Partitions by congruent sets and optimal positions

Ergodic Theory and Dynamical Systems 31 (2011), pp.613-629

(doi:10.1017/S0143385709001175)

Yu-Mei XUE* and Teturo KAMAE†

Abstract

Let X be a metrizable space with a continuous group or semi-group action G. Let D be a nonempty subset of X. Our problem is how to choose a fixed number of sets in $\{g^{-1}D;\ g \in G\}$, say $\sigma^{-1}D$ with $\sigma \in \tau$, to maximize the cardinality of the partition $\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \tau\})$ generated by them. Let

$$p^*_{X,G,D}(k) = \sup_{\tau \subset G,\ \#\tau = k} \#\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \tau\}) \ \ (k = 1, 2, \cdots).$$

An infinite subset Σ of G is called an optimal position of the triple (X, G, D) if

$$\#\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \tau\}) = p^*_{X,G,D}(k),$$

holds for any $k = 1, 2, \cdots$ and $\tau \subset \Sigma$ with $\#\tau = k$. In this paper, we discuss examples of the triple (X, G, D) admitting or not admitting an optimal position. Let $X = G = \mathbb{R}^n$ $(n \geq 1)$, where the action $g \in G$ to $x \in X$ is the translation $x - g$. If D is the n-dimensional unit ball, then

$$p^*_{X,G,D}(k) = 2\sum_{i=0}^{n} \binom{k-1}{i}$$

holds and the triple (X, G, D) admits an optimal position. In fact, if $n \geq 2$ and Σ is an infinite subset of G such that for some δ with $0 < \delta < 1$, $\Sigma \subset \{x \in \mathbb{R}^n;\ ||x|| = \delta\}$, and that any subset of Σ with cardinality $n+1$ is not on a hyper-plane, then Σ is an optimal position of the triple (X, G, D). We determined the primitive factor of the uniform sets coming from these optimal positions. We also show that in the above setting with $n = 2$ and the unit square D' in place of the unit disk D, the maximal pattern complexity is unchanged and $p^*_{X,G,D'}(k) = k^2 - k + 2$, but there is no optimal position.

*School of Mathematics and System Sciences & LMIB, Beijing University of Aeronautics and Astronautics , Beijing 100191, PR China (yxue@buaa.edu.cn)

†5-9-6 Satakedai, Suita, 565-0855 Japan (kamae@apost.plala.or.jp)

1 Introduction

Let X be a metrizable space with a continuous group or semi-group action G. For a family of subsets $A_1, A_2, \cdots, A_k$ of X, $\mathbb{P}(\{A_i;\ i = 1, 2, \cdots, k\})$ denotes the *partition* of X generated by these sets, that is, the family of nonempty sets of the form

$$A_1^{i_1} \cap A_2^{i_2} \cap \cdots \cap A_k^{i_k} \quad (i_1, i_2, \cdots, i_k \in \{0, 1\}),$$

where for a set $A \subset X$, we denote $A^1 = A$ and $A^0 = X \setminus A$.

Let D be a nonempty subset of X. Our problem is how to choose a fixed number of sets in $\{g^{-1}D;\ g \in G\}$ to maximize the cardinality of the partition generated by them. Define the *maximal pattern complexity* function $p^*_{X,G,D}$ of the triple (X, G, D) by

$$p^*_{X,G,D}(k) = \sup_{\tau \subset G,\ \#\tau = k} \#\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \tau\}) \quad (k = 1, 2, \cdots),$$

where # implies the number of elements in a set.

An infinite subset Σ of G is called an *optimal position* of the triple (X, G, D) if

$$\#\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \tau\}) = p^*_{X,G,D}(k),$$

holds for any $k = 1, 2, \cdots$ and $\tau \subset \Sigma$ with $\#\tau = k$.

In this paper, we discuss examples of the triple (X, G, D) admitting or not admitting an optimal position. Let $X = G = \mathbb{R}^n$ $(n \geq 1)$, where the action $g \in G$ to $x \in X$ is the translation $x - g$. If D is the n-dimensional unit ball, then

$$p^*_{X,G,D}(k) = 2 \sum_{i=0}^{n} \binom{k-1}{i}$$

holds and the triple (X, G, D) admits an optimal position. In fact, if $n \geq 2$ and Σ is an infinite subset of G such that for some δ with $0 < \delta < 1$, $\Sigma \subset \{x \in \mathbb{R}^n;\ ||x|| = \delta\}$, and that any subset of Σ with cardinality $n+1$ is not on a hyper-plane, then Σ is an optimal position of the triple (X, G, D). Also, if $n = 1$ and $\Sigma \subset (-1, 1)$, then Σ is an optimal position.

On the other hand, if in the above with $n = 2$, D is replaced by the unit square $D' = [0, 1] \times [0, 1]$, then we have the same maximal pattern complexity

$$p^*_{X,G,D'}(k) = p^*_{X,G,D}(k) = k^2 - k + 2,$$

but the optimal position does not exit.

If the optimal position Σ for the triple (X, G, D) exists, then the name set Ω which is the closure of $\{\omega_x;\ x \in X\} \subset \{0, 1\}^\Sigma$, where $\omega_x \in \{0, 1\}^\Sigma$ is such that $\omega_x(\sigma) = 1$ if and only if $x \in \sigma^{-1}D$, satisfies the property that $\#\Omega|_S$ depends only on $\#S$ for any finite set $S \subset \Sigma$, where $\Omega|_S = \{\omega|_S;\ \omega \in \Omega\}$

and $\omega|_S \in \{0,1\}^S$ is the restriction of $\omega \in \{0,1\}^\Sigma$ to S. Such a set is called a *uniform set*. In fact, we have $\#\Omega|_S = p^*_{X,G,D}(\#S)$.

