:\mathbb{A}^+\to\mathbb{A}^+$.

A *weighted substitution* (σ,τ) on $\mathbb{A}$ is a mapping $\mathbb{A}\to\mathbb{A}^+\times(0,1)^+$ such that $|\sigma(a)| = |\tau(a)|$ and $\sum_{i<|\tau(a)|}\tau(a)_i = 1$ for any $a\in\mathbb{A}$. Note that σ is a substitution on $\mathbb{A}$. We define $\tau^n:\mathbb{A}\to(0,1)^+$ $(n=2,3,...)$ (depending on σ) inductively by

$$\tau^n(a)_k = \tau(a)_i\tau^{n-1}(\sigma(a)_i)_j$$

for any $a\in\mathbb{A}$ and i,j,k with

$$0\le i<|\sigma(a)|,\ 0\le j<|\sigma^{n-1}(\sigma(a)_i)|,\ k=\sum_{h<i}|\sigma^{n-1}(\sigma(a)_h)|+j.$$

Then, (σ^n,τ^n) is also a weighted substitution for $n=2,3,\cdots$.

A substitution σ on $\mathbb{A}$ is called *primitive* if there exists a positive integer n such that for any $a,a'\in\mathbb{A}$, $\sigma^n(a)_i = a'$ holds for some i with $0\le i<|\sigma^n(a)|$.

For a weighted substitution (σ,τ) on $\mathbb{A}$, we always assume that

$$\textit{the substitution } \sigma \textit{ is primitive.} \qquad (4)$$

We define the *base set* $B(\sigma,\tau)$ as the closed, multiplicative subgroup of $\mathbb{R}_+$ generated by the set

$$\left\{ \begin{array}{c} \tau^n(a)_i \ ; \quad a \in \mathbb{A}, \quad n = 0, 1, \cdots \text{ and } 0 \le i < |\sigma^n(a)| \\ \text{such that } \sigma^n(a)_i = a \end{array} \right\}.$$

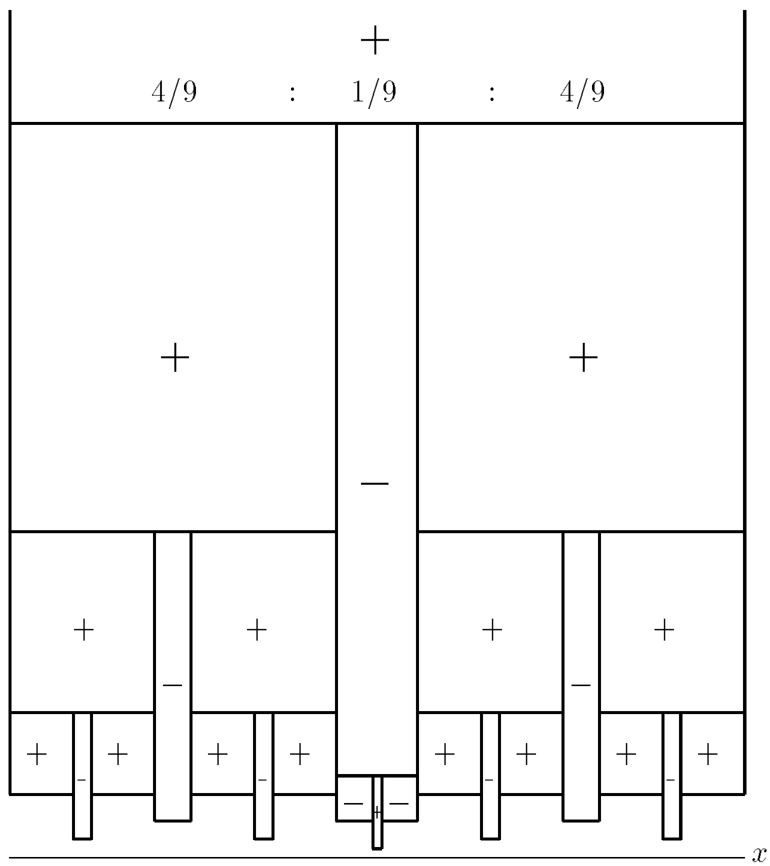

Figure 3: the weighted substitution in Example 1

Example 1. Let $\mathbb{A} = \{+,-\}$ and (σ,τ) be a weighted substitution such that

$$\begin{array}{ccl} + & \to & (+,4/9)(-,1/9)(+,4/9) \\ - & \to & (-,4/9)(+,1/9)(-,4/9), \end{array}$$

where we express a weighted substitution (σ, τ) by

$$a \to (\sigma(a)_0, \tau(a)_0)(\sigma(a)_1, \tau(a)_1) \cdots \ (a \in \mathbb{A}).$$

Then, $4/9 \in B(\sigma, \tau)$ since $\sigma(+)_0 = +$ and $\tau(+)_0 = 4/9$. Moreover, $1/81 \in B(\sigma, \tau)$ since $\sigma^2(+)_4 = +$ and $\tau^2(+)_4 = 1/81$. Since $4/9$ and $1/81$ do not have a common multiplicative base, we have $B(\sigma, \tau) = \mathbb{R}_+$. This weighted substitution is discussed in the following sections. The repetition of this weighted substitution starting at $+$ is shown in Figure 3 by colored tiles. The substituted word of a color is represented as the sequence of colors of the connected tiles in below in order from left. The horizontal (additive) sizes of tiles are proportional to the weights and the vertical (multiplicative) sizes are the inverse of the weights.

Let $G := B(\sigma, \tau)$. Then, there exists a function $g : \mathbb{A} \to \mathbb{R}_+$ such that

$$g(\sigma(a)_i)G = g(a)\tau(a)_i G \tag{5}$$

for any $a \in \mathbb{A}$ and $0 \le i < |\sigma(a)|$. Note that if $G = \mathbb{R}_+$, then we can take $g \equiv 1$. In the other case, we can define g by $g(a_0) = 1$ and $g(a) := \tau^n(a_0)_i$ for some n and i such that $\sigma^n(a_0)_i = a$, where a_0 is any fixed element in $\mathbb{A}$.

Let (σ, τ) be a weighted substitution satisfying (4). Let $G = B(\sigma, \tau)$. Let g satisfy (5). Let $\Omega(\sigma, \tau, g)'$ be the set of all elements ω in $\Omega(\mathbb{A})$ such that for any $((x_1, x_2) \times (y_1, y_2), a) \in \omega$, we have

(I) $y_1 \in g(a)G$, and

(II) $(R^i, \sigma(a)_i) \in \omega$ holds for $i = 0, 1, \cdots, |\sigma(a)| - 1$, where

$$\begin{aligned} R^i \ &:= \ (x_1 + (x_2 - x_1)\sum_{j=0}^{i-1} \tau(a)_j, \ x_1 + (x_2 - x_1)\sum_{j=0}^{i} \tau(a)_j) \\ &\quad \times (\tau(a)_i y_1, \ y_1). \end{aligned}$$

A vertical line $\gamma := \{x\} \times (-\infty, \infty)$ is called a *separating line* of $\omega \in \Omega(\sigma, \tau, g)'$ if for any $(R, a) \in \omega$, $R \cap \gamma = \emptyset$. Let $\Omega(\sigma, \tau, g)''$ be the set of all $\omega \in \Omega(\sigma, \tau, g)'$ which do not have a separating line and $\Omega(\sigma, \tau, g)$ be the closure of $\Omega(\sigma, \tau, g)''$. Then, the action of $G \times \mathbb{R}$ on $\Omega(\sigma, \tau, g)$ satisfies ($\sharp$1). We usually denote $\Omega(\sigma, \tau, 1)$ simply by $\Omega(\sigma, \tau)$.

Remark 1. [8] A nontrivial primitive substitution $\sigma : \mathbb{A} \to \mathbb{A}^+$, where "nontrivial" means $\sum_{a\in\mathbb{A}} |\sigma(a)| \geq 2$, is considered as a weighted substitution in a canonical way. Let

$$M := (\sharp\{0 \leq i < |\sigma(a)|;\ \sigma(a)_i = a'\})_{a,a'\in\mathbb{A}}$$

be the associate matrix. Let λ be the maximum eigen-value of M and $\xi := (\xi_a)_{a\in\mathbb{A}}$ be a positive column vector such that $M\xi = \lambda\xi$. Define weight τ by

$$\tau(a)_i = \frac{\xi_{\sigma(a)_i}}{\lambda\xi_a},$$

which is called the *natural weight* of σ. Thus, we get a weighted substitution (σ, τ) which admits weight 1. We modify (σ, τ) if necessary in the following way. If there exists $a \in \mathbb{A}$ with $|\sigma(a)| = 1$, so that $a \to (a', 1)$ is a part of (σ, τ), then we replace all the occurrences of a in the right hand side of "$\to$" by a' and remove a from $\mathbb{A}$ together with the rule $a \to (a', 1)$ from (σ, τ). We continue this process until no $a \in \mathbb{A}$ satisfies $|\sigma(a)| = 1$. After that if there exist $a,\ a' \in \mathbb{A}$ such that $(\sigma(a), \tau(a)) = (\sigma(a'), \tau(a'))$, then we identify them.

For example, the 2-*adic expansion substitution* $1 \to 12,\ 2 \to 12$ corresponds to the weighted substitution $1 \to (1, 1/2)(1, 1/2)$. The *Thue-Morse substitution* $1 \to 12,\ 2 \to 21$ corresponds to the weighted substitution $1 \to (1, 1/2)(2, 1/2),\ 2 \to (2, 1/2)(1, 1/2)$. The *Fibonacci substitution* $1 \to 12,\ 2 \to 1$ corresponds to the weighted substitution $1 \to (1, \lambda^{-1})(1, \lambda^{-2})$, where $\lambda = (1 + \sqrt{5})/2$.

The weighted substitution (σ, τ) obtained in this way satisfies that $B(\sigma, \tau) = \{\lambda^n;\ n \in \mathbb{Z}\}$ and that g in (5) can be defined by $g(a) = \xi_a\ (a \in \mathbb{A})$. Dynamical systems coming from substitutions are discussed by many authors (see [2], for example). Our weighted substitutions are a generalization of them.

Let (σ, τ) be a weighted substitution on $\mathbb{A}$ satisfying (4). Let g satisfy (5). Consider $\Omega(\sigma, \tau, g)$. We call the tile R^i in (II) the i-th *child* of the tile $(x_1, x_2) \times (y_1, y_2)$, and $(x_1, x_2) \times (y_1, y_2)$ the *mother* of R^i. Note that the vertical size $S(R^i)$ of R^i coincides with the inverse of the weight $\tau(a)_i$. If R_j is a child of R_{j+1} for $j = 0, 1, \cdots, k-1$. Then, the tile R_0 is called a k-th *descendant* of the tile R_k. If R_0 is the i-th tile among the set of the k-th descendants of R_k counting as $0, 1, 2, \cdots$ from left, we call R_0 the (k,i)-*descendant* of the tile R_k. In this case, we also say that R_k is the k-th *ancestor* of R_0.

Theorem 2. *The space $\Omega(\sigma,\tau,g)$ is a numeration system with $G = B(\sigma,\tau)$.*

Proof. We have already proved ($\sharp$1) and ($\sharp$2) in Theorem 3 of [8]. Here we prove ($\sharp$3). Let $\Omega := \Omega(\sigma,\tau,g)$ and μ_Ω be the equilibrium measure. Since μ_Ω is the unique invariant probability measure under the additive action, it is also invariant under the multiplicative action.

By Goodman [4], it is sufficient to prove that for any $\lambda \in G$ with $\lambda \neq 1$, the transformation $\omega \mapsto \lambda\omega$ on Ω has the metrical entropy $|\log \lambda|$ under μ_Ω, while it has the metrical entropy less than $|\log \lambda|$ under any other G-invariant probability measure.

Lemma 1. *Let*

$$\begin{aligned} \Sigma &:= \{\omega \in \Omega;\ \omega \textit{ has a separating line}\} \\ \Sigma_0 &:= \{\omega \in \Omega;\ y\textit{-axis is the separating line of } \omega\}. \end{aligned}$$

Then, we have

(i) $\Sigma \setminus \Sigma_0$ is dissipative with respect to the G-action, so that $\nu(\Sigma \setminus \Sigma_0) = 0$ for any G-invariant probability measure ν on Ω.

(ii) For any $\omega \in \Sigma_0$, ω restricted to the right quarter plane $(0,\infty)\times(0,\infty)$ and to the left quarter plane $(-\infty,0)\times(0,\infty)$ are cyclic individually with respect to the G-action. Hence, $\overline{G\omega}$ with respect to the G-action is either cyclic or conjugate to a 2-dimensional irrational rotation with a multiplicative time parameter.

(iii) Σ_0 is a finite union of minimal and equicontinuous sets with respect to the G-action. In fact, there is a mapping from the set of pairs $a \in \mathbb{A}$ and i with $0 \le i < i+1 < |\sigma(a)|$ onto the set of minimal sets in Σ_0.

Proof. (i) If the line $x = u$ is the separating line of $\omega \in \Omega$, then $x = \lambda u$ is the separating line of $\lambda\omega$. Hence, $\Sigma \setminus \Sigma_0$ is dissipative.

(ii) Let $\omega \in \Sigma_0$. Denote by ω^+ the restriction of ω to the right quarter plane $(0,\infty)\times(0,\infty)$, while by ω^- the restriction of ω to the left quarter plane $(-\infty,0)\times(0,\infty)$. Let $(R_i^{\pm})_{i\in\mathbb{Z}}$ be the sequence of tiles in $dom(\omega)$ such that $R_i^{\pm}$ intersects with the upper half lines of $x=\pm 0$, and $R_i^{\pm}$ is a child of $R_{i+1}^{\pm}$ for any $i\in\mathbb{Z}$ ($\pm$ respectively). Let $a_i^{\pm} := \omega(R_i^{\pm})$ be the colors of $R_i^{\pm}$ ($\pm$ respectively). Define mappings $\sigma_{\pm}$ from $\mathbb{A}$ to $\mathbb{A}$ by $\sigma_+(a)=\sigma(a)_0$ and $\sigma_-(a)=\sigma(a)_{|\sigma(a)|-1}$. Since $\sigma_{\pm}(a_i^{\pm}) = a_{i-1}^{\pm}$ ($i\in\mathbb{Z}$) ($\pm$ respectively), the sequence $(a_i^{\pm})_{i\in\mathbb{Z}}$ is periodic, which also implies that the vertical sizes $S(R_i^{\pm})$ of $R_i^{\pm}$, which coincide with the inverses of the weights $\tau(a_{i+1})_{\pm}$, are also periodic in $i\in\mathbb{Z}$ with the period, say $r^{\pm}$ which is the minimum period of $(a_i^{\pm})_{i\in\mathbb{Z}}$ ($\pm$ respectively). Then, $\lambda^+ := \tau^{r^+}(a_0^+)_0^{-1}$ is the minimum (multiplicative) cycle of ω^+, while $\lambda^- := \tau^{r^-}(a_0^-)^{-1}_{|\sigma^{r^-}(a_0^-)|-1}$ is the minimum (multiplicative) cycle of ω^-, that is, $\lambda\omega^{\pm}=\omega^{\pm}$ holds for $\lambda=\lambda^{\pm}$ and $\lambda^{\pm}$ is the minimum among $\lambda>1$ with this property ($\pm$ respectively).