For $\Omega \subset \{0,1\}^{\mathbb{N}}$ and an infinite set $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$, we denote $\Omega[\mathcal{N}] = \{\omega[\mathcal{N}];\ \omega \in \Omega\}$, where $\omega[\mathcal{N}] \in \{0,1\}^{\mathbb{N}}$ is such that $\omega[\mathcal{N}](n) = \omega(N_n)$ $(n \in \mathbb{N})$. We call a nonempty closed set $\Omega \subset \{0,1\}^{\mathbb{N}}$ a *super-stationary* set if $\Omega[\mathcal{N}] = \Omega$ holds for any infinite subset $\mathcal{N}$ of $\mathbb{N}$.

It is known [1] that for any uniform set $\Omega \subset \{0,1\}^\Sigma$, there exists an injection $\psi : \mathbb{N} \to \Sigma$ such that $\Omega \circ \psi = \{\omega \circ \psi \in \{0,1\}^{\mathbb{N}};\ \omega \in \Omega\}$ is a super-stationary set. In this case, we say that Ω has a *primitive factor* $[\Omega \circ \psi]$, where $[\Omega \circ \psi]$ is the *isomorphic* class containing $\Omega \circ \psi$ in the sense that 2 nonempty closed subsets Ω and Θ of $\{0,1\}^{\mathbb{N}}$ are said to be isomorphic if there is an isometric bijection between them. The notion of primitive factor is introduced in [3] to distinguish pattern Sturmian words of different types.

The class of super-stationary sets is characterized in terms of *super-subword* in [2]. For $\xi = \xi_1\xi_2\cdots\xi_k \in \{0,1\}^k$ and $\omega \in \{0,1\}^{\mathbb{N}}$, ξ is called a super-subword of ω, denoted by $\xi \ll \omega$, if there exist $l_1, l_2, \cdots, l_k$ such that $0 \le l_1 < l_2 < \cdots < l_k$ and $\omega(l_1)\omega(l_2)\cdots\omega(l_k) = \xi$. The set of infinite words with prohibited super-subword ξ is denoted by $\mathcal{P}(\xi) = \{\omega \in \{0,1\}^{\mathbb{N}};\ \xi \ll \omega$ does not hold$\}$. Then, the class of super-stationary sets other than $\{0,1\}^{\mathbb{N}}$ coincides with the class of sets of the form $\bigcup_{\xi\in\Xi} \mathcal{P}(\xi)$ with nonempty finite sets $\Xi \subset \bigcup_{k=1}^{\infty}\{0,1\}^k$.

For the name set $\Omega \subset \{0,1\}^\Sigma$ with respect to the above optimal position Σ for the triple (X, G, D) with $X = G = \mathbb{R}^n$ and D the n-dimensional unit ball, we prove that Ω has a unique primitive factor $[\mathcal{P}((01)^l 0) \cup \mathcal{P}((10)^l 1)]$ if $n = 2l$ is even, and $[\mathcal{P}((01)^{l+1}) \cup \mathcal{P}((10)^{l+1})]$ if $n = 2l+1$ is odd.

The basic terminology and notations follow from [1, 2]. For further related results, refer [3, 4, 5, 6, 7, 8, 9].

2 Basic lemmas

Definition 1. The Euclidean space $\mathbb{R}^{n+1}$ is sometimes considered as the vector space. Let $\mathcal{S}^n = \{x \in \mathbb{R}^{n+1};\ ||x|| = 1\}$ $(n \ge 1)$ be the n-dimensional unit sphere, where for $x = (x_1, \cdots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$, we denote $||x|| = \sqrt{x_1^2 + \cdots + x_n^2 + x_{n+1}^2}$. For $v \in \mathbb{R}^{n+1}$ with $||v|| = 1$ and $c \in \mathbb{R}$ with $-1 < c < 1$, we define a *disk* on $\mathcal{S}^n$ by

$$D_n(v, c) := \{x \in \mathcal{S}^n;\ (x, v) \ge c\},$$

where (,) is the inner product. A one-point set in $\mathcal{S}^n$ is considered as a *degenerate disk*. Let $\mathcal{D}_n$ be the family of disks on $\mathcal{S}^n$. Also let $\overline{\mathcal{D}}_n$ be the union of $\mathcal{D}_n$ with the family of degenerate disks.

For $A = D_n(v, c)$, ∂A denotes the boundary. That is, $\partial A = \{x \in \mathcal{S}^n;\ (x, v) = c\}$, which is the $(n-1)$-dimensional sphere with center cv and radius $\sqrt{1-c^2}$.

In particular, a 0-dimensional sphere is considered as a two-points set, and a (-1)-dimensional sphere is considered as the empty set.

Lemma 2. *For* $A := \bigcap_{i=1}^{k} \partial D_n(v_i, c_i)$ *with* $n \geq 1$ *and* $k \geq 1$, *one of the following* 3 *cases occurs:*
(1) $A = \emptyset$,
(2) $\#A = 1$, *and*
(3) A *is an* $(n-d)$*-dimensional sphere with* $d = 1, 2, \cdots, \min\{k, n\}$.

Moreover, this condition (3) *is equivalent to the following condition* (4).
(4) $\dim\{v_1, v_2, \cdots, v_k\} = d$ *and* $\mathbf{c}\mathbf{A}^{-1}\mathbf{c}' < 1$ *hold for any* $h_1, h_2, \cdots, h_d$ *such that* $1 \leq h_1 < h_2 < \cdots < h_d \leq k$ *and* $v_{h_1}, \cdots, v_{h_d}$ *are linearly independent, where* $\mathbf{A} := ((v_{h_i}, v_{h_j}))_{i,j=1,\cdots,d}$ *and* $\mathbf{c} := (c_{h_1}, \cdots, c_{h_d})$.