Therefore, ω is cyclic with respect to the G-action if λ^+ and λ^- have a common multiplicative base. In this case, the minimum cycle of ω is the minimum positive number x such that $x=(\lambda^+)^n=(\lambda^-)^m$ holds for some positive integers n,m. Otherwise, the G-action to $\overline{G\omega}$ is conjugate to an 2-dimensional irrational rotation with a multiplicative time parameter.

(iii) We use the notations in the proof of (ii). Take any pair (a,i) with $a\in\mathbb{A}$ and $0\le i<i+1<|\sigma(a)|$. Take any $\omega'\in\Omega$ having a tile $R\in dom(\omega')$ with $\omega'(R)=a$ such that the y-axis passes in between the i-th child of R and the $i+1$-th child of R. Let $\psi(a,i)$ be the set of limit points of $\lambda\omega'$ as $\lambda\in G$ tends to ∞. Note that this does not depend on the choice of ω'. Then, $\psi(a,i)$ is a closed G-invariant subset of Σ_0. Moreover, since the sequence $(\sigma_-^n(\sigma(a)_i), \sigma_+^n(\sigma(a)_{i+1}))_{n=0,1,2,\cdots}$ enter into a cycle after some time, $\psi(a,i)$ is minimal and equicontinuous with respect to the G-action.

To prove that the mapping ψ is onto, take any $\omega\in\Sigma_0$. There exists $\omega_n\in\Omega(\sigma,\tau,g)''$ which converges to ω as $n\to\infty$. We may assume that there exists a pair (a,i) such that for any $n=1,2,\cdots$, there exists $R\in dom(\omega_n)$ with $a=\omega_n(R)$ such that the y-axis separetes the i-th child of R and the $i+1$-th child of R. Then, $\omega\in\psi(a,i)$, which proves that ψ is a mapping from the set of pairs (a, i) with $a \in \mathbb{A}$ and $0 \le i < i+1 < |\sigma(a)|$ onto the set of minimal sets in Σ_0 with respect to the G-action. $\square$

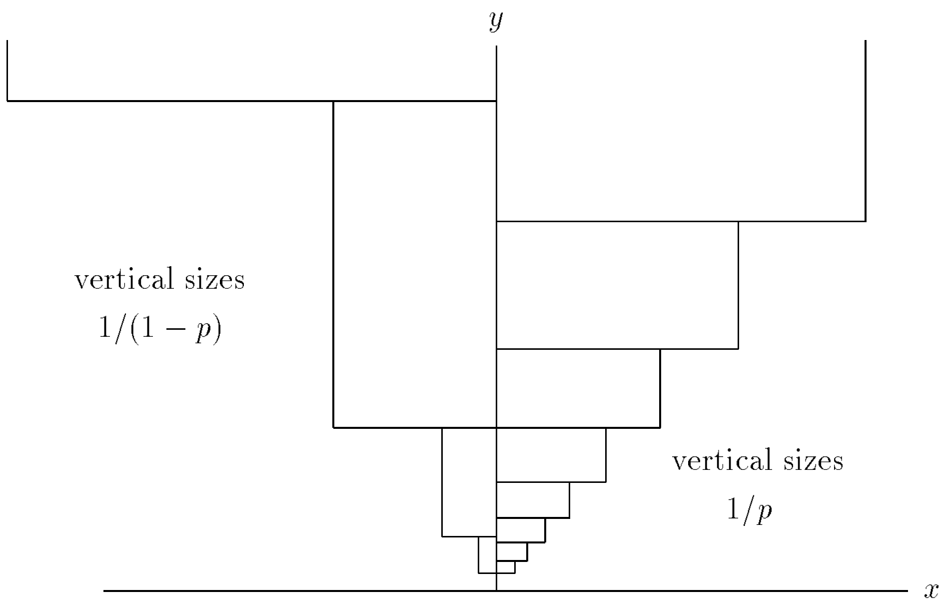

Figure 4: an element in Σ_0 in Example 2

Example 2. Let p with $0 < p < 1$ satisfy that $\log p/\log(1-p)$ is irrational. Let (σ, τ) be a weighted substitution on $\mathbb{A} = \{1\}$ such that $1 \to (1, p)(1, 1-p)$. Then, $B(\sigma, \tau) = \mathbb{R}_+$ holds. Let $\Omega = \Omega(\sigma, \tau)$. In this case, elements in Σ_0 are not periodic, but almost periodic as shown in Figure 4. Then, the dynamical system $(\Sigma_0,\ \lambda\ (\lambda \in \mathbb{R}_+))$ is isomorphic to $((\mathbb{R}/\mathbb{Z})^2,\ T_\lambda\ (\lambda \in \mathbb{R}_+))$ with

$$T_\lambda(x, y) = (x + \log\lambda/\log(1/p),\ y + \log\lambda/\log(1/(1-p))).$$

Lemma 2. *It holds that* $h_{\mu_\Omega}(\lambda) = |\log\lambda|$ *for any* $\lambda \in G$. *Let* $\lambda \neq 1$ *and* ν *be any other* λ*-invariant probability measure on* Ω*, then* $h_\nu(\lambda) < |\log\lambda|$.

Proof. To prove the lemma, it is sufficient to prove the statements for $\lambda > 1$. Take any G-invariant probability measure ν on Ω which attains the topological entropy of the multiplication by $\lambda_1 \in G$ with $\lambda_1 > 1$, that is, $h_\nu(\lambda_1) = \log \lambda_1$. We assume also that the G-action to Ω is ergodic with respect to ν. Then by Lemma 1, either $\nu(\Sigma_0) = 1$ or $\nu(\Omega \setminus \Sigma) = 1$. In the former case, $h_\nu(\lambda) = 0$ holds for any $\lambda \in G$ since the G-action on Σ_0 is equicontinuous by Lemma 1, which contradicts with the assumption. Thus, we have $\nu(\Omega \setminus \Sigma) = 1$.

For $\omega \in \Omega$, let $R_0(\omega) \in dom(\omega)$ be such that $R_0(\omega) = (x_1, x_2) \times (y_1, y_2)$ with $x_1 \leq 0 < x_2$ and $y_1 \leq 1 < y_2$. Take $a_0 \in \mathbb{A}$ such that

$$\nu(\{\omega \in \Omega;\ \omega(R_0(\omega)) = a_0\}) > 0.$$

Take $b_0 := \max\{b \leq 1;\ b \in g(a_0)G\}$ (see (5)). Let

$$\begin{aligned} \Omega_1 \ &:= \ \{\ \omega \in \Omega;\ \text{the set } \{\lambda \in G;\ \lambda\omega(R_0(\lambda\omega)) = a_0\} \\ &\qquad\qquad \text{is unbounded at 0 and } \infty \text{ simultaneously } \} \\ \Omega_0 \ &:= \ \{\ \omega \in \Omega_1;\ R_0(\omega) = (x_1, x_2) \times (y_1, y_2) \\ &\qquad\qquad \text{with } y_1 = b_0 \text{ and } \omega(R_0(\omega)) = a_0\ \}. \end{aligned}$$

For $\omega \in \Omega_0$, let $\lambda_0(\omega)$ be the smallest $\lambda \in G$ with $\lambda > 1$ such that $\lambda\omega \in \Omega_0$. Define a mapping $\Lambda : \Omega_0 \to \Omega_0$ by $\Lambda(\omega) := \lambda_0(\omega)\omega$.

For $k = 0, 1, 2, \cdots$ and $i = 0, 1, \cdots, |\sigma^k(a_0)| - 1$, let

$$\begin{aligned} P(k,i) \ &:= \ \{\ \omega \in \Omega_0;\ \lambda_0(\omega)^{-1} R_0(\lambda_0(\omega)\omega) \\ &\qquad\qquad \text{is the } (k,i)\text{-descendant of } R_0(\omega)\} \end{aligned}$$

(see Figure 5) and let

$$\mathcal{P} := \{\ P(k,i);\ k = 1, 2, \cdots,\ 0 \leq i < |\sigma^k(a_0)|\ \}$$

be a measurable partition of Ω_0. Note that $\lambda_0(\omega) = \tau^k(a_0)_i^{-1}$ if $\omega \in P(k,i)$.

Since $\nu(\Omega_1) = 1$ by the ergodicity and

$$\Omega_1 = \bigcup_{P(k,i) \in \mathcal{P}} \ \bigcup_{\substack{1 \leq \lambda < \tau^k(a_0)_i^{-1} \\ \lambda \in G}} \lambda P(k,i),$$

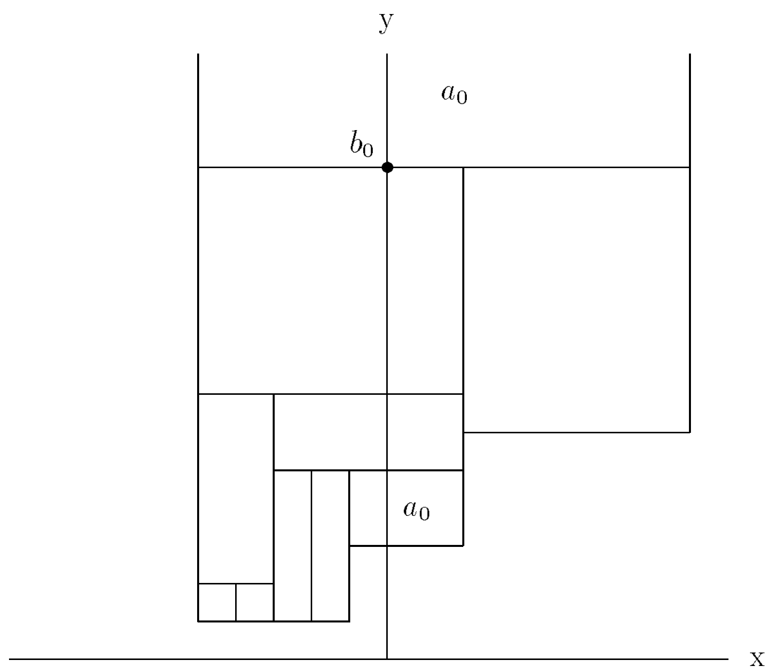

Figure 5: $\omega \in P(3,4)$ with $\lambda_0(\omega) = 13/3$

there exists a unique Λ-invariant probability measure ν_0 on Ω_0 such that for any Borel set $B \subset \Omega$, we have

$$\nu(B) = C(\nu)^{-1} \sum_{P(k,i)\in\mathcal{P}} \int_{b_0}^{b_0 \tau^k(a_0)_i^{-1}} \nu_0(\lambda^{-1}B \cap P(k,i)) d\lambda/\lambda$$

with

$$C(\nu) := \sum_{P(k,i)\in\mathcal{P}} -\log \tau^k(a_0)_i \; \nu_0(P(k,i)) < \infty \tag{6}$$

if $G = \mathbb{R}_+$ and

$$\nu(B) = C(\nu)^{-1} \sum_{P(k,i)\in\mathcal{P}} \sum_{\substack{\lambda\in G \\ b_0 \le \lambda < b_0 \tau^k(a_0)_i^{-1}}} \nu_0(\lambda^{-1}B \cap P(k,i))$$

with

$$C(\nu) := \sum_{P(k,i)\in\mathcal{P}} (-\log \tau^k(a_0)_i / \log \beta) \; \nu_0(P(k,i)) < \infty \tag{7}$$

if $G = \{\beta^n;\ n \in \mathbb{Z}\}$ with $\beta > 1$.

Since

$$\sum_{P(k,i)\in\mathcal{P}} \tau^k(a_0)_i = 1 \text{ and } \sum_{P(k,i)\in\mathcal{P}} \nu_0(P(k,i)) = 1,$$

we have

$$\begin{aligned} H_{\nu_0}(\mathcal{P}) &:= -\sum_{P(k,i)\in\mathcal{P}} \log \nu_0(P(k,i)) \cdot \nu_0(P(k,i)) \\ &\leq -\sum_{P(k,i)\in\mathcal{P}} \log \tau^k(a_0)_i \cdot \nu_0(P(k,i)) \qquad (8) \end{aligned}$$

by the convexity of $-\log x$. The equality in (8) holds if and only if

$$\nu_0(P(k,i)) = \tau^k(a_0)_i \quad (\forall P(k,i) \in \mathcal{P}). \qquad (9)$$

By (6)(7)(8), we have

$$H_{\nu_0}(\mathcal{P}) = -\sum_{P(k,i)\in\mathcal{P}} \log \nu_0(P(k,i)) \cdot \nu_0(P(k,i)) < \infty.$$

For any $\omega,\ \omega' \in \Omega_0$ such that $\Lambda^k(\omega)$ and $\Lambda^k(\omega')$ belong to the same element in $\mathcal{P}$ for $k = 0, 1, 2, \cdots$, the horizontal position of $R_0(\omega)$, say (x_1, x_2), coincides with that of $R_0(\omega')$. Therefore, ω and ω' restricted to $(x_1, x_2) \times (0, b_0)$ coincide. In the same way, if $\Lambda^k(\omega)$ and $\Lambda^k(\omega')$ belong to the same element in $\mathcal{P}$ for any $k \in \mathbb{Z}$, then $R_0 := R_0(\omega) = R_0(\omega')$ holds and all the ancestors of R_0 in ω and ω' coincide as well as their colors. Therefore, ω and ω' restricted to the region covered by the ancestors of R_0 coincide. Hence, if ω or ω' does not have the separating lines, then $\omega = \omega'$ holds.

Since $\nu(\Sigma) = 0$, we have $\nu_0(\Sigma \cap \Omega_0) = 0$. Hence, the above argument implies that $\mathcal{P}$ is a generator of the system (Ω_0, ν, Λ). Thus, $h_{\nu_0}(\Lambda) = h_{\nu_0}(\Lambda, \mathcal{P})$. It follows from (8) that

$$\begin{aligned} h_{\nu_0}(\Lambda) &= h_{\nu_0}(\Lambda, \mathcal{P}) \\ &\leq H_{\nu_0}(\mathcal{P}) \\ &\leq -\sum_{P(k,i)\in\mathcal{P}} \log \tau^k(a_0)_i \cdot \nu_0(P(k,i)). \qquad (10) \end{aligned}$$

The equality in the above that

$$h_{\nu_0}(\Lambda) = -\sum_{P(k,i)\in\mathcal{P}} \log \tau^k(a_0)_i \cdot \nu_0(P(k,i))$$

holds if and only if $(\Lambda^n\mathcal{P})_{n\in\mathbb{Z}}$ is an independent sequence with respect to ν_0 satisfying (9).