Proof Let $\dim\{v_1, v_2, \cdots, v_k\} = d$. Assume without loss of generality that $v_1, v_2, \cdots, v_d$ are linearly independent so that $h_1 = 1, h_2 = 2, \cdots, h_d = d$. Let

$$W = \{x \in \mathbb{R}^{n+1};\ (x, v_i) = c_i \text{ for } i = 1, 2, \cdots, d\}.$$

Then, either $\{x \in \mathbb{R}^{n+1};\ (x, v_i) = c_i \text{ for } i = 1, 2, \cdots, k\} = W$ or $= \emptyset$. In the latter case, we have $A = \emptyset$. Assume the former case.

Let $w \in W$ be such that $||w|| = \min\{||x||;\ x \in W\}$. Then, it holds that

$$\begin{cases} w = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_d v_d \\ (w, v_i) = c_i \quad (i = 1, 2, \cdots, d). \end{cases}$$

Hence, we have

$$\sum_{j=1}^{d} (v_i, v_j)\lambda_j = c_i \quad (i = 1, 2, \cdots, d),$$

and $\mathbf{A}\lambda' = \mathbf{c}'$. where $\lambda = (\lambda_1, \lambda_2, \cdots \lambda_d)$. Therefore, $\lambda' = \mathbf{A}^{-1}\mathbf{c}'$.

Denoting $\mathbf{A} = (a_{ij})_{i,j=1,\cdots,d}$ and $\mathbf{A}^{-1} = (a^{ij})_{i,j=1,\cdots,d}$, we have

$$w = \sum_j a^{1j} c_j v_1 + \sum_j a^{2j} c_j v_2 + \cdots + \sum_j a^{dj} c_j v_d. \tag{1}$$

Hence,

$$\begin{aligned} ||w||^2 &= \sum_i \sum_{i'} \sum_j \sum_{j'} a^{ij} c_j a^{i'j'} c_{j'} (v_i, v_{i'}) \\ &= \sum_i \sum_{i'} \sum_j \sum_{j'} a^{ij} c_j a^{i'j'} c_{j'} a_{ii'} \\ &= \sum_{i'} \sum_j \sum_{j'} c_j a^{i'j'} c_{j'} \sum_i a^{ij} a_{ii'} \\ &= \sum_{i'} \sum_j \sum_{j'} c_j a^{i'j'} c_{j'} \delta_{i'j} \\ &= \sum_{i'} \sum_{j'} c_{i'} a^{i'j'} c_{j'} = \mathbf{c}\mathbf{A}^{-1}\mathbf{c}'. \end{aligned}$$

Note that W is the $(n-d+1)$-dimensional plane with $w \in W$ orthogonal to $\vec{Ow}$, where O is the origin. Moreover, $A = W \cap \{x \in \mathbb{R}^{n+1};\ ||x|| = 1\}$. There are 3 cases.

Case 1: If $||w|| > 1$, then $A = \emptyset$.

Case 2: If $||w|| = 1$, then $\#A = 1$.

Case 3: If $||w|| < 1$, then A is an $(n-d)$-dimensional sphere.

Thus, we complete the proof. □

Definition 3. Let $\mathcal{A} \subset \mathcal{D}_n$ with $\#\mathcal{A} \geq 1$ and $n \geq 1$. We call $\mathcal{A}$ a *general position* if for any subset $\{D_n(v_i, c_i);\ i = 1, 2, \cdots, l\}$ of $\mathcal{A}$ with cardinality l such that $1 \leq l \leq n+1$, it holds that either $\bigcap_{i=1}^{l} \partial D_n(v_i, c_i)$ is an $(n-l)$-dimensional sphere, or the empty set, where a 0-dimensional sphere is a two-points set and (-1)-dimensional sphere is the empty set. It is called *intersecting* if the former case always holds.

Definition 4. For a family $\mathcal{A}$ of subsets of $\mathcal{S}^n$, $\mathbb{P}(\mathcal{A})$ denotes the partition of $\mathcal{S}^n$ generated by the sets in $\mathcal{A}$. That is,

$$\mathbb{P}(\mathcal{A}) = \{\bigcap_{A \in \mathcal{A}} A^{i_A} \neq \emptyset;\ i_A \in \{0, 1\} \text{ is chosen for each } A \in \mathcal{A}\},$$

where we denote $A^1 = A$ and $A^0 = A^c = \mathcal{S}^n \setminus A$. We also denote by $\mathbb{Q}(\mathcal{A})$ the set of connected components of $\mathcal{S}^n \setminus \bigcup_{A \in \mathcal{A}} \partial A$.

Lemma 5. *Let $\mathcal{A} \subset \mathcal{D}_n$ with $n \geq 1$ and $\#\mathcal{A} \geq 1$ be a general position. Then, every C in $\mathbb{P}(\mathcal{A})$ satisfies that $\overline{C} = \overline{C^I}$, where C^I is the set of interior points of C, and $\overline{C}$ is the closure of C.*

Proof Let $x_0 \in C \in \mathbb{P}(\mathcal{A})$ and $C = \bigcap_{A \in \mathcal{E}} A \cap \bigcap_{A \in \mathcal{A} \setminus \mathcal{E}} A^c$ for $\mathcal{E} \subset \mathcal{A}$. Let

$$\mathcal{C} := \{A \in \mathcal{E};\ x_0 \in \partial A\} = \{D_n(v_i, c_i);\ i = 1, 2, \cdots, d\}.$$

If $d = 0$, then x_0 is an interior point of C. Assume that $d \geq 1$.