Since

$$\begin{aligned} h_\nu(\lambda_1)/\log\lambda_1 &= \frac{h_{\nu_0}(\Lambda)}{\int_{\Omega_0}\lambda_0(\omega)d\nu_0(\omega)} \\ &= \frac{h_{\nu_0}(\Lambda)}{-\sum_{P(k,i)\in\mathcal{P}}\log\tau^k(a_0)_i\cdot\nu_0(P(k,i))}\ , \end{aligned}$$

$h_\nu(\lambda_1) \le \log\lambda_1$ follows from (10), while the equality holds if and only if $(\Lambda^n\mathcal{P})_{n\in\mathbb{Z}}$ is an independent sequence with respect to ν_0 satisfying (9). Let this probability measure be μ. Then, it is not difficult to prove that μ is invariant under the additive action. Hence, the uniqueness of such measure ([8]) proves $\mu = \mu_\Omega$, which completes the proof of Lemma 2 and Theorem 2. □

The following Theorem 3 follows from a known result about the spectrum of unitary operators corresponding to the affine action (Lemma 11.6 of [13], for example). Nevertheless, we give the proof for to be self-contained.

Theorem 3. *Let Ω be a numeration system with $G = \mathbb{R}_+$, that is, with the multiplicative $\mathbb{R}_+$-action. Then, the additive action on the probability space Ω with respect to μ_Ω has a pure Lebesgue spectrum.*

Proof. Let U_t ($t\in\mathbb{R}$) and V_λ ($\lambda\in\mathbb{R}_+$) be the groups of the unitary operators on $L^2(\Omega,\mu_\Omega)$ defined by

$$(U_tf)(\omega) = f(\omega+t)\ ,\ (V_\lambda f)(\omega) = f(\lambda\omega).$$

Let

$$U_t = \int_{-\infty}^{\infty} e^{itu}dE_u \quad (t\in\mathbb{R})$$

be the spectral decomposition of U_t ([3]). Since $U_t V_\lambda = V_\lambda U_{\lambda t}$, we have $dE_u V_\lambda = V_\lambda dE_{\lambda^{-1}u}$.

Take any $f \in L^2(\Omega, \mu_\Omega)$ with $\int f d\mu_\Omega = 0$ and $\int |f|^2 d\mu_\Omega = 1$. Let $m(f)$ be the measure on $\mathbb{R}$ defined by

$$m(f)(S) = \int_S \| dE_u f \|^2$$

for any Borel set $S \subset \mathbb{R}$. Then, $m(f)$ is a probability measure with $m(f)(\{0\}) = 0$. Since $dE_u V_\lambda = V_\lambda dE_{\lambda^{-1}u}$, we have

$$\begin{aligned} m(V_\lambda f)(S) &= \int_S \| dE_u V_\lambda f \|^2 = \int_S \| V_\lambda dE_{\lambda^{-1}u} f \|^2 \\ &= \int_S \| dE_{\lambda^{-1}u} f \|^2 = \int_{\lambda^{-1}S} \| dE_u f \|^2 = m(f)(\lambda^{-1}S). \end{aligned}$$

Moreover, we have

$$\begin{aligned} |(f, V_\lambda f)| &= |\int (dE_u f, dE_u V_\lambda f)| \\ &\leq \int \| dE_u f \| \| dE_u V_\lambda f \| = \int \sqrt{dm(f)}\sqrt{dm(V_\lambda f)}. \end{aligned}$$

Since $\lim_{\lambda \to 1} |(f, V_\lambda f)| = 1$, we have

$$\lim_{\lambda \to 1} \int \sqrt{dm(f)}\sqrt{dm(f) \circ \lambda^{-1}} = \lim_{\lambda \to 1} \int \sqrt{dm(f)}\sqrt{dm(V_\lambda f)} = 1.$$

It follows from this that $m(f)$ is absolutely continuous by the following well known argument (see [12], for example).

Suppose to the contrary that $m := m(f)$ is not absolutely continuous. Take a Borel set $S \subset \mathbb{R}$ such that S has Lebesgue measure 0 while $\delta := m(S) > 0$. Denoting $\rho(\lambda) := \int \sqrt{dm}\sqrt{dm \circ \lambda^{-1}}$, we have

$$\begin{aligned} 2(1 - \rho(\lambda)) &= \int \left(\sqrt{dm} - \sqrt{dm \circ \lambda^{-1}}\right)^2 \\ &\geq \left(\sqrt{m(S)} - \sqrt{m \circ \lambda^{-1}(S)}\right)^2 = \left(\sqrt{\delta} - \sqrt{m(\lambda^{-1}S)}\right)^2. \end{aligned}$$

Since $2(1-\rho(\lambda)) \to 0$ as $\lambda \to 1$, there exists $\epsilon > 0$ such that for any λ with $1-2\epsilon \le \lambda \le 1+2\epsilon$, $m(\lambda^{-1}S) > \delta/2$ holds. Hence,

$$2\delta\epsilon \le \int_{1-2\epsilon}^{1+2\epsilon} m(\lambda^{-1}S)d\lambda = \int\int_{1-2\epsilon}^{1+2\epsilon} 1_S(\lambda u)d\lambda dm(u).$$

This implies that the set of $u \in \mathbb{R}$ such that $\int_{1-2\epsilon}^{1+2\epsilon} 1_S(\lambda u)d\lambda \ge \delta\epsilon$ has the measure at least $\delta/4$ with respect to m. Since $m(\{0\}) = 0$, this implies that there exists $u \ne 0$ such that $\int_{1-2\epsilon}^{1+2\epsilon} 1_S(\lambda u)d\lambda \ge \delta\epsilon$. Thus, S has Lebesgue measure at least $|u|\delta\epsilon$, which contradicts the assumption that S has Lebesgue measure 0. Thus, $m = m(f)$ is absolutely continuous. □

4 ζ-function

Let $\Omega := \Omega(\sigma,\tau,g)$ satisfying (4) and (5). For $\alpha \in \mathbb{C}$, we define the associated matrices on the suffix set $\mathbb{A}\times\mathbb{A}$ as follows:

$$\begin{aligned} M_\alpha &:= \left(\sum_{i;\sigma(a)_i=a'} \tau(a)_i^\alpha\right)_{a,a'\in\mathbb{A}} \qquad (11)\\ M_{\alpha,+} &:= \left(1_{\sigma(a)_0=a'}\ \tau(a)_0^\alpha\right)_{a,a'\in\mathbb{A}} \\ M_{\alpha,-} &:= \left(1_{\sigma(a)_{|\sigma(a)|-1}=a'}\ \tau(a)_{|\sigma(a)|-1}^\alpha\right)_{a,a'\in\mathbb{A}} \end{aligned}$$

Let Θ be the set of *closed orbits* of Ω with respect to the action of G. That is, Θ is the family of subsets ξ of Ω such that $\xi = G\omega$ for some $\omega \in \Omega$ with $\lambda\omega = \omega$ for some $\lambda \in G$ with $\lambda > 1$. We call λ as above a *multiplicative cycle* of ξ. The minimum multiplicative cycle of ξ is denoted by $c(\xi)$. Note that $c(\xi)$ exists since $\lambda\omega \ne \omega$ for any $\omega \in \Omega$ and $\lambda \in G$ with $1 < \lambda < \min\{\tau(a)_i^{-1};\ a \in \mathbb{A},\ 0 \le i < |\tau(a)|\}$.

We say that $\xi \in \Theta$ has a *separating line* if $\omega \in \xi$ has a separating line. Note that in this case, the separating line is necessarily the y-axis and is in common among $\omega \in \xi$. Denote by Θ_0 the set of $\xi \in \Theta$ with the separating line.

Let

$$L(\sigma) := \{(a,k,i);\ a \in \mathbb{A},\ \ k = 1,2,\cdots \text{ and } \\ 0 \le i < |\sigma^k(a)| \text{ with } \sigma^k(a)_i = a\}.$$

For (a,k,i) and (a,k',i') in $L(\sigma)$, define the product by

$$(a,k,i)(a,k',i') = (a, k+k',\ \sum_{j=0}^{i-1} |\sigma^{k'}(\sigma^k(a)_j)| + i').$$

We say that $(a,k,i) \in L(\sigma)$ is *irreducible* if $(a,k,i) = (a,k',i')^h$ does not hold for any $h = 2,3,\cdots$ and $(a,k',i') \in L(\sigma)$.

Let $\xi \in \Theta \setminus \Theta_0$ and $\omega \in \xi$. Then, there exists $R = (x_1,x_2) \times (y_1,y_2) \in dom(\omega)$ with $x_1 < 0 < x_2$. Since $c(\xi)\omega = \omega$, there exists a descendant R' of R such that $\omega(R') = \omega(R) =: a$ and $R = c(\xi)R'$. Let R' be the (k,i)-descendant of R.

Lemma 3. *For any $\xi \in \Theta \setminus \Theta_0$ with the above setting, the following statements hold.*
(i) $1 \le i < |\sigma^k(a)| - 1$.
(ii) (a,k,i) is in $L(\sigma)$ and irreducible.
(iii) $c(\xi) = \tau^k(a)_i^{-1}$.
Conversely, any triple $(a,k,i) \in L(\sigma)$ satisfying $(i)(ii)$ determines $\xi \in \Theta \setminus \Theta_0$ and (iii) follows.

Proof. (iii) holds since $c(\xi)R' = R$ and R' is the (k,i)-descendant of R.

Since R' is the (k,i)-descendant of R such that $\tau^k(a)_i^{-1}R' = R$, we have

$$\frac{-x_1}{x_2} = \frac{\sum_{j \le i-1} \tau^k(a)_j}{\sum_{j \ge i+1} \tau^k(a)_j}\ . \tag{12}$$

Since $0 < -x_1/x_2 < \infty$, (i) follows.

If $(a,k,i) = (a,k',i')^\ell$ with $\ell \ge 2$, then (k',i')-descendant R'' of R also satisfies $\tau^{k'}(a)_{i'}^{-1}R'' = R$ and that $\tau^{k'}(a)_{i'}^{-1} = c(\xi)^{1/\ell}$ becomes a cycle of ξ, contradicting the minimality of $c(\xi)$. Thus, (ii) follows.

Let us prove the last statement. Assume that $(a,k,i) \in L(\sigma)$ is irreducible satisfying (i). Take any $\omega' \in \Omega$ and $R \in dom(\omega')$ with

$\omega'(R) = a$. Take $t \in \mathbb{R}$ such that $R + t =: (x_1, x_2) \times (y_1, y_2)$ satisfies the equation (12). Then $x_1 < 0 < x_2$, and $\tau^k(a)_i^{-n}(\omega' + t)$ converges as $n \to \infty$ to, say $\omega \in \Omega$ which satisfies that $\tau^k(a)_i^{-1}\omega = \omega$. Thus, we have $\xi := G\omega \in \Theta \setminus \Theta_0$ which is determined by $(a, k, i) \in L(\sigma)$. (iii) follows by the irreducibility of (a, k, i). $\square$

Let $\xi \in \Theta \setminus \Theta_0$ and $\omega \in \xi$. Let R and R' satisfy that $R = c(\xi)R'$ and that R' is the (k, i)-descendant of R. Let $R_0, R_1, R_2, \cdots$ be the sequence of tiles in ω intersecting with the y-axis such that R_{i+1} is the mother of R_i $(i = 0, 1, 2, \cdots)$ and $R_0 = R'$. Let R_j be the (k, i_j)-descendant of R_{j+k} for any $j = 0, 1, \cdots, k-1$. Then, $\omega(R_{j+k}) = \omega(R_j)$ holds and the triple $(\omega(R_{j+k}), k, i_j)$ $(j = 0, 1, \cdots, k-1)$ satisfies the condition $(i)(ii)$ in Lemma 3 determining ξ in the sense of Lemma 3. In fact, these k triples are different from each other such that they are all that determine ξ, while k is common among them which we denote by $k(\xi)$.

Lemma 4. *For $k = 1, 2, \cdots$, it holds that*

$$\begin{aligned} & tr(M_\alpha^k) - tr(M_{\alpha,+}^k) - tr(M_{\alpha,-}^k) \\ = & \sum_{\substack{\xi \in \Theta \setminus \Theta_0 \\ k(\xi) | k}} k(\xi) c(\xi)^{-\frac{k}{k(\xi)}\alpha}. \end{aligned} \tag{13}$$

Proof. Note that

$$tr(M_\alpha^k) - tr(M_{\alpha,+}^k) - tr(M_{\alpha,-}^k) = \sum_{\substack{a \in \mathbb{A}, 1 \leq i < |\sigma^k(a)| - 1 \\ \sigma^k(a)_i = a}} \tau^k(a)_i^\alpha.$$

For any $\xi \in \Theta \setminus \Theta_0$ with $k(\xi)|k$, there exist exactly $k(\xi)$ number of triples $(a_j, k(\xi), i_j) \in L(\sigma)$ $(j = 0, 1, \cdots, k(\xi) - 1)$ determining ξ so that $\tau^{k(\xi)}(a_j)_{i_j} = c(\xi)^{-1}$ follows. Therefore,

$$\begin{aligned}
\sum_{\substack{a\in\mathbb{A}, 1\le i<|\sigma^k(a)|-1 \\ \sigma^k(a)_i=a}} \tau^k(a)_i^\alpha &= \sum_{\substack{(a,k,i)\in L(\sigma) \\ 1\le i<|\sigma^k(a)|-1}} \tau^k(a)_i^\alpha \\
&= \sum_{k'|k} \sum_{\substack{(a,k',i)\in L(\sigma):irreducible \\ 1\le i<|\sigma^{k'}(a)|-1}} \tau^{k'}(a)_i^{(k/k')\alpha} \\
&= \sum_{k'|k} \sum_{\substack{\xi\in\Theta\setminus\Theta_0 \\ k(\xi)=k'}} k' c(\xi)^{-(k/k')\alpha} \\
&= \sum_{\substack{\xi\in\Theta\setminus\Theta_0 \\ k(\xi)|k}} k(\xi) c(\xi)^{-\frac{k}{k(\xi)}\alpha}.
\end{aligned}$$

□

The following lemma follows from Lemma 1.

Lemma 5. *The set Θ_0 is a finite set. In fact, we have*

$$\sharp\Theta_0 \le \sum_{a\in\mathbb{A}} (|\sigma(a)|-1).$$

Lemma 6. *For $\alpha \in \mathbb{C}$ with $\mathcal{R}(\alpha) > 1$, where $\mathcal{R}(\alpha)$ is the real part of α, we have*

$$\sum_{\xi\in\Theta} |c(\xi)^{-\alpha}| < \infty.$$

Proof. By Lemma 5, it is sufficient to prove that

$$\sum_{\xi\in\Theta\setminus\Theta_0} |c(\xi)^{-\alpha}| < \infty.$$

Since

$$\max_{a\in\mathbb{A}} \sum_{0\le i<|\tau(a)|} \tau(a)_i^{\mathcal{R}(\alpha)} < 1,$$

there exists δ with $0 < \delta < 1$ such that the maximal eigen-value of $M_{\mathcal{R}(\alpha)}$ is less than δ. Hence, by (13) we have

$$\begin{aligned} \sum_{\xi\in\Theta\setminus\Theta_0} |c(\xi)^{-\alpha}| &= \sum_{\xi\in\Theta\setminus\Theta_0} c(\xi)^{-\mathcal{R}(\alpha)} \\ &\leq \sum_{k=1}^{\infty} \sum_{\substack{\xi\in\Theta\setminus\Theta_0 \\ k(\xi)|k}} k(\xi) c(\xi)^{-\frac{k}{k(\xi)}\mathcal{R}(\alpha)} \\ &\leq \sum_{k=1}^{\infty} tr(M^k_{\mathcal{R}(\alpha)}) \leq \sum_{k=1}^{\infty} C\delta^k < \infty. \end{aligned}$$

□

Define the ζ-function of G-action to Ω by

$$\zeta_\Omega(\alpha) := \prod_{\xi\in\Theta}(1 - c(\xi)^{-\alpha})^{-1}, \tag{14}$$

where the infinite product converges for any $\alpha \in \mathbb{C}$ with $\mathcal{R}(\alpha) > 1$ by Lemma 6. It is extended to the whole complex plane by the analytic extension.

Theorem 4. *We have*

$$\zeta_\Omega(\alpha) = \frac{\det(I - M_{\alpha,+})\det(I - M_{\alpha,-})}{\det(I - M_\alpha)}\ \zeta_{\Sigma_0}(\alpha),$$

where

$$\zeta_{\Sigma_0}(\alpha) := \prod_{\xi\in\Theta_0}(1 - c(\xi)^{-\alpha})^{-1}$$

is a finite product with respect to $\xi \in \Theta_0$.

Proof. By the definition of $\zeta_\Omega(\alpha)$ and (13), for any α with $\mathcal{R}(\alpha) > 1$, it holds that

$$\begin{aligned}
\zeta_\Omega(\alpha) &= \zeta_{\Sigma_0}(\alpha)\exp\left(\sum_{\xi\in\Theta\setminus\Theta_0} -\log(1-c(\xi)^{-\alpha})\right) \\
&= \zeta_{\Sigma_0}(\alpha)\exp\left(\sum_{\xi\in\Theta\setminus\Theta_0}\sum_{k=1}^{\infty}(1/k)c(\xi)^{-k\alpha}\right) \\
&= \zeta_{\Sigma_0}(\alpha)\exp\left(\sum_{k=1}^{\infty}\sum_{\substack{\xi\in\Theta\setminus\Theta_0\\ k(\xi)|k}}(k(\xi)/k)c(\xi)^{-\frac{k}{k(\xi)}\alpha}\right) \\
&= \zeta_{\Sigma_0}(\alpha)\exp\left(\sum_{k=1}^{\infty}(1/k)(tr(M_\alpha^k)-tr(M_{\alpha,+}^k)-tr(M_{\alpha,-}^k))\right) \\
&= \zeta_{\Sigma_0}(\alpha)\exp\left(tr(\sum_{k=1}^{\infty}(1/k)(M_\alpha^k-M_{\alpha,+}^k-M_{\alpha,-}^k))\right) \\
&= \zeta_{\Sigma_0}(\alpha)\exp\left(tr(-\log(I-M_\alpha)+\log(I-M_{\alpha,+})\right. \\
&\qquad\qquad\qquad\qquad\left.+\log(I-M_{\alpha,-}))\right) \\
&= \frac{\det(I-M_{\alpha,+})\det(I-M_{\alpha,-})}{\det(I-M_\alpha)}\,\zeta_{\Sigma_0}(\alpha)
\end{aligned}$$

□

Example 3. For the 2-adic expansion substitution (σ,τ) defined in Remark 1, $1\to(1,1/2)(1,1/2)$, define $\Omega:=\Omega(\sigma,\tau)$ with $G=\{2^n; n\in\mathbb{Z}\}$, $g\equiv 1$ (see (5)). Then, we have

$$M_\alpha = 2(1/2)^\alpha\ ,\quad M_{\alpha,+}=M_{\alpha,-}=(1/2)^\alpha.$$

Moreover, we have $\Theta_0=\{G\omega_0\}$ with ω_0 shown in Figure 6. Since $c(G\omega_0)=2$ and $\zeta_{\Sigma_0}(\alpha)=(1-2^{-\alpha})^{-1}$, we have

$$\begin{aligned}
\zeta_\Omega(\alpha) &= \zeta_{\Sigma_0}(\alpha)\frac{(1-(1/2)^\alpha)^2}{1-2(1/2)^\alpha} \\
&= \frac{1-(1/2)^\alpha}{1-2(1/2)^\alpha}
\end{aligned}$$

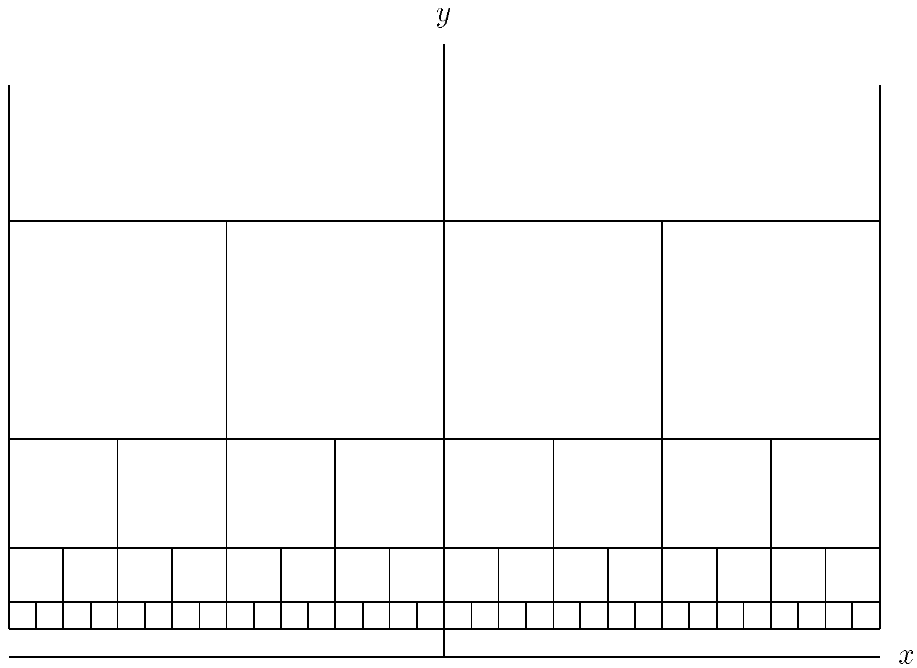

Figure 6: ω_0 in the 2-adic expansion

Example 4. Consider the weighted substitution (σ, τ) in Example 2, that is $1 \to (1, p)(1, 1-p)$. Then, we have

$$M_\alpha = p^\alpha + (1-p)^\alpha \ , \ M_{\alpha,+} = p^\alpha \ , \ M_{\alpha,-} = (1-p)^\alpha.$$

Since $\Theta_0 = \emptyset$, we have

$$\zeta_\Omega(\alpha) = \frac{(1-p^\alpha)(1-(1-p)^\alpha)}{1-p^\alpha-(1-p)^\alpha}.$$

Example 5. For the Thue-Morse substitution (σ, τ) defined in Remark 1, $1 \to (1, 1/2)(2, 1/2)$, $2 \to (2, 1/2)(1, 1/2)$, define $\Omega :=$

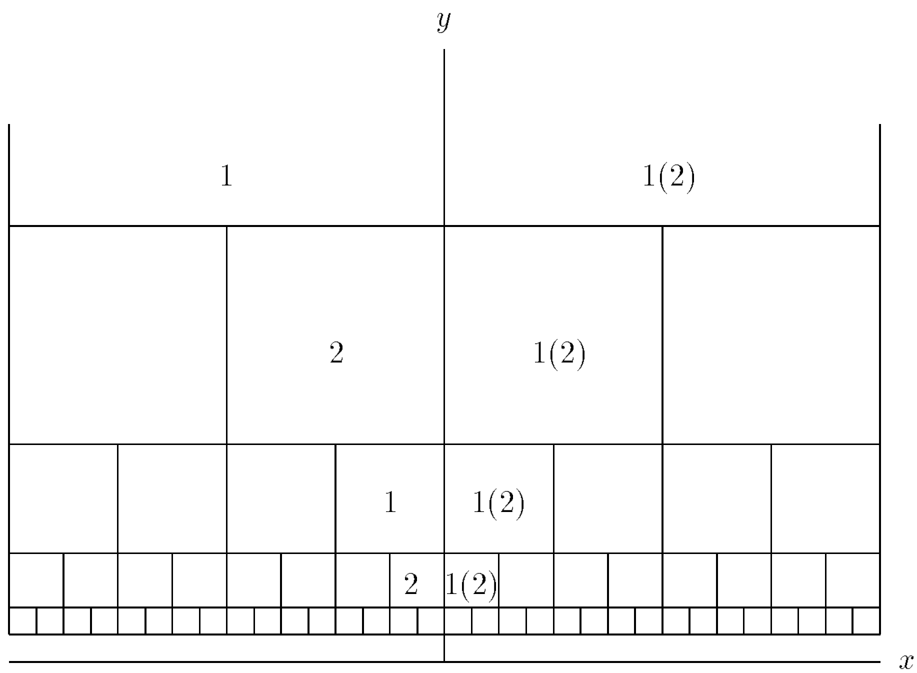

Figure 7: $\omega_1(\omega_2$, respectively) in the Thue-Morse substitution

$\Omega(\sigma,\tau)$ with $G=\{2^n; n\in\mathbb{Z}\}$, $g\equiv 1$. Then, we have

$$\begin{aligned} M_\alpha &= \begin{pmatrix} (1/2)^\alpha & (1/2)^\alpha \\ (1/2)^\alpha & (1/2)^\alpha \end{pmatrix} \\ M_{\alpha,+} &= \begin{pmatrix} (1/2)^\alpha & 0 \\ 0 & (1/2)^\alpha \end{pmatrix} \\ M_{\alpha,-} &= \begin{pmatrix} 0 & (1/2)^\alpha \\ (1/2)^\alpha & 0 \end{pmatrix} \end{aligned}$$

Moreover, we have $\Theta_0=\{G\omega_1, G\omega_2\}$ with ω_1 and ω_2 shown in Figure 7. Since $c(G\omega_1)=c(G\omega_2)=4$ and $\zeta_{\Sigma_0}(\alpha)=(1-4^{-\alpha})^{-2}$, we have

$$\begin{aligned} \zeta_\Omega(\alpha) &= \zeta_{\Sigma_0}(\alpha)\frac{(1-(1/2)^\alpha)^2(1-(1/2)^{2\alpha})}{1-2(1/2)^\alpha} \\ &= \frac{1-(1/2)^\alpha}{(1+(1/2)^\alpha)(1-2(1/2)^\alpha)} \end{aligned}$$

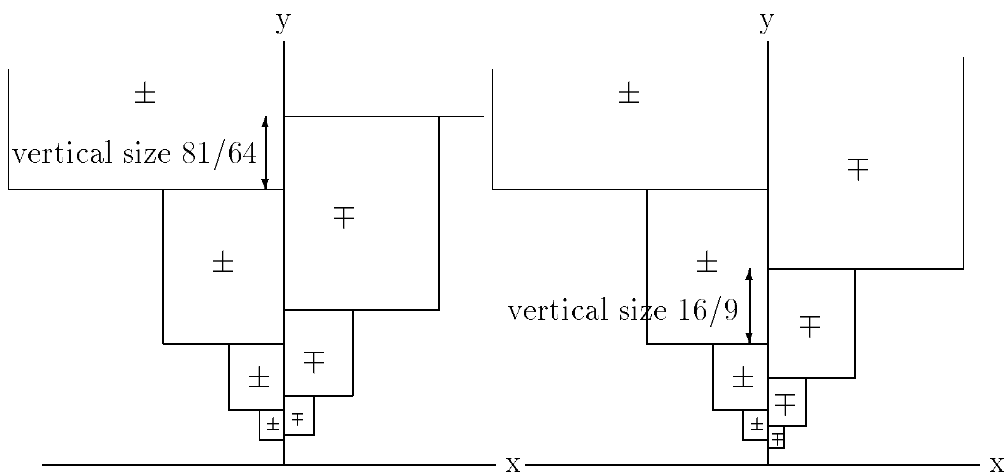

Figure 8: 4 elements in Θ_0 in Example 6 ($\pm$, respectively)

Example 6. For the weighted substitution (σ, τ) in Example 1

$$\begin{aligned} + &\rightarrow (+, 4/9)(-, 1/9)(+, 4/9) \\ - &\rightarrow (-, 4/9)(+, 1/9)(-, 4/9), \end{aligned}$$

define $\Omega := \Omega(\sigma, \tau)$, where since $B(\sigma, \tau) = \mathbb{R}$ holds, taking $g \equiv 1$, we have

$$\begin{aligned} M_\alpha &= \begin{pmatrix} 2(4/9)^\alpha & (1/9)^\alpha \\ (1/9)^\alpha & 2(4/9)^\alpha \end{pmatrix} \\ M_{\alpha,+} &= M_{\alpha,-} = \begin{pmatrix} (4/9)^\alpha & 0 \\ 0 & (4/9)^\alpha \end{pmatrix} \end{aligned}$$

There are 4 elements in Θ_0 determined as in (iii) of Lemma 1 by the pairs $(+,\ 0), (+,\ 1), (-,\ 0), (-,\ 1)$. All of them are different as shown in Figure 8. The $\pm$ corresponds to the $\pm$ in the first coordinate, while 0, 1 corresponds to the vertical gap from the left side tiles to the right side tiles. It is 81/64 for 0 (left side in Figure 8) and 16/9 for 1 (right side in Figure 8). These 4 elements have the same multiplicative cycle 9/4 . Hence, we have

$$\zeta_\Omega(\alpha) = \frac{1}{(1-2(4/9)^\alpha - (1/9)^\alpha)(1-2(4/9)^\alpha + (1/9)^\alpha)} .$$

Theorem 5.
(i) $\zeta_\Omega(\alpha) \neq 0$ *if* $\mathcal{R}(\alpha) \neq 0$.
(ii) In the region $\mathcal{R}(\alpha) \neq 0$, α *is a pole of* $\zeta_\Omega(\alpha)$ *with multiplicity* k *if and only if it is a zero of* $\det(I - M_\alpha)$ *with multiplicity* k *for any* $k = 1, 2, \cdots$.
(iii) 1 *is a simple pole of* $\zeta_\Omega(\alpha)$.

Proof. Since $c(\xi) > 1$ for any $\xi \in \Theta$, $1 - c(\xi)^{-\alpha} \neq 0$ if $\mathcal{R}(\alpha) \neq 0$. Hence, $\zeta_{\Sigma_0}(\alpha)$ has neither pole nor zero in the region $\mathcal{R}(\alpha) \neq 0$.

For α with $\mathcal{R}(\alpha) \neq 0$, suppose that $\det(I - M_{\alpha,+}) = 0$, so that $M_{\alpha,+}\xi = \xi$ holds for some nonzero vector $\xi = (\xi_a)_{a\in\mathbb{A}}$. Take $a_0 \in \mathbb{A}$ such that $\xi_{a_0} \neq 0$. Then, for some $k \geq 0$ and $\ell \geq 1$, $\sigma^k(a_0)_0 = \sigma^{k+\ell}(a_0)_0 =: a_1$ holds. Since $\xi_{a_1} = \tau^k(a_0)_0^\alpha \xi_{a_0} \neq 0$ and $\tau^\ell(a_1)_0^\alpha \xi_{a_1} = \xi_{a_1}$, we have $\tau^\ell(a_1)_0^\alpha = 1$. This is impossible since $0 < \tau^\ell(a_1)_0 < 1$.