Since $\mathcal{A}$ is a general position, $d \leq n$ holds, and $v_1, v_2, \cdots, v_d$ are linearly independent with $\mathbf{c}\mathbf{A}^{-1}\mathbf{c}' < 1$, where $\mathbf{A} := ((v_i, v_j))_{i,j=1,\cdots,d}$ and $\mathbf{c} := (c_1, \cdots, c_d)$. Since $v_1, v_2, \cdots, v_d$ are linearly independent,

$$\bigcap_{i=1}^{d} \{x \in \mathbb{R}^{n+1};\ (x, v_i) > c_i\}$$

is the interior of an n-dimensional polygon. Moreover, since $\mathbf{c}\mathbf{A}^{-1}\mathbf{c}' < 1$,

$$\bigcap_{A \in \mathcal{C}} A^I = \mathcal{S}^n \cap \bigcap_{i=1}^{d} \{x \in \mathbb{R}^{n+1};\ (x, v_i) > c_i\}$$

is a nonempty open set in $\mathcal{S}^n$. Since x_0 belongs to an open set

$$\bigcap_{A \in \mathcal{E} \setminus \mathcal{C}} A^I \cap \bigcap_{A \in \mathcal{A} \setminus \mathcal{E}} A^c,$$

$\bigcap_{A\in\mathcal{E}} A^I \cap \bigcap_{A\in\mathcal{A}\setminus\mathcal{E}} A$ is a nonempty open set contained in C, and x_0 is in its closure, which completes the proof. $\square$

Lemma 6. *For $\mathcal{E} \subset \overline{\mathcal{D}}_n$ with $n \geq 1$ and $\#\mathcal{E} = k \geq 1$, there exists a general position $\mathcal{A}$ such that $\#\mathcal{A} = \#\mathcal{E}$ and $\#\mathbb{P}(\mathcal{A}) \geq \#\mathbb{P}(\mathcal{E})$.*

Proof Denote each degenerate disk, say $\{x\} \in \mathcal{E}$, by $D(x,1)$. Let $\mathcal{E} = \{D_n(v_i, c_i),\ i = 1, 2, \cdots, k\}$. For each $C \in \mathbb{P}(\mathcal{E})$, choose $x_C \in C$. There exists $\epsilon > 0$ such that if $C = \bigcap_{i\in\mathcal{C}} D_n(v_i, c_i) \bigcap \bigcap_{i\in\{1,2,\cdots,k\}\setminus\mathcal{C}} D_n(v_i, c_i)^c$, then

$$x_C \in \bigcap_{i\in\mathcal{C}} D_n(v_i, c_i - \epsilon)^I \bigcap \bigcap_{i\in\{1,2,\cdots,k\}\setminus\mathcal{C}} D_n(v_i, c_i - \epsilon)^c.$$

Hence, each x_C is an interior point of the corresponding element of the partition $\mathbb{P}(\{D_n(v_i, c_i - \epsilon);\ i = 1, 2, \cdots, k\})$.

We can also move v_i's slightly to u_i's, keeping x_C's separated so that $\mathcal{A} := \{D_n(u_i, c_i - \epsilon),\ i = 1, 2, \cdots, k\}$ is a general position. Then, $\#\mathcal{A} = k$ and $\#\mathbb{P}(\mathcal{A}) \geq \#\mathbb{P}(\mathcal{E})$, which complete the proof. $\square$

Lemma 7. *Let $\mathcal{A} \subset \mathcal{D}_n$ be a general position. Then, $\#\mathbb{P}(\mathcal{A}) \leq \#\mathbb{Q}(\mathcal{A})$ holds.*

Proof If C and C' are distinct elements in $\mathbb{P}(\mathcal{A})$. By Lemma 5, $C^I \neq \emptyset$ and $C'^I \neq \emptyset$. Take $x \in C^I$ and $x' \in C'^I$. There exists $A = D_n(v, c) \in \mathcal{A}$ such that either $C \subset A$ and $C' \cap A = \emptyset$ or $C' \subset A$ and $C \cap A = \emptyset$. Assume the former case without loss of generality. Then, $(x, v) > c$ and $(x', v) < c$. Thus, x and x' belong to different elements in $\mathbb{Q}(\mathcal{A})$. This implies that $\#\mathbb{P}(\mathcal{A}) \leq \#\mathbb{Q}(\mathcal{A})$. $\square$

Definition 8. Let $A = D_n(v, c)$ with $n \geq 2$. Let $\mathcal{D}_n^A$ be the set of $B \in \mathcal{D}_n$ such that $\partial A \cap \partial B$ is an $(n-2)$-dimensional sphere. Let $E_{v,c} := \{x \in \mathbb{R}^{n+1};\ (x, v) = c\}$ and the space of elements $x - cv$ with $x \in E_{v,c}$ is identified with $\mathbb{R}^n$. Let $\phi_A : E_{v,c} \to \mathbb{R}^n$ be such that $\phi_A(x) = \dfrac{x - cv}{\sqrt{1 - c^2}}$.

Lemma 9. *Let $A = D_n(v, c)$.*
(1) *The map $\phi_A : E_{v,c} \to \mathbb{R}^n$ is an affine map such that for $D_n(w, d) \in \mathcal{D}_n^A$, $\phi_A(D_n(w, d) \cap E_{v,c}) = D_{n-1}(w', d')$ with*

$$\begin{cases} w' = \frac{w-(v,w)v}{\sqrt{1-(v,w)^2}} \\ d' = \frac{d-c(v,w)}{\sqrt{1-c^2}\sqrt{1-(v,w)^2}} \end{cases}$$

(2) *Let $\mathcal{A} \subset \mathcal{D}_n$ be a general position with $\#\mathcal{A} \geq 2$ and $A \in \mathcal{A}$. Then $\{\phi_A(B \cap E_{v,c});\ B \in \mathcal{A} \cap \mathcal{D}_n^A\}$ is a general position in $\mathcal{D}^{n-1}$.*

Proof (1) Let $B = D_n(w, d) \in \mathcal{D}_n^A$. It is clear that $w' \in \mathbb{R}^n$ and $||w'|| = 1$. Since $\partial A \cap \partial B$ is an $(n-2)$-dimensional sphere, v and w are linearly independent and $\mathbf{cA}^{-1}\mathbf{c}' < 1$ by Lemma 2, where $\mathbf{c} = (c, d)$ and $\mathbf{A} = \begin{pmatrix} 1 & (v,w) \\ (v,w) & 1 \end{pmatrix}$. Therefore,

$$\mathbf{cA}^{-1}\mathbf{c}' = \frac{c^2 + d^2 - 2cd(v,w)}{1 - (v,w)^2} < 1,$$

and hence,

$$c^2 + d^2 - 2cd(v,w) < 1 - (v,w)^2.$$

This implies that

$$0 \leq c^2(v,w)^2 + d^2 - 2cd(v,w) < (1-c^2)(1-(v,w)^2),$$

and hence,

$$d'^2 = \frac{c^2(v,w)^2 + d^2 - 2cd(v,w)}{(1-c^2)(1-(v,w)^2)} < 1.$$

Thus, we have $-1 < d' < 1$.

Let $y \in D_n(w,d) \cap E_{v,c}$. Then, we have $(y,v) = c$ and $(y,w) \geq d$. For $y' = \phi_A(y)$, we have

$$\begin{aligned} (y', w') &= \left(\frac{y - cv}{\sqrt{1-c^2}}, \frac{w - (v,w)v}{\sqrt{1-(v,w)^2}} \right) \\ &\geq \frac{d - c(v,w)}{\sqrt{1-c^2}\sqrt{1-(v,w)^2}} = d'. \end{aligned}$$

Therefore, $\phi_A(D_n(v,c) \cap E_{v,c}) \subset D_{n-1}(w', d')$.

Conversely, let $y' \in \mathbb{R}^n$ satisfy that $||y'|| = 1$ and $(y', w') \geq d'$. Note that the space $\mathbb{R}^n$ is identified with the space in $\mathbb{R}^{n+1}$ orthogonal to v. Hence, $(y', v) = 0$ holds. Let $y = \sqrt{1-c^2}y' + cv \in E_{v,c}$. Then, we have $\phi_A(y) = y'$. Moreover,

$$\begin{aligned} (y,v) &= \sqrt{1-c^2}(y',v) + c(v,v) = c \\ (y,w) &= \sqrt{1-c^2}(y',w) + c(v,w) \\ &= \sqrt{1-c^2}\left(y', \sqrt{1-(v,w)^2}w' + (v,w)v\right) + c(v,w) \\ &\geq \sqrt{1-c^2}\sqrt{1-(v,w)^2}d' + c(v,w) \\ &= \sqrt{1-c^2}\sqrt{1-(v,w)^2}\frac{d - c(v,w)}{\sqrt{1-c^2}\sqrt{1-(v,w)^2}} + c(v,w) \\ &= d - c(v,w) + c(v,w) = d. \end{aligned}$$

Therefore, $y' \in \phi_A(D_n(v,c) \cap E_{v,c})$, and hence,

$$\phi_A(D_n(v,c) \cap E_{v,c}) \supset D_{n-1}(v', d'),$$

which completes the proof of (1).

(2) follows from (1) and the definition of a general position. □

Lemma 10. *Let $\mathcal{A} \subset \mathcal{D}_n$ be a general position with $n \geq 1$ and $\#\mathcal{A} = k \geq 1$. Then, $\#\mathbb{Q}(\mathcal{A}) \leq 2\sum_{i=0}^{n} \binom{k-1}{i}$, where we define $\binom{k}{i} = 0$ if $k < i$. Moreover, the equality holds if $\mathcal{A}$ is intersecting.*

Proof Let $\mathcal{A} \subset \mathcal{D}_n$ be a general position with $n \geq 1$ and $\#\mathcal{A} = k \geq 1$. If $k = 1$, then $\#\mathbb{Q}(\mathcal{A}) = 2$ and our lemma holds. Moreover, if $n = 1$, then ∂A is a two-point set for any $A \in \mathcal{A}$, and $\partial A \cap \partial B = \emptyset$ for any distinct elements $A, B \in \mathcal{A}$ since $\mathcal{A}$ is a general position. Therefore, $\mathcal{S}^1 \setminus \cup_{A\in\mathcal{A}} \partial A$ has $2k$ connected components. Thus, our lemma holds for $n = 1$, since

$$\#\mathbb{P}(\mathcal{A}) = 2k = 2\sum_{i=0}^{1} \binom{k-1}{i} \quad (k = 1, 2, \cdots).$$

We use induction on k. Let $k \geq 2$ and assume that our lemma holds with $1, 2, \cdots, k-1$ in place of k. Assume also that $n \geq 2$. Take $A \in \mathcal{A}$. Then, $\{\phi_A(B \cap E_{v,c});\ B \in \mathcal{A} \cap \mathcal{D}_n^A\}$ is a general position by Lemma 9, which is intersecting if $\mathcal{A}$ is intersecting. Let $l = \#(\mathcal{A} \cap \mathcal{D}_n^A)$. Then, $l \leq k-1$ and the equality holds if $\mathcal{A}$ is intersecting.