Thus, $\det(I - M_{\alpha,+})$ has no zero in the region $\mathcal{R}(\alpha) \neq 0$. In the same way, $\det(I - M_{\alpha,-})$ has no zero in the region $\mathcal{R}(\alpha) \neq 0$. These facts with Theorem 4 prove $(i)(ii)$ of Theorem 5.

(iii) Since $M_1{}^t(1, \cdots, 1) = {}^t(1, \cdots, 1)$, 1 is an eigen-value of M_1, hence is a zero of $\det(I - M_\alpha)$. We prove that it is a simple zero. Let $\mathbb{A} = \{a_1, \cdots, a_r\}$ and $\mathbb{A}' := \mathbb{A} \setminus \{a_1\}$. For a matrix $M = (m_{ij})_{i\in I, j\in J}$ and $I' \subset I, J' \subset J$, let $M[I', J'] := (m_{i,j})_{i\in I', j\in J'}$. Since 1 is the maximum eigen-value of M_1 and σ is primitive, there exists a positive row vector $(\xi_1, \xi_2, \cdots, \xi_r)$ with $\xi_1 = 1$ such that $(\xi_1, \xi_2, \cdots, \xi_r)(I - M_1) = (0, \cdots, 0)$. Therefore, since

$$\det(I - M_\alpha) = \det \begin{pmatrix} 1 - \sum_{i=0}^{|\tau(a_1)|-1} \tau(a_1)_i^\alpha & \\ \vdots & (I - M_\alpha)[\mathbb{A}, \mathbb{A}'] \\ 1 - \sum_{i=0}^{|\tau(a_r)|-1} \tau(a_r)_i^\alpha & \end{pmatrix},$$

we have

$$
\begin{aligned}
&\frac{d}{d\alpha}\det(I-M_\alpha)|_{\alpha=1} \\
&= \det\begin{pmatrix} \sum_i -\tau(a_1)_i \log\tau(a_1)_i & \\ \vdots & (I-M_1)[\mathbb{A},\mathbb{A}'] \\ \sum_i -\tau(a_r)_i \log\tau(a_r)_i & \end{pmatrix} \\
&= \det\begin{pmatrix} \sum_{i,j} -\xi_j\tau(a_j)_i \log\tau(a_j)_i & 0 \quad \cdots \quad 0 \\ \sum_i -\tau(a_2)_i \log\tau(a_2)_i & \\ \vdots & (I-M_1)[\mathbb{A}',\mathbb{A}'] \\ \sum_i -\tau(a_r)_i \log\tau(a_r)_i & \end{pmatrix} \\
&= \left(\sum_{i,j} -\xi_j\tau(a_j)_i \log\tau(a_j)_i\right)\det((I-M_1)[\mathbb{A}',\mathbb{A}']).
\end{aligned}
$$

We have $\sum_{i,j} -\xi_j\tau(a_j)_i \log\tau(a_j)_i > 0$ and $\det((I-M_1)[\mathbb{A}',\mathbb{A}']) \neq 0$ since the spectral radius of $M_1[\mathbb{A}',\mathbb{A}']$ is strictly less than 1. Hence, $\frac{d}{d\alpha}\det(I-M_\alpha)|_{\alpha=1} \neq 0$ and 1 is a simple zero of $\det(I-M_\alpha)$. By (ii), it is a simple pole of ζ_Ω. □

Theorem 6. *For $\Omega=\Omega(\sigma,\eta,g)$, if $B(\sigma,\tau)=\{\lambda^n;\ n\in\mathbb{Z}\}$ with $\lambda>1$, then there exist polynomials $p,q\in\mathbb{Z}[z]$ such that $\zeta_\Omega(\alpha)=p(\lambda^\alpha)/q(\lambda^\alpha)$. Conversely, if $\zeta_\Omega(\alpha)=p(\lambda^\alpha)/q(\lambda^\alpha)$ holds for some polynomials $p,q\in\mathbb{Z}[z]$ and $\lambda>1$, then $B(\sigma,\tau)=\{\lambda^{kn};\ n\in\mathbb{Z}\}$ for some positive integer k.*

Proof. Assume that $B(\sigma,\eta)=\{\lambda^n;\ n\in\mathbb{Z}\}$ with $\lambda>1$. Let g satisfies (5). Then for any $a\in\mathbb{A}$ and i with $0\leq i<|\sigma(a)|$, there exists $r(a)_i\in\mathbb{Z}$ such that $\tau(a)_i=\frac{g(\sigma(a)_i)}{g(a)}\lambda^{r(a)_i}$. Hence, we have

$$
\begin{aligned}
M_\alpha &= \Lambda_\alpha^{-1}\left(\sum_{\substack{0\leq i<|\sigma(a)|\\ \sigma(a)_i=a'}}(\lambda^\alpha)^{r(a)_i}\right)_{a,a'\in\mathbb{A}}\Lambda_\alpha \\
M_{\alpha,+} &= \Lambda_\alpha^{-1}\left(1_{\sigma(a)_0=a'}(\lambda^\alpha)^{r(a)_0}\right)_{a,a'\in\mathbb{A}}\Lambda_\alpha \\
M_{\alpha,-} &= \Lambda_\alpha^{-1}\left(1_{\sigma(a)_{|\sigma(a)|-1}=a'}(\lambda^\alpha)^{r(a)_{|\sigma(a)|-1}}\right)_{a,a'\in\mathbb{A}}\Lambda_\alpha,
\end{aligned}
$$

where $\Lambda_\alpha = (g(a)^\alpha 1_{a'=a})_{a,a'\in\mathbb{A}}$ is a diagonal matrix. Therefore, $\det(I - M_\alpha)$, $\det(I - M_{\alpha,+})$ and $\det(I - M_{\alpha,-})$ are polynomials in λ^α divided possibly by $(\lambda^\alpha)^n$ for some positive integer n. Since $c(\xi) = \lambda^n$ for some positive integer n for any $\xi \in \Theta_0$ and Θ_0 is a finite set, $\zeta_{\Sigma_0}(\alpha)^{-1}$ is a polynomial in $\lambda^{-\alpha}$. Thus, $\zeta_\Omega(\alpha) = p(\lambda^\alpha)/q(\lambda^\alpha)$ for some polynomials $p, q \in \mathbb{Z}[z]$.

Conversely, assume that $\zeta_\Omega(\alpha) = p(\lambda^\alpha)/q(\lambda^\alpha)$ for some polynomials $p, q \in \mathbb{Z}[z]$. Then we have

$$\Pi_{\xi\in\Theta}(1 - c(\xi)^{-\alpha}) = q(\lambda^\alpha)/p(\lambda^\alpha)$$

on $\mathcal{R}(\alpha) > 1$. Comparing their expansions as Dirichlet series in α, it holds that $c(\xi) \in \{\lambda^n;\ n \in \mathbb{Z}\}$ for any $\xi \in \Theta$ ([6]). Since $B(\sigma,\tau)$ is generated by $\{c(\xi);\ \xi \in \Theta\}$, we have $B(\sigma,\tau) \subset \{\lambda^n;\ n \in \mathbb{Z}\}$. Thus, $B(\sigma,\tau) = \{\lambda^{kn};\ n \in \mathbb{Z}\}$ for some positive integer k. □

Theorem 7. *If $B(\sigma,\tau) = \{\lambda^n;\ n \in \mathbb{Z}\}$, then λ is an algebraic number.*

Proof. Let $\Omega := \Omega(\sigma,\tau,g)$. By Theorem 6, there exist polynomials $p, q \in \mathbb{Z}[z]$ such that $\zeta_\Omega(\alpha) = p(\lambda^\alpha)/q(\lambda^\alpha)$. By Theorem 5, 1 is a pole of $\zeta_\Omega(\alpha)$. Hence $q(\lambda) = 0$, which implies that λ is algebraic. □

5 β-expansion system

Let β be an algebraic integer with $\beta > 1$ such that 1 has the following periodic β-expansion

$$\begin{aligned} &1 = (b_1 0^{i_1-1} b_2 0^{i_2-1} \cdots b_k 0^{i_k-1})^\infty \\ &b_1, b_2, \cdots, b_k \in \{1, 2, \cdots, \lfloor\beta\rfloor\} \\ &i_1, i_2, \cdots, i_k \in \{1, 2, \cdots\}, \end{aligned}$$

where $(\quad)^\infty$ implies the infinite time repetition of $(\quad)$. Let $n := i_1 + i_2 + \cdots + i_k \geq 1$ and assume that n is the minimum period of the above sequence. Since the above sequence is the expansion of 1, we have the solution of the following equation in $a_1, a_2, \cdots, a_{k+1}$ with $a_1 = a_{k+1} = 1$ and $0 < a_j < 1$ $(j = 2, \cdots, k)$:

$$a_j = b_j \beta^{-1} + a_{j+1}\beta^{-i_j}\ (j = 1, 2, \cdots, k).$$

Let $\mathbb{A} := \{1, 2, \cdots, k\}$ and define a weighted substitution (σ, τ) by

$$\begin{gathered} j \to (1,\ (1/a_j)\beta^{-1})^{b_j}\ (j+1,\ (a_{j+1}/a_j)\beta^{-i_j}) \\ (j = 1, 2, \cdots, k-1) \\ k \to (1,\ (1/a_k)\beta^{-1})^{b_k}\ (1,\ (a_{k+1}/a_k)\beta^{-i_k}) \end{gathered}$$

where $(\ ,\ )^k$ implies the k-time repetition of $(\ ,\ )$. Then, σ is primitive and $B(\sigma, \tau) = \{\beta^n;\ n \in \mathbb{Z}\}$. Define $g : \mathbb{A} \to \mathbb{R}_+$ by $g(j) := a_j$. Then, g satisfies (5) and $\Omega(\sigma, \tau, g)$ is a numeration system by Theorem 2. We denote $\Omega(\beta) := \Omega(\sigma, \tau, g)$ and $\Omega(\beta)$ is called the *β-expansion system.*

The β-expansion system is studied by many authors, for example, S. Ito and Y. Takahashi ([7]) where the ζ-function is obtained. Here, we give the formula again as a corollary of Theorem 4. There is a little difference between them. In [7], the ζ-function is for the restriction $\{\omega^+;\ \omega \in \Omega\}$ to the right-quarter space, while ours is for Ω itself, so that ours is the product of the former and $\frac{1-\beta^{\alpha}}{1-\beta^{-n\alpha}}$.

Theorem 8. *We have*

$$\zeta_{\Omega(\beta)}(\alpha) = \frac{1 - \beta^{-\alpha}}{1 - \sum_{j=1}^{k} b_j \beta^{-(i_1 + \cdots + i_{j-1} + 1)\alpha} - \beta^{-n\alpha}}.$$

Proof. We have

$$\det(I - M_\alpha) =$$
$$\begin{vmatrix} 1 - b_1(\frac{1}{a_1\beta})^\alpha & -(\frac{a_2}{a_1\beta^{i_1}})^\alpha & 0 & 0 & \cdots & 0 & 0 \\ -b_2(\frac{1}{a_2\beta})^\alpha & 1 & \ddots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ -b_{k-1}(\frac{1}{a_{k-1}\beta})^\alpha & 0 & 0 & 0 & \cdots & 1 & -(\frac{a_k}{a_{k-1}\beta^{i_{k-1}}})^\alpha \\ -b_k(\frac{1}{a_k\beta})^\alpha - (\frac{a_{k+1}}{a_k\beta^{i_k}})^\alpha & 0 & 0 & 0 & \cdots & 0 & 1 \end{vmatrix}$$
$$= 1 - \sum_{j=1}^{k} b_j \beta^{-(i_1 + \cdots + i_{j-1} + 1)\alpha} - \beta^{-n\alpha},$$

$$\det(I - M_{\alpha,+}) = \begin{vmatrix} 1-(\frac{1}{a_1\beta})^\alpha & 0 & \cdots & \cdots & 0 \\ -(\frac{1}{a_2\beta})^\alpha & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -(\frac{1}{a_{k-1}\beta})^\alpha & 0 & \cdots & 1 & 0 \\ -(\frac{1}{a_k\beta})^\alpha & 0 & \cdots & 0 & 1 \end{vmatrix} = 1 - \beta^{-\alpha}$$

and

$$\det(I - M_{\alpha,-}) = \begin{vmatrix} 1 & -(\frac{a_2}{a_1\beta^{i_1}})^\alpha & 0 & \cdots & 0 \\ 0 & 1 & -(\frac{a_3}{a_2\beta})^\alpha & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -(\frac{a_k}{a_{k-1}\beta^{i_{k-1}}})^\alpha \\ -(\frac{a_{k+1}}{a_k\beta^{i_k}})^\alpha & 0 & \cdots & 0 & 1 \end{vmatrix} = 1 - \beta^{-n\alpha}.$$

Let $\xi \in \Theta_0$ and $\omega \in \xi$. Let $R_i^\pm$ $(i \in \mathbb{Z})$ be the sequence of tiles in $dom(\omega)$ from up to down intersecting with the line $y = \pm 0$ and $(0, \pm 0) \in R_0^\pm$ ($\pm$ respectively). Then, we have

$$\begin{aligned} \omega(R_{-1}^+)\omega(R_0^+)\omega(R_1^+)\omega(R_2^+) &= \cdots\cdots 11111 \cdots\cdots \\ \cdots\omega(R_{-1}^-)\omega(R_0^-)\omega(R_1^-)\omega(R_2^-)\cdots &= \cdots 12 \cdots k12 \cdots k \cdots . \end{aligned}$$

Hence, the minimum multiplicative cycle of ω on the right half plane is β, while on the left half plane is β^n. Thus, $c(\xi) = \beta^n$. Moreover, the upper half tiling and the lower half tiling are synchronized so that the horizontal position of any tile R_i^- with $\omega(R_i^-) = 1$ coincides with that of R_j^+ for some j. Therefore, ξ is the unique element in Θ_0. Hence, $\zeta_{\Sigma_0}(\alpha) = 1 - \beta^{-n\alpha}$.

Combining these results using Theorem 4, we have

$$\zeta_{\Omega(\beta)}(\alpha) = \frac{1 - \beta^{-\alpha}}{1 - \sum_{j=1}^k b_j \beta^{-(i_1 + \cdots + i_{j-1} + 1)\alpha} - \beta^{-n\alpha}}.$$

□

Example 7. Let $\beta = (1+\sqrt{5})/2$ be the golden number. Then, the expansion of 1 is $(10)^\infty$. Therefore, $\mathbb{A} = \{1\}$ and (σ, τ) is

$$1 \to (1, \beta^{-1})(1, \beta^{-2}).$$

By Theorem 8, we have

$$\zeta_{\Omega(\beta)}(\alpha) = \frac{1-\beta^{-\alpha}}{1-\beta^{-\alpha}-\beta^{-2\alpha}}.$$

Example 8. Let us consider the β-expansion system with $\beta > 1$ such that $\beta^3 - \beta^2 - \beta - 1 = 0$. Then the expansion of 1 is $(110)^\infty$ and the corresponding weighted substitution is

$$\begin{array}{rcl} 1 & \to & (1,\ \beta^{-1})(2,\ \beta^{-2}+\beta^{-3}) \\ 2 & \to & (1,\ \dfrac{\beta}{\beta+1})(1,\ \dfrac{1}{\beta+1}) \end{array}$$

By Theorem 8, we have

$$\zeta_{\Omega(\beta)}(\alpha) = \frac{1-\beta^{-\alpha}}{1-\beta^{-\alpha}-\beta^{-2\alpha}-\beta^{-3\alpha}}.$$

We will discuss this example in the next section.