Let $C \in \mathbb{Q}(\mathcal{A} \setminus \{A\})$. We count the number of elements in $\mathbb{Q}(\mathcal{A})$, $\mathbb{Q}(\mathcal{A} \setminus \{A\})$ and $\mathbb{Q}(\{\phi_A(B \cap E_{v,c});\ B \in \mathcal{A} \cap \mathcal{D}_n^A\})$ coming from C, say p, q and r. There are 2 cases.

Case 1: If $C \cap \partial A = \emptyset$, then $p = 1$, $q = 1$ and $r = 0$.

Case 2: If $C \cap \partial A \neq \emptyset$, then $C \cap \partial A$ is divided into connected components, which are elements in $\mathbb{Q}(\{\phi_A(B \cap E_{v,c});\ B \in \mathcal{A} \cap \mathcal{D}_n^A\})$. Let them be $\ell_1, \ell_2, \cdots, \ell_t$. Then, ℓ_1 divides C into 2 connected components, say C_1 and C_2, and ℓ_2 divides one of C_1 and C_2 into 2 connected components, and hence, C is divided into 3 connected components by $\ell_1 \cup \ell_2$. In the same way, C is divided into $t+1$ connected components by $\ell_1 \cup \ell_2 \cup \cdots \cup \ell_t$. These connected components are elements of $\mathbb{Q}(\mathcal{A})$.

Thus, $p = t+1$, $q = 1$ and $r = t$.

Hence, using the induction hypothesis, we have

$$\begin{aligned}
\#\mathbb{Q}(\mathcal{A}) &= \sum_C p = \sum_C (q + r) \\
&= \#\mathbb{Q}(\mathcal{A} \setminus \{A\}) + \#\mathbb{Q}(\{\phi_A(B \cap E_{v,c});\ B \in \mathcal{A} \cap \mathcal{D}_n^A\}) \\
&\leq 2\sum_{i=0}^{n} \binom{k-2}{i} + 2\sum_{i=0}^{n-1} \binom{l-1}{i} \qquad (2) \\
&\leq 2\sum_{i=0}^{n} \binom{k-2}{i} + 2\sum_{i=0}^{n-1} \binom{k-2}{i} \qquad (3) \\
&= 2\sum_{i=0}^{n} \binom{k-1}{i}.
\end{aligned}$$

In the above, if $\mathcal{A}$ is intersecting, then $\{\phi_A(B \cap E_{v,c});\ B \in \mathcal{A} \cap \mathcal{D}_n^A\}$ is also intersecting, and hence, the equality in (2) holds by the induction hypothesis. Moreover, since $l = k-1$ holds in this case, the equality in (3) holds. Thus, the equality holds if $\mathcal{A}$ is intersecting. □

Definition 11. Let $\mathcal{A} \subset \mathcal{S}^n$ be a general position with $n \geq 1$ and $\#\mathcal{A} = k \geq 1$. It is called *connected* if any C in $\mathbb{P}(\mathcal{A})$ is connected.

Lemma 12. *For any $\mathcal{A} \subset \mathcal{D}^n$ with $n \geq 1$ and $\#\mathcal{A} \geq 1$, we have*

$$\#\mathbb{P}(\mathcal{A}) \leq 2\sum_{i=0}^{n} \binom{k-1}{i}.$$

Moreover, the equality holds if $\mathcal{A}$ is a connected, intersecting general position.

Proof The inequality follows from Lemma 6, 7 and 10. The equality in the case that $\mathcal{A}$ is a connected, intersecting general position follows from Lemma 10, since $\#\mathbb{P}(\mathcal{A}) = \#\mathbb{Q}(\mathcal{A})$. □

3 Existence of an optimal position

Let $B_n(v,r) = \{x \in \mathbb{R}^n;\ ||x-v|| \leq r\}$ be the n-dimensional ball with center $v \in \mathbb{R}^n$ and radius $r > 0$. Let $\mathcal{B}_n$ be the set of all such balls.

Definition 13. A nonempty finite subset $\mathcal{A}$ of $\mathcal{B}_n$ is called a *general position* if for any subset $\{B_n(v_i,r_i);\ i = 1,2,\cdots,l\}$ of $\mathcal{A}$ with cardinality l such that $1 \leq l \leq n+1$, it holds that either $\bigcap_{i=1}^{l} \partial B_n(v_i,r_i)$ is an $(n-l)$-dimensional sphere, or the empty set, where a 0-dimensional sphere is a two-point set and a (-1)-dimensional sphere is defined to be the empty set. It is called *intersecting* if the former case holds always. Moreover, a general position $\mathcal{A}$ is called *connected* if any C in $\mathbb{P}(\mathcal{A})$ is connected.

A bounded region of $\mathbb{R}^n$ can be imbedded into a small region of $\mathcal{S}^n$ by a map which is close enough to a homothetic map. By this map, unit balls in $\mathbb{R}^n$ are mapped to small, slightly perturbed disks with almost uniform sizes in $\mathcal{S}^n$. Moreover, a general position $\mathcal{A} \subset \mathcal{B}_n$ corresponds to a general position $\mathcal{A}' \subset \mathcal{D}_n$ having the same configuration as $\mathcal{A}$.

This correspondence gives the following theorem.