6 homogeneous cocycles and fractals

Let $\Omega := \Omega(\sigma, \tau, g)$ satisfy (4) and (5).

A continuous function $F : \Omega \times \mathbb{R} \to \mathbb{C}$ is called a *cocycle* on Ω if

$$F(\omega, t+s) = F(\omega, t) + F(\omega + t, s) \tag{15}$$

holds for any $\omega \in \Omega$ and $s, t \in \mathbb{R}$. A cocycle F on Ω is called α-*homogeneous* if

$$F(\lambda\omega, \lambda t) = \lambda^\alpha F(\omega, t)$$

for any $\omega \in \Omega$, $\lambda \in G$ and $t \in \mathbb{R}$, where α is a given complex number. A cocycle $F(\omega, t)$ on Ω is called *adapted* if there exists a function $\Xi : \mathbb{A} \times \mathbb{R}_+ \to \mathbb{C}$ such that

$$F(\omega, x_2) - F(\omega, x_1) = \Xi(\omega(R), x_2 - x_1) \tag{16}$$

for any $\omega \in \Omega$ and tile $R := (x_1, x_2) \times (y_1, y_2) \in dom(\omega)$.

In [8], nonzero adapted α-homogeneous cocycles on Ω with $0 < \alpha < 1$ is characterized. In fact, we have

Theorem 9. *A nonzero adapted α-homogeneous cocycle on Ω is characterized by* (16) *with α and Ξ satisfying that $\mathcal{R}(\alpha) > 0$ and there exists a nonzero vector $\xi = (\xi_a)_{a \in \mathbb{A}}$ such that $M_\alpha \xi = \xi$ (see* (11)*) and $\Xi(\omega(R), x_2 - x_1) = (x_2 - x_1)^\alpha \xi_{\omega(R)}$ for any tile $R := (x_1, x_2) \times (y_1, y_2) \in dom(\omega)$. Hence, a nonzero adapted α-homogeneous cocycle exists if and only if $\mathcal{R}(\alpha) > 0$ and α is a pole of $\zeta_\Omega(\alpha)$.*

Proof. The last part of the theorem follows from Theorem 5. The condition that $\mathcal{R}(\alpha) > 0$ is necessary for the convergence of $F(\omega, t)$ with (16) for a general $t \in \mathbb{R}$. □

It is known [8] that

Theorem 10. *Let μ be the unique invariant probability measure on Ω under the additive action. Let $0 < \alpha < 1$. For a nonzero α-homogeneous cocycle F on Ω, we have the following results.*

(i) There exists a constant C such that

$$|F(\omega, t) - F(\omega, s)| \le C|t - s|^\alpha$$

for any $\omega \in \Omega$ and $s, t \in \mathbb{R}$. That is, the functions $F(\omega, t)$ on t for $\omega \in \Omega$ are uniformly α-Hölder continuous.

(ii) For any $\omega \in \Omega$ and $t \in \mathbb{R}$,

$$\limsup_{s \downarrow 0} \frac{1}{s^\alpha} |F(\omega, t + s) - F(\omega, t)| > 0$$

holds. That is, for any $\omega \in \Omega$ the function $F(\omega, \cdot)$ is nowhere locally α'-Hölder continuous for any $\alpha' > \alpha$. In particular, $F(\omega, \cdot)$ is nowhere differentiable.

(iii) The stochastic process $F(\omega, t)$ with time parameter $t \in \mathbb{R}$ and random element $\omega \in \Omega$ with respect to μ has a strictly ergodic stationary increment having 0-entropy.

(iv) $F(\omega, \lambda t)$ has the same law as $\lambda^\alpha F(\omega, t)$ for any $\lambda \in G$. Hence, the process $F(\omega, t)$ is α-self similar if $G = \mathbb{R}_+$.

(v) $\int F(\omega, t) d\mu(\omega) = 0$ for any $t \in \mathbb{R}$.

Example 9. Take Ω in Example 1 and 6. Since

$$M_{1/2} = \begin{pmatrix} 4/3 & 1/3 \\ 1/3 & 4/3 \end{pmatrix},$$

$\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $M_{1/2}$ with eigen-value 1. Let F be the 1/2-homogeneous adapted cocycle on Ω defined by the equation:

$$F(\omega, x_2) - F(\omega, x_1) = \pm(x_2 - x_1)^{1/2}$$

if there exists a tile $(x_1, x_2) \times (y_1, y_2)$ in ω with color $\pm$, respectively (see Theorem 9).

Then, $F(\omega, t)$ is a 1/2-selfsimilar process with respect to the unique invariant measure μ under the additive action, called N-process, which is discussed in the next section.

Consider the family of functions $(F(\omega+n, 1))_{n=0,1,2,\cdots}$ on the probability space (Ω, μ). Since

$$\begin{aligned} & nE[F(\omega,1)^2] = E[(n^{1/2}F(\omega,1))^2] = E[F(n\omega,n)^2] = E[F(\omega,n)^2] \\ = \ & E[F(\omega,1)^2] + E[(F(\omega,n) - F(\omega,1))^2] \\ & \qquad +2E[F(\omega,1)(F(\omega,n) - F(\omega,1))] \\ = \ & E[F(\omega,1)^2] + E[F(\omega+1,n-1)^2] \\ & \qquad +2E[F(\omega,1)(F(\omega,n) - F(\omega,1))] \\ = \ & E[F(\omega,1)^2] + (n-1)E[F(\omega,1)^2] \\ & \qquad +2E[F(\omega,1)(F(\omega,n) - F(\omega,1))], \end{aligned}$$

we have

$$E[F(\omega,1)(F(\omega,n) - F(\omega,1))] = 0$$

for any $n = 1, 2, \cdots$. Therefore,

$$\begin{aligned} & E[F(\omega,1)(F(\omega+n,1)] \\ = \ & E[F(\omega,1)(F(\omega,n+1) - F(\omega,n))] \\ = \ & E[F(\omega,1)(F(\omega,n+1) - F(\omega,1))] \\ & \qquad -E[F(\omega,1)(F(\omega,n) - F(\omega,1))] \\ = \ & 0 \end{aligned}$$

for any $n = 1, 2, \cdots$. Hence, for any $n < m$,

$$E[F(\omega + n, 1)(F(\omega + m, 1)] = E[F(\omega, 1)(F(\omega + m - n, 1)] = 0.$$

This implies that the family of functions $(F(\omega + n, 1))_{n=0,1,\cdots}$ is non-correlated.

Let $\mathcal{I}(\Omega)$ be the set of $\omega \in \Omega$ such that there exists $(x_1, x_2) \times (y_1, y_2) \in dom(\omega)$ satisfying that $x_1 = 0$ and $y_1 \leq 1 < y_2$. An element $\omega \in \mathcal{I}(\Omega)$ is called an *integer* in Ω. Let

$$\mathcal{II}(\Omega) := \{(\omega, t) \in \mathcal{I}(\Omega) \times \mathbb{R};\ \omega + t \in \mathcal{I}(\Omega)\}.$$

A continuous function $F : \mathcal{II}(\Omega) \to \mathbb{C}$ is called a cocycle on $\mathcal{I}(\Omega)$ if (15) is satisfied for any $\omega \in \mathcal{I}(\Omega)$ and $t, s \in \mathbb{R}$ such that $(\omega, t) \in \mathcal{II}(\Omega)$ and $(\omega, t + s) \in \mathcal{II}(\Omega)$.

A cocycle F on $\mathcal{I}(\Omega)$ is called adapted if there exists a function $\Xi : \mathbb{A} \times \mathbb{R}_+ \to \mathbb{C}$ such that (16) is satisfied for any $\omega \in \mathcal{I}(\Omega)$ and tile $(x_1, x_2) \times (y_1, y_2) \in dom(\omega)$ with $y_2 > 1$. Let $\alpha \in \mathbb{C}$. A cocycle F on $\mathcal{I}(\Omega)$ is called α-homogeneous if

$$F(\lambda\omega, \lambda t) = \lambda^\alpha F(\omega, t)$$

for any $(\omega, t) \in \mathcal{II}(\Omega)$ and $\lambda \in G$ with $(\lambda\omega, \lambda t) \in \mathcal{II}(\Omega)$. Note that if $(\omega, t) \in \mathcal{II}(\Omega)$, then for any $\lambda \in G$ with $\lambda > 1$, $(\lambda\omega, \lambda t) \in \mathcal{II}(\Omega)$ holds.

A cocycle F on $\mathcal{I}(\Omega)$ is called a *coboundary* on $\mathcal{I}(\Omega)$ if there exists a continuous function $G : \mathcal{I}(\Omega) \to \mathbb{R}^k$ such that

$$F(\omega, t) = G(\omega + t) - G(\omega)$$

for any $(\omega, t) \in \mathcal{II}(\Omega)$.

The following theorem is proved in [10].

Theorem 11. *A nonzero adapted α-homogeneous cocycle on $\mathcal{I}(\Omega)$ with $\mathcal{R}(\alpha) < 0$ is characterized by* (16) *with Ξ satisfying that there exists a nonzero vector $\xi = (\xi_a)_{a \in \mathbb{A}}$ such that $M_\alpha \xi = \xi$ (see* (11)*) and $\Xi(\omega(R), x_2 - x_1) = (x_2 - x_1)^\alpha \xi_{\omega(R)}$ for any tile $R := (x_1, x_2) \times (y_1, y_2) \in dom(\omega)$ with $y_2 > 1$. Hence, a nonzero adapted α-homogeneous cocycle on $\mathcal{I}(\Omega)$ with $\mathcal{R}(\alpha) < 0$ exists if and only if α is a pole of $\zeta_\Omega(\alpha)$. Moreover, any cocycle as this is a coboundary.*

Example 10. Let us consider the β-expansion system in Example 8. Denote $\Omega := \Omega(\beta)$. The associated matrix is

$$M_\alpha = \begin{pmatrix} \beta^{-\alpha} & (\beta^{-2}+\beta^{-3})^\alpha \\ \frac{\beta^\alpha+1}{(\beta+1)^\alpha} & 0 \end{pmatrix}$$

Let γ be one of the complex solutions of the equation $z^3 - z^2 - z - 1 = 0$. Then, $|\gamma| < 1$. Let $\alpha \in \mathbb{C}$ be such that $\gamma = \beta^\alpha$. Then, $\mathcal{R}(\alpha) < 0$. Since we have

$$M_\alpha \begin{pmatrix} 1 \\ \delta \end{pmatrix} = \begin{pmatrix} 1 \\ \delta \end{pmatrix}$$

with $\delta := \dfrac{\beta^\alpha + 1}{(\beta+1)^\alpha}$, there exists an α-homogeneous adapted cocycle F on $\mathcal{I}(\Omega)$ satisfying that

$$F(\omega, x_2) - F(\omega, x_1) = \begin{cases} (x_2 - x_1)^\alpha & (\omega(R) = 1) \\ \delta(x_2 - x_1)^\alpha & (\omega(R) = 2) \end{cases}$$

if there exists $R := (x_1, x_2) \times (y_1, y_2) \in dom(\omega)$ with $y_2 > 1$.

For $\omega \in \mathcal{I}(\Omega)$, let $R_0(\omega)$ be the tile $(x_0, \tilde{x}_0) \times (y_0, \tilde{y}_0) \in dom(\omega)$ such that $x_0 = 0$ and $y_0 \leq 1 < \tilde{y}_0$. For $i = 0, 1, 2, \cdots$, let R_i be the i-th ancestor of $R_0(\omega)$. Let $R_i =: (x_i, \tilde{x}_i) \times (y_i, \tilde{y}_i)$. Let

$$G(\omega) := \sum_{i=0}^{\infty} (x_i - x_{i+1})^\alpha.$$

Since if $x_i > x_{i+1}$, then there exists a tile $(x_{i+1}, x_i) \times (y_{i+1}, \beta y_{i+1})$ with color 1 in ω, we have

$$F(\omega, x_i) - F(\omega, x_{i+1}) = (x_i - x_{i+1})^\alpha$$

for any $i = 0, 1, \cdots$.

Take any $t \in \mathbb{R}$ such that $(\omega, t) \in \mathcal{II}(\Omega)$. Let $(R'_i)_{i=1,2,\cdots}$ and $(x'_i)_{i=0,1,\cdots}$ be the sequences as above for $\omega + t$ instead of ω. Then, there exist $i_0 \geq 1$, $j_0 \geq 1$ such that $R'_{i_0+k} = R_{j_0+k} - t$ for any $k = 0, 1, \cdots$. Then, since $x'_{j_0+k} = x_{i_0+k} - t$ for any $k = 0, 1, \cdots$, we have

$$
\begin{aligned}
& G(\omega+t)-G(\omega) \\
= & \sum_{i=0}^{j_0-1}(x'_i-x'_{i+1})^\alpha-\sum_{i=0}^{i_0-1}(x_i-x_{i+1})^\alpha \\
= & -F(\omega+t,x'_{j_0})+F(\omega,x_{i_0})=F(\omega,t).
\end{aligned}
$$

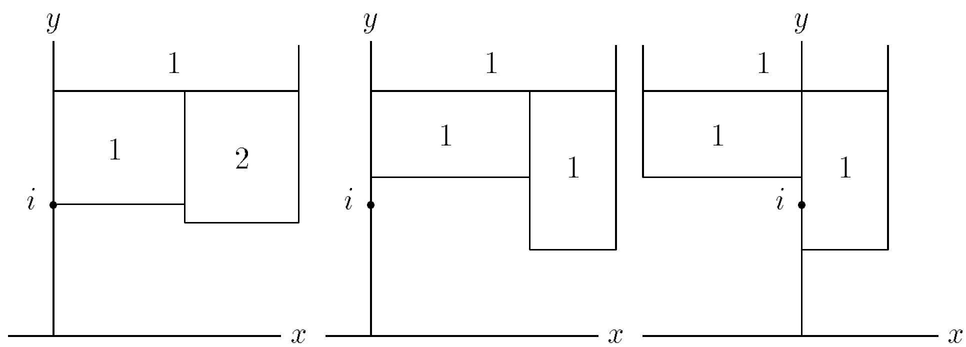

Figure 9: $\beta\mathcal{I}(\Omega)^1,\ \beta\mathcal{I}(\Omega)^2,\ \beta^2\mathcal{I}(\Omega)^2+\beta$

Thus, the α-homogeneous cocycle F is a coboundary with coboundary function G. The set $G(\mathcal{I}(\Omega))$ is known as Rauzy fractal. For $\omega\in\mathcal{I}(\Omega)$, let

$$
\mathcal{I}(\Omega)^a=\{\omega\in\mathcal{I}(\Omega);\ \omega(R_0(\omega))=a\}\quad(a=1,2)
$$

and

$$
G_a=G(\mathcal{I}(\Omega)^a)\quad(a=1,2).
$$

Considering the children of the tile $R_0(\omega)$, we have a set equation that

$$
\begin{aligned}
\mathcal{I}(\Omega)^1 &= \beta\mathcal{I}(\Omega)^1\cup\beta\mathcal{I}(\Omega)^2\cup(\beta^2\mathcal{I}(\Omega)^2+\beta) \\
\mathcal{I}(\Omega)^2 &= \beta\mathcal{I}(\Omega)^1+1.
\end{aligned}
$$

Hence, the following set equation holds:

$$
\begin{aligned}
G_1 &= \gamma G_1\cup\gamma G_2\cup(\gamma^2G_2+\gamma) \\
G_2 &= \gamma G_1+1.
\end{aligned}
$$

$G_1\cup G_2$ is shown in Figure 10.

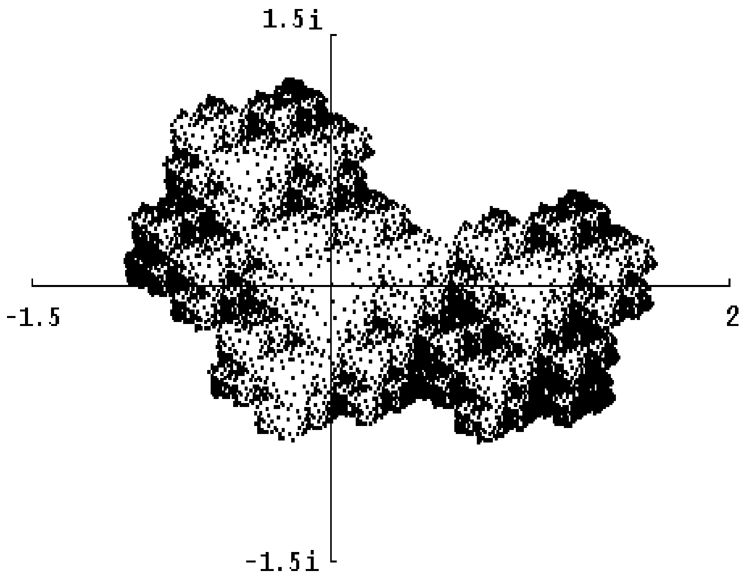

Figure 10: $G(\mathcal{I}(\Omega))$

Remark 2. The above Rauzy fractal is usually introduced by the substitution $1 \to 12,\ 2 \to 13,\ 3 \to 1$ (S. Ito and P. Arnoux [1]). We modify it canonically to the weighted substitution in Example 8 (see Remark 1). The set equation is usually denoted as

$$\begin{aligned} G_1 &= \gamma G_1 \cup \gamma G_2 \cup \gamma G_3 \\ G_2 &= \gamma G_1 + 1 \\ G_3 &= \gamma G_2 + 1, \end{aligned}$$

which is equivalent to ours.

7 N-process

We consider the *N-process* defined in Example 9. It is defined as a (1/2)-homogeneous cocycle F on the space $\Omega = \Omega(\sigma, \tau)$ with the weighted substitution (σ, τ) in Example 1. Hence, F is defined by

$$F(\omega, x_2) - F(\omega, x_1) = \pm(x_2 - x_1)^{1/2} \tag{17}$$

if there is a tile $(x_1, x_2) \times (y_1, y_2) \in dom(\omega)$, where $\pm$ corresponds to the color of the tile.

Take $\omega_0 \in \Omega$ which has a tile $R_0 := (0,1) \times (1, 9/4)$ with $\omega_0(R_0) = +$. Then, $F(\omega_0, 1) = F(\omega_0, 1) - F(\omega_0, 0) = 1$ by (17). Since R_0 has 3 children $R_{1,0}, R_{1,1}, R_{1,2}$ with colors $+, -, +$ and the vertical sizes $4/9, 1/9, 4/9$, we have $F(\omega_0, 4/9) = 2/3$ and $F(\omega_0, 5/9) - F(\omega_0, 4/9) = -1/3$ by (17). Hence, $F(\omega_0, 5/9) = 1/3$.

Since there is a one-to-one correspondence between the descendants of R_0 and those of $R_{1,0}$ keeping the color given by

$$(x_1, x_2) \times (y_1, y_2) \to ((4/9)x_1, (4/9)x_2) \times ((4/9)y_1, (4/9)y_2).$$

Hence by (17),

$$F(\omega_0, (4/9)x_2) - F(\omega_0, (4/9)x_1) = (2/3)(F(\omega_0, x_2) - F(\omega_0, x_1))$$

holds if $(x_1, x_2) \times (y_1, y_2) \in dom(\omega_0)$ and $(x_1, x_2) \subset [0, 1]$. By the continuity of $F(\omega_0, t)$ in t, this implies that

$$F(\omega_0, (4/9)t) = (2/3)F(\omega_0, t)$$

for any $t \in [0, 1]$. By the similar correspondence keeping the color between the descendants of R_0 and those of $R_{1,2}$, we have

$$F(\omega_0, (5 + 4t)/9) - F(\omega_0, 5/9) = (2/3)F(\omega_0, t)$$

for any $t \in [0, 1]$. By the similar correspondence changing the color between the descendants of R_0 and those of $R_{1,1}$, we have

$$F(\omega_0, (4 + t)/9) - F(\omega_0, 4/9) = -(1/3)F(\omega_0, t)$$

for any $t \in [0, 1]$.

Hence, the graph Γ of the function $F(\omega_0, t)$ on $t \in [0, 1]$ satisfies the set equation $\Gamma = \psi(\Gamma)$, where for a compact set R,

$$\psi(R) := \psi_0(R) \cup \psi_1(R) \cup \psi_2(R)$$

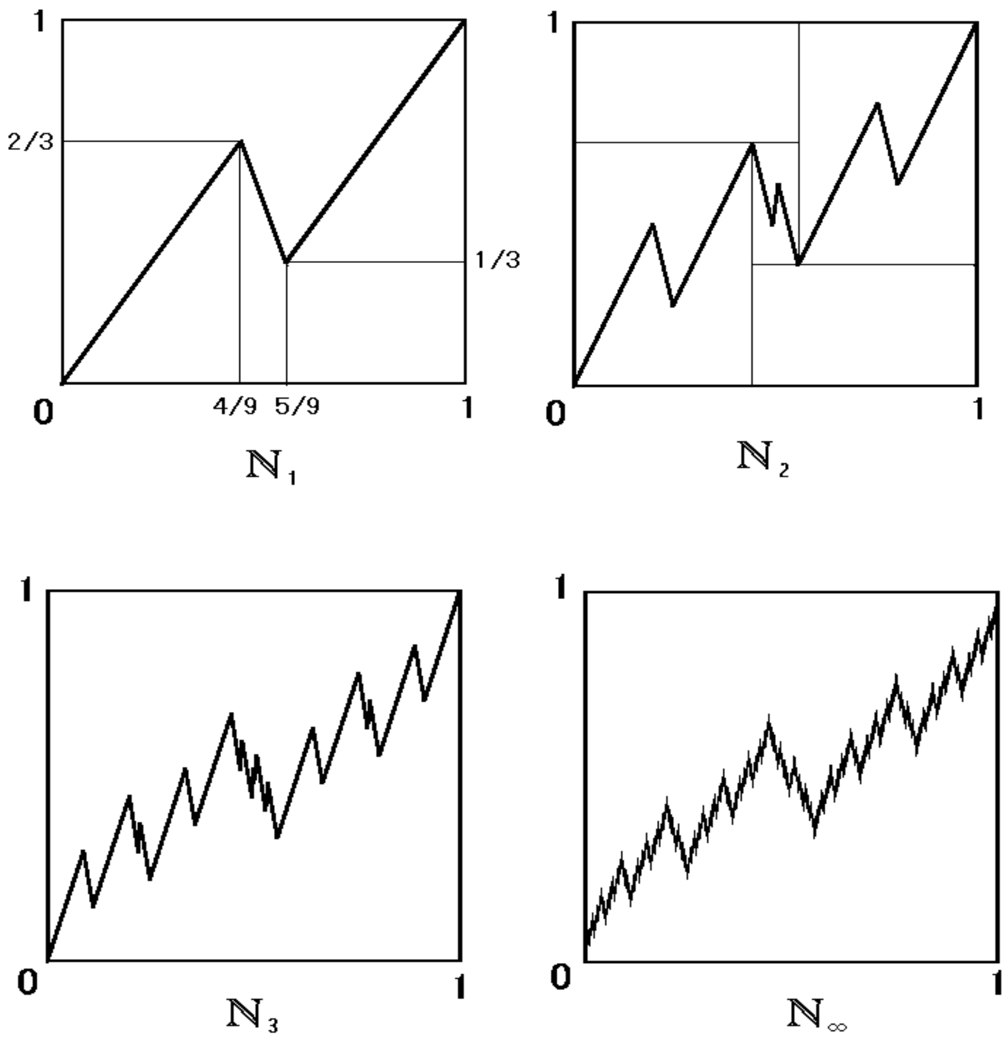

Figure 11: N_1, N_2, N_3 and N_∞

with the functions $\psi_0, \psi_1, \psi_2 : \mathbb{H} \to \mathbb{R}^2$ given by

$$\begin{aligned}
\psi_0(x, y) &= ((4/9)x, (2/3)y) \\
\psi_1(x, y) &= ((x + 4)/9, (-y + 2)/3) \\
\psi_2(x, y) &= ((4x + 5)/9, (2y + 1)/3).
\end{aligned}$$

Then, Γ is obtained as the limit in the sense of Hausdorff metric of $\psi^n(\Gamma_{N_0})$ as $n \to \infty$, where Γ_{N_0} denotes the graph of the function $N_0(t) := t$ $(t \in [0, 1])$. For $n = 1, 2, \cdots$, define a function N_n on $[0, 1]$ by $\Gamma_{N_n} = \psi^n(\Gamma_{N_0})$. Then, N_1 is a continuous piecewise linear function whose graph consists of 3 line segments

$$\begin{aligned}
\psi_0(\Gamma_{N_0}) &= \{(x, y);\ y = (3/2)x,\ 0 \le x \le 4/9\} \\
\psi_1(\Gamma_{N_0}) &= \{(x, y);\ y = -3x + 2,\ 4/9 \le x \le 5/9\} \\
\psi_2(\Gamma_{N_0}) &= \{(x, y);\ y = (3/2)x - (1/2),\ 5/9 \le x \le 1\}.
\end{aligned}$$

as seen in Figure 11.

Since $\Gamma_{N_2} = \psi(\Gamma_{N_1})$, N_2 is a continuous piecewise linear function on $[0, 1]$ obtained by replacing 3 line segments in Γ_{N_1} by self-affine images of Γ_{N_1} or Γ_{-N_1} keeping the 2 end points fixed. Then, the graph of N_2 consists of 9 line segments. In the same way, we obtain N_n from N_{n-1} for $n = 3, 4, \cdots$. Then, N_n is a continuous piecewise linear function on $[0, 1]$ whose graph consists of 3^n line segments.

Let Ξ_n be the set of closed intervals which are the projection to the horizontal axis of the 3^n line segments consisting of the graph of N_n. Note that Ξ_n is the set of closed intervals which are the projection to the vertical axis of the descendents of the n-th generation of the tile R_0 in ω_0. Let Δ_n be the set of the end points of Ξ_n. Let $\Xi = \cup_{n=0}^{\infty} \Xi_n$ and $\Delta = \cup_{n=0}^{\infty} \Delta_n$. Denote $N_\infty(t) := F(\omega_0, t)$ $(t \in [0, 1])$. Then, the function $N_\infty(t)$ is the pointwise limit of $N_n(t)$ as $n \to \infty$.

Theorem 12.

(i) *For any* s, t *with* $0 \le s < t \le 1$, $|N_\infty(t) - N_\infty(s)| \le |t - s|^{1/2}$ *holds. Moreover, the equality holds if and only if* $[s, t] \in \Xi$.

(ii) *For any Borel set* $R \subset [0, 1]$,

$$\int_{N_\infty(t) \in R} dt = \int_R \big(2 - |4t - 2|\big)\, dt.$$

(iii) The Hausdorff dimension of the graph Γ_{N_∞} *of the function* N_∞ *satisfies that* $\text{H-dim}\Gamma_{N_\infty} = 3/2$*. In fact, the* $(3/2)$*-dimensional Hausdorff measure of it is positive and finite.*

(iv) $N_\infty(t)$ *is locally minimal or maximal at* $t = t_0$ *if and only if* $t_0 \in \Delta$*. In this case, there exists* $\delta > 0$ *such that*

$$(1/\sqrt{5})|h|^{1/2} \le |N_\infty(t_0+h) - N_\infty(t_0)| \le |h|^{1/2}$$

holds for any h *with* $t_0 + h \in [0,1]$ *and* $|h| < \delta$*.*

(v) For any $t_0 \in [0,1]$ *with* $t_0 \notin \Delta$ *and* $\epsilon > 0$*, it holds that*

$$\begin{aligned} & \text{H-dim}\{s \in (t_0 - \epsilon, t_0);\ N_\infty(s) = N_\infty(t_0)\} \\ = \ & \text{H-dim}\{s \in (t_0, t_0 + \epsilon);\ N_\infty(s) = N_\infty(t_0)\} = 1/2. \end{aligned}$$

In fact, the $(1/2)$*-dimensional Hausdorff measures of the above sets are positive and finite.*

Proof. (i) is proved in [9].

(ii) Let $\mathcal{L}$ be a bounded operator on the Banach space $C([0,1])$ defined by

$$(\mathcal{L}f)(t) := (4/9)f((2/3)t) + (1/9)f((2-t)/3) + (4/9)f((1+2t)/3)$$

for any $t \in [0,1]$ and $f \in C([0,1])$.

Let ν and τ be the probability measures on $[0,1]$ defined by

$$\begin{aligned} \nu(S) &= \int_{N_\infty(t)\in S} dt \\ \int_S d\tau &= \int_S \big(2 - |4t-2|\big)dt \end{aligned}$$

for any Borel set $S \subset [0,1]$. We prove that $\nu = \tau$.

For any $f \in C([0,1])$, we have

$$\begin{aligned}
\int \mathcal{L} f d\nu &= \int (\mathcal{L} f)(N_\infty(t))dt \\
&= \int (4/9) f((2/3)N_\infty(t))dt \\
&\quad + \int (1/9) f((2 - N_\infty(t))/3)dt \\
&\qquad + \int (4/9) f((1 + 2N_\infty(t))/3)dt \\
&= \int (4/9) f(N_\infty((4/9)t))dt \\
&\quad + \int (1/9) f(N_\infty((t+4)/9))dt \\
&\qquad + \int (4/9) f(N_\infty((4t+5)/9))dt \\
&= \int_0^{4/9} f(N_\infty(t))dt + \int_{4/9}^{5/9} f(N_\infty(t))dt + \int_{5/9}^{1} f(N_\infty(t))dt \\
&= \int f(N_\infty(t))dt = \int f d\nu.
\end{aligned}$$

Thus, ν is invariant under $\mathcal{L}$. We prove that τ is also invariant under $\mathcal{L}$.