Theorem 14. *For any $\mathcal{A} \subset \mathcal{B}_n$ with $n \geq 1$ and $\#\mathcal{A} = k \geq 1$, we have*

$$\#\mathbb{P}(\mathcal{A}) \leq 2\sum_{i=0}^{n} \binom{k-1}{i}.$$

Moreover, the equality holds if $\mathcal{A}$ is a connected and intersecting general position.

Theorem 15. *Take $n \geq 2$ and any δ with $0 < \delta < 1$. Let*

$$\mathcal{A} = \{B_n(v_1, 1), B_n(v_2, 1), \cdots, B_n(v_k, 1)\}$$

with $||v_i|| = \delta$ for any $i = 1, 2, \cdots, k$. Assume that if $k \leq n$, then $v_1, v_2, \cdots, v_k$ are not on a plane of dimension $k-2$, and if $k \geq n+1$, then any subset of $v_1, v_2, \cdots, v_k$ with cardinality $n+1$ is not on a hyper-plane. Then, $\mathcal{A}$ is a connected and intersecting general position.

Proof Take any subset of $\mathcal{A}$ with cardinality l such that $2 \leq l \leq n$. Without loss of generality, let it be $\{B_n(v_1, 1), B_n(v_2, 1), \cdots, B_n(v_l, 1)\}$. By our assumption, $v_1, v_2, \cdots, v_l$ are not on a plane of dimension $l-2$, but exactly on a $(l-1)$-dimensional plane. Hence, they are on a $(l-2)$-dimensional sphere, say with center $u \in \mathbb{R}^n$ and radius ρ, such that $||u|| < \delta$ and $\rho = \sqrt{\delta^2 - ||u||^2}$. Hence, $\bigcap_{i=1}^{l} \partial B_n(v_i, 1)$ is a $(n-l)$-dimensional sphere

$$\{x \in \mathbb{R}^n;\ ||x-u|| = \sqrt{1-\delta^2 + ||u||^2},\ (x-u, v_i - u) = 0\ (i = 1, \cdots, l-1)\}.$$

If $k \geq n+1$, then take any subset of $\mathcal{A}$ with cardinality $n+1$. Let it be

$$\{B_n(v_1, 1), B_n(v_2, 1), \cdots, B_n(v_{n+1}, 1)\}$$

without loss of generality. Suppose that $\bigcap_{i=1}^{n+1} \partial B_n(v_i, 1)$ is nonempty. Let w be an element of it. Then, we have

$$\begin{aligned}\{v_1, v_2, \cdots, v_{n+1}\} &\subset \{x \in \mathbb{R}^n;\ ||x|| = \delta\} \cap \{x \in \mathbb{R}^n;\ ||x - w|| = 1\} \\ &\subset \{x \in \mathbb{R}^n;\ (x, w) = (||w||^2 + \delta^2 - 1)/2\},\end{aligned}$$

which contradicts our assumption.

Thus, $\mathcal{A}$ is an intersecting general position.

To prove that $\mathcal{A}$ is connected, take any $C \in \mathbb{P}(\mathcal{A})$. Let

$$C = \bigcap_{A \in \mathcal{E}} A \cap \bigcap_{A \in \mathcal{A} \setminus \mathcal{E}} A^c$$

with some $\mathcal{E} \subset \mathcal{A}$. Without loss of generality, assume that

$$\mathcal{E} = \{B(v_i, 1);\ i = 1, 2, \cdots, d\}$$

with $0 \leq d \leq k$. For $i, j = 1, 2, \cdots, k$ with $i \neq j$, let

$$R_{i,j} = \{x \in \mathbb{R}^n;\ ||x - v_i|| < ||x - v_j||\},$$

which is an open half space bounded by the hyper-plane $\partial R_{i,j}$ containing the origin O. Denote

$$\mathcal{R} = \bigcap_{i \leq d,\ j > d} R_{i,j}.$$

Then, $\mathcal{R}$ is an open polygonal cone with focus O containing C^I.

Let $\mathcal{L}$ be the set of unit vectors u in $\mathbb{R}^n$ such that $\{tu;\ t > 0\} \subset \mathcal{R}$. Define functions F and f from $\mathcal{L}$ to $\mathbb{R}_+$ such that

$$\begin{aligned} F(u) &= \inf\{t > 0;\ ||tu - v_i|| > 1 \text{ for some } i = 1, \cdots, d\} \\ f(u) &= \sup\{t > 0;\ ||tu - v_i|| \leq 1 \text{ for some } i = d+1, \cdots, k\}. \end{aligned}$$

Then, it is clear that $f(u) < F(u)$ for any $u \in \mathcal{L}$ and

$$C^I = \{x \in \mathbb{R}^n;\ x = tu \text{ for some } u \in \mathcal{L} \text{ and } t \text{ with } f(u) < t < F(u)\}.$$

Moreover, since F and f are continuous function, C^I is connected, and hence, C is connected since $C^I \subset C \subset \overline{C^I}$. Thus, $\mathcal{A}$ is connected. □

Corollary 16. *Let $X = G = \mathbb{R}^n$ $(n \geq 1)$ and the action $g \in G$ to $x \in X$ is the translation $x - g$. If $D = B_n(0,1)$ is the n-dimensional unit ball, then*

$$p^*_{X,G,D}(k) = 2\sum_{i=0}^{n} \binom{k-1}{i}.$$

Moreover, if $n \geq 2$ and Σ is an infinite subset of G such that for some δ with $0 < \delta < 1$, $\Sigma \subset \{x \in \mathbb{R}^n;\ ||x|| = \delta\}$, and that any subset of Σ with cardinality $n+1$ is not on a hyper-plane, then Σ is an optimal position of the triple (X, G, D). Also, if $n = 1$ and $\Sigma \subset (-1, 1)$, then Σ is an optimal position.