In fact,

$$
\begin{aligned}
&\int \mathcal{L} f d\tau = \int \big(2-|4t-2|\big)(\mathcal{L}f)(t)dt \\
= &\int \big(2-|4t-2|\big)(4/9)f((2/3)t)dt \\
&\quad + \int \big(2-|4t-2|\big)(1/9)f((2-t)/3)dt \\
&\qquad + \int \big(2-|4t-2|\big)(4/9)f((1+2t)/3)dt \\
= &\int_0^{2/3} \big(2-|6t-2|\big)(4/9)f(t)(3/2)dt \\
&\quad + \int_{1/3}^{2/3} \big(2-|12t-6|\big)(1/9)f(t)3dt \\
&\qquad + \int_{1/3}^{1} \big(2-|6t-4|\big)(4/9)f(t)(3/2)dt \\
= &\int_0^{1/3} 6t(2/3)f(t)dt \\
&+ \int_{1/3}^{1/2} ((4-6t)(2/3)+(-4+12t)(1/3)+(-2+6t)(2/3))f(t)dt \\
&\quad + \int_{1/2}^{2/3} \big((4-6t)(2/3)+(8-12t)(1/3)+(-2+6t)(2/3)\big)f(t)dt \\
&\quad + \int_{2/3}^{1} (6-6t)(2/3)f(t)dt \\
= &\int \big(2-|4t-2|\big)f(t)dt = \int f d\tau.
\end{aligned}
$$

By the definition of the operator $\mathcal{L}$, we have

$$(\mathcal{L}f)'(t) = (8/27)f'((2/3)t) - (1/27)f'((2-t)/3) + (8/27)f'((1+2t)/3)$$

for any $f \in C^1([0,1])$. This implies that $\| (\mathcal{L}f)' \| \leq (17/27) \| f' \|$. Therefore, we have $\| (\mathcal{L}^n f)' \| \leq (17/27)^n \| f' \|$, and hence, $(\mathcal{L}^n f)'$ converges to 0. This implies that there exists a subsequence $\{n'\} \subset$

$\{n\}$ such that $\mathcal{L}^{n'}f$ converges to a constant, say c. Hence, we have

$$\int f d\nu = \lim \int \mathcal{L}^{n'} f d\nu = c = \lim \int \mathcal{L}^{n'} f d\tau = \int f d\tau$$

for any $f \in C^1([0,1])$. This implies that $\nu = \tau$.

(iii) Let $\Gamma := \Gamma_{N_\infty(t)}$. Since N_∞ is uniformly 1/2-Hölder continuous, the (3/2)-dimensional Hausdorff measure of Γ is finite. We prove that it is positive.

Let ν_Γ be the probability measure supported by Γ such that for any Borel set $S \subset [0,1]$,

$$\nu_\Gamma((S \times [0,1]) \cap \Gamma) := \int_S dt.$$

Then by (ii), it holds that

$$\nu_\Gamma([0,1] \times S) = \int_S \big(2 - |4t-2|\big) dt. \tag{18}$$

Take any cover $\mathcal{U} = \{U_i;\ i = 1, 2, \cdots\}$ of Γ. Let $I_{3/2}(\mathcal{U}) := \sum_i d(U_i)^{3/2}$, where $d(U)$ denotes the diameter of the set U. Since for any set U, we can find a closed rectangle U' such that $U \subset U'$ and $d(U') \leq 2\sqrt{2}d(U)$, we can replace each set $U \in \mathcal{U}$ by a closed rectangle as this, so that we have a cover $\mathcal{U}'$ of Γ consisting of closed rectangles such that $I_{3/2}(\mathcal{U}') \leq 5I_{3/2}(\mathcal{U})$. Since for any interval $[a,b] \subset [c,d]$, we can find at most 2 intervals I_1, I_2 in Ξ such that $I_1 \cup I_2 \supset [a,b]$ and $|I_1| + |I_2| \leq 9(b-a)$. We replace each rectangle $[a,b] \times [c,d] \in \mathcal{U}'$ as above by $I_1 \times [c,d]$ and $I_2 \times [c,d]$. Furthermore, we replace them again by $I_i \times ([c,d] \cap N_\infty(I_i))$ $(i = 1,2)$. Let $\mathcal{V} = \{V_i;\ i = 1, 2, \cdots\}$ be the cover of Γ obtained by this procedure from $\mathcal{U}$. Then, we have

$$I_{3/2}(\mathcal{V}) \leq 27 I_{3/2}(\mathcal{U}') \leq 135 I_{3/2}(\mathcal{U}).$$

Take any of $V_i =: I \times [c,d] \in \mathcal{V}$. By the assumption, $I \in \Xi$ and $[c,d] \subset N_\infty(I) =: [C,D]$. Let $N_\infty(I) =: [C,D]$. The graph of Γ

restricted to $I \times [C, D]$ is the image of Γ by the mapping $(x, y) \mapsto (a + |I|x, b \pm |I|^{1/2}y)$ with some a, b and $\pm$. Hence by (18), we have

$$\begin{aligned} \nu_\Gamma(V_i) &= \nu_\Gamma(I \times [c, d]) \\ &= |I|\nu_\Gamma\big([0,1] \times [\frac{c-C}{D-C}, \frac{d-C}{D-C}]\big) \\ &\leq 2|I|\frac{x_2 - x_1}{D - C} \\ &= 2|I|\frac{x_2 - x_1}{|I|^{1/2}} \\ &= 2|I|^{1/2}(x_2 - x_1) \leq 2d(V_i)^{3/2}. \end{aligned} \tag{19}$$

Thus, adding the above inequality, we have

$$I_{3/2}(\mathcal{V}) = \sum_i d(V_i)^{3/2} \geq (1/2) \sum_i \nu_\Gamma(V_i) \geq (1/2)\nu_\Gamma(\Gamma) = 1/2,$$

so that $I_{3/2}(\mathcal{U}) \geq 1/270$, which completes the proof.

(iv) Let $t_0 \in \Delta \setminus \{0, 1\}$. By (i),

$$|N_\infty(t_0 + h) - N_\infty(t_0)| \leq |h|^{1/2}$$

holds for any h with $t_0 + h \in [0, 1]$. Therefore, it is sufficient to prove that there exists $\delta > 0$ such that

$$(1/\sqrt{5})|h|^{1/2} \leq |N_\infty(t_0 + h) - N_\infty(t_0)|$$

holds for any h with $t_0 + h \in [0, 1]$ and $|h| < \delta$. There are intervals I and J in some Γ_n such that $\{t_0\} = I \cap J$ and I is in the left side of J. Then, either the piecewise linear function N_n is increasing in I and decreasing in J or N_n is decreasing in I and increasing in J. Without loss of generality, we assume the latter. Then, we have

$$\begin{aligned} N_\infty(t_0 - |I|s) - N_\infty(t_0) &= |I|^{1/2}(1 - N_\infty(1 - s)) \\ N_\infty(t_0 + |J|s) - N_\infty(t_0) &= |J|^{1/2}N_\infty(s). \end{aligned}$$

for any $s \in [0, 1]$ by the set equation $\psi(\Gamma) = \Gamma$. Therefore, with $h = |J|s$

$$|N_\infty(t_0 + h) - N_\infty(t_0)| \geq \frac{1}{\sqrt{5}}|h|^{1/2}$$

follows from the statement that

$$N_\infty(s) \geq \frac{s^{1/2}}{\sqrt{5}}$$

for any $s \in (0,1]$ and with $h = -|I|s$

$$|N_\infty(t_0+h) - N_\infty(t_0)| \geq \frac{1}{\sqrt{5}}|h|^{1/2}$$

follows from the statement that

$$1 - N_\infty(1-s) \geq \frac{s^{1/2}}{\sqrt{5}}$$

for any $s \in (0,1]$. By the symmetry of the graph of N_∞ with respect to $(1/2, 1/2)$, the second statement follows from the first statement that

$$N_\infty(s) \geq \frac{s^{1/2}}{\sqrt{5}}$$

for any $s \in (0,1]$.

We prove this inequality. Note that the equality holds for $s = 5/9$. Suppose that

$$N_\infty(s)/s^{1/2} < \frac{1}{\sqrt{5}}$$

holds for some $s \in (0,1]$. Since

$$c_0 := \inf_{s\in(0,1]} N_\infty(s)/s^{1/2} = \min_{s\in[4/9,1]} N_\infty(s)/s^{1/2} = \min_{s\in[5/9,1]} N_\infty(s)/s^{1/2},$$

there exists $s_0 \in [5/9, 1]$ which attains the minimum. Then,

$$\begin{aligned} c_0 {s_0}^{1/2} &= N_\infty(s_0) \\ &= (1/3) + N_\infty(s_0 - (5/9)) \\ &> c_0(5/9)^{1/2} + c_0(s_0 - (5/9))^{1/2} \\ &\geq c_0 s_0^{1/2} \end{aligned}$$

which is a contradiction. Thus, $N_\infty(s) \geq \frac{s^{1/2}}{\sqrt{5}}$ for any $s \in [0,1]$, which completes the proof of (iv).

(v) We prove that for any $x \in (0,1)$, the $(1/2)$-dimensional Hausdorff measure of $(N_\infty)_x$ is positive and finite, where we denote $(N_\infty)_x := \{t \in [0,1];\ N_\infty(t) = x\}$. Let

$$\delta := (2 - |4x-2|) \wedge (2 - |2(x+1)-2|) > 0.$$

We prove that for any cover $\mathcal{U}$ of $(N_\infty)_x$, $I_{1/2}(\mathcal{U}) \geq (1/10)\delta$. We may assume that $\mathcal{U}$ consists of open intervals, so that there exits a finite subcover of $\mathcal{U}$. Therefore, we may assume that $\mathcal{U}$ is a finite cover consisting of closed intervals. Moreover, by the same argument as in the proof of (iii), it is sufficient to prove that for any

$$\mathcal{V} := \{I_i;\ i = 1,2,\cdots,K\} \subset \Xi,$$

$I_{1/2}(\mathcal{V}) \geq (1/2)\delta$. Let $\epsilon > 0$ satisfy that $x + 2\epsilon < 1$ and $|I_i| > \epsilon$ for any $i = 1,2,\cdots,K$. By the same argument as in (19), we have

$$\nu_\Gamma(I_i \times [x, x+\epsilon]) \leq 2|I_i|\frac{\epsilon}{|I_i|^{1/2}} \leq 2\epsilon|I_i|^{1/2}.$$

Adding the above inequality, we have

$$\nu_\Gamma(\Gamma \cap ([0,1] \times [x, x+\epsilon])) \leq 2\epsilon I_{1/2}(\mathcal{V}),$$

Hence by (18) and the definition of δ, we have

$$I_{1/2}(\mathcal{V}) \geq (2\epsilon)^{-1}\nu_\Gamma([0,1] \times [x, x+\epsilon]) \geq (1/2)\delta,$$

which completes the proof that Hausdorff measure of $(N_\infty)_x$ is positive.

To prove that it is finite, for any sufficiently small $\epsilon > 0$ and $t \in (N_\infty)_x$, we take $V(t) \in \Xi$ such that $t \in V(t)$ and $2\epsilon \leq |V(t)| < 18\epsilon$. Then, there exists a finite subcover $\mathcal{V} := \{V_i :\ i = 1,2,\cdots,K\}$ of $\{V(t);\ t \in (N_\infty)_x\}$ such that $V_i \cap V_j$ is at most one point for any $i \neq j$.

For any $i = 1,2,\cdots,K$, by the same argument as in (19), we have

$$\begin{aligned} & \nu_\Gamma(V_i \times [x - \epsilon^{1/2}, x + \epsilon^{1/2}]) \\ \geq\ & |V_i| \int_0^{(\epsilon/|V_i|)^{1/2}} (2 - |4t-2|)dt \\ \geq\ & |V_i| 2\epsilon/|V_i| \geq 2\epsilon \geq (2/5)\epsilon^{1/2}|V_i|^{1/2}. \end{aligned}$$

Adding this inequality together with (18), we have

$$\begin{aligned} 4\epsilon^{1/2} &\geq \nu_\Gamma([0,1]\times[x-\epsilon^{1/2}, x+\epsilon^{1/2}]) \\ &= \nu_\Gamma(\Gamma\cap([0,1]\times[x-\epsilon^{1/2}, x+\epsilon^{1/2}])) \\ &\geq (2/5)\epsilon^{1/2} I_{1/2}(\mathcal{V}). \end{aligned}$$

Hence, we have $I_{1/2}(\mathcal{V}) \leq 10$, which completes the proof that (1/2)-dimensional Hausdorff measure of $(N_\infty)_x$ is finite.

The statements in (iv) follows from this. □

Consider the stochastic process $(\mathbf{N}_t)_{t\in\mathbb{R}}$ defined by $\mathbf{N}_t(\omega) = F(\omega, t)$, where ω comes from the probability space (Ω, μ), μ being the unique invariant probability measure invariant under the additive action. This process was called the N-process and studied in [9]. A prediction theory based on the N-process was developed. A process $Y_t = H(\mathbf{N}_t, t)$, where the function $H(x, s)$ is an unknown function which is twice continuously differentiable in x and once continuously differentiable in s and $H_x(x, s) > 0$ is considered. The aim is to predict the value Y_c from the observation $Y_J := \{Y_t;\ t \in J\}$, where $J = [a, b]$ and $a < b < c$.

Theorem 13. ([9]) *There exists an estimator $\hat{Y}_c$ which is a measurable function of the observation Y_J such that*

$$\mathrm{E}[(\hat{Y}_c - Y_c)^2] = O((c-b)^2)$$

as $c \downarrow b$.

Open problems:

(1) The systems of the geodesic and horocycle flows on the upper half plane $\mathbb{H}/G$ devided by cocompact discrete groups $G \in PSL_2(\mathbb{R})$ are numeration systems which are not isomorphic to those coming from weighted substitutions (private communications by Prof. Benjamin Weiss). How to characterize the numeration systems coming from weighted substitutions?

(2) Is the condition $B(\sigma, \tau) = \mathbb{R}_+$ necessary for the $\mathbb{R}$-action of a numeration system coming from weighted substitutions with respect

to the unique invariant probability measure to be weakly mixing? When doess it have a discrete spectrum?
(3) What is the multiplicity of the pure Lebesgue spectrum possessed by the $\mathbb{R}$-action of a numeration system coming from a weighted substitution with $B(\sigma, \tau) = \mathbb{R}_+$ with respect to the unique invariant probability measure?
(4) When does a numeration system admit an additive group structure consistent with the $(\mathbb{R}, G)$-action?

Acknowledgement: The author would like to thank the anonymous referee for useful suggestions. The metric in (11) and the proof of Lemma 4 which simplified the original one are suggested by the referee. The author thanks Prof. Sigeki Akiyama and Prof. Pierre Arnoux for their useful discussions with the author. The author also thanks Prof. Benjamin Weiss and Prof. Hillel Furstenberg for teaching him the references about Theorem 3.

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