4 Non-existence of an optimal position

Let $X = G = \mathbb{R}^2$ and the action $g \in G$ to $x \in X$ is the translation $x - g$. If $D = [0,1] \times [0,1]$ is the unit square, then the same result as for the unit disk holds for the maximal pattern complexity, that is, $p^*_{X,G,D}(k) = k^2 - k + 2$. But in this case, an optimal position does not exist.

To get $p^*_{X,G,D}(k) = k^2 - k + 2$, we put translations of the unit square, say $D_1,\ D_2,\ \cdots$, one by one in the following way, where we let V_n denote the union of the vertices of $D_1,\ D_2,\ \cdots,\ D_n$.

(i) D_1 is put anywhere. D_2 is put so that one of its vertices is in the interior of D_1. Hence, $D_1 \cap D_2$ is a rectangle such that just two of its vertices situated diagonally are elements in V_2.

(ii) Assume that D_i $(i = 1, 2, \cdots, 2n)$ have been already put so that $\cap_{i=1}^{2n} D_i$ is a rectangle such that just two of its vertices situated diagonally, say u and v, are elements in V_{2n}. Put D_{2n+1} so that one of its vertices is in the interior of $\cap_{i=1}^{2n} D_i$ and it does not contain any of u and v. Then, $\cap_{i=1}^{2n+1} D_i$ is a rectangle such that just one of its vertices, say w, belongs to V_{2n+1}. Put D_{2n+2} so that one of its vertices is in the interior of $\cap_{i=1}^{2n+1} D_i$ and it contains

w. Then, $\cap_{i=1}^{2n+2} D_i$ is a rectangle such that just two of its vertices situated diagonally belongs to V_{2n+2}.

(iii) Repeat (ii) with $n+1$ instead of n.

This choice of the translations of the unit square, called $\mathcal{A}$, realizes the maximal pattern complexity $p^*_{X,G,D}(k) = k^2 - k + 2$. On the other hand, the triple (X, G, D) does not admit an optimal position. We prove them in the following lemmas.

Lemma 17. *For the above triple* (X, G, D)*, it holds that* $p^*_{X,G,D}(k) = k^2 - k + 2$*, and the above* $\mathcal{A} := \{D_i;\ i = 1, 2, \cdots, k\}$ *satisfies that* $\#\mathbb{P}(\mathcal{A}) = \#\mathbb{Q}(\mathcal{A}) = k^2 - k + 2 \quad (k = 1, 2, \cdots)$.

Proof Let $\mathcal{B}$ consist of k translations of D such that any two of their boundaries intersect exactly at 2 points. Then, we have $\#\mathbb{Q}(\mathcal{B}) = k^2 - k + 2$. For any $\mathcal{E} \subset \{D + g;\ g \in \mathbb{R}^2\}$ with $\#\mathcal{E} = k$, by the same arguments as the proofs of Lemma 6 and 7, there exists $\mathcal{B}$ as above such that $\#\mathbb{P}(\mathcal{E}) \leq \#\mathbb{P}(\mathcal{B}) \leq \#\mathbb{Q}(\mathcal{B}) = k^2 - k + 2$. Hence, we have $p^*_{X,G,D}(k) \leq k^2 - k + 2$.

By the above argument, we have $\#\mathbb{P}(\mathcal{A}) \leq \#\mathbb{Q}(\mathcal{A}) = k^2 - k + 2$. To prove $\#\mathbb{P}(\mathcal{A}) = k^2 - k + 2$, it is sufficient to prove $\#\mathbb{P}(\mathcal{A}) = \#\mathbb{Q}(\mathcal{A})$. Hence, it is sufficient to prove that any element in $\mathbb{P}(\mathcal{A})$ is connected. We prove this by induction on k. This is clear for $k = 1, 2$. For $k = 3$, this holds since we have the right configuration in Figure 1, but not the left by our construction.

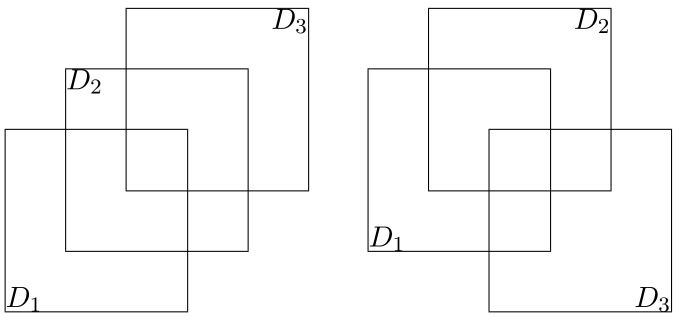

Figure 1: We have the right configuration, not the left.

Assume that this is true for $1, 2, \cdots, k-1$ for some $k \geq 4$. Since one of the vertices, say left and down, of D_k is in the interior of $\cap_{i=1}^{k-1} D_i$, only 2 neighboring edges of D_k, intersect with $\cup_{i=1}^{k-1} D_i$. Let $C \in \mathbb{P}(\{D_i;\ i = 1, 2, \cdots, k-1\})$. By the induction hypothesis, C is connected. If $C \cap \partial D_k$ is connected, then ∂D_k divides C into 2 connected region belonging to distinct elements in $\mathbb{P}(\{D_i;\ i = 1, 2, \cdots, k\})$. Therefore, it is sufficient to prove that this is the only possible case.

Suppose that $C \cap \partial D_k$ is not connected. Then, C should be a region extending vertically and horizontally at the same time and intersecting with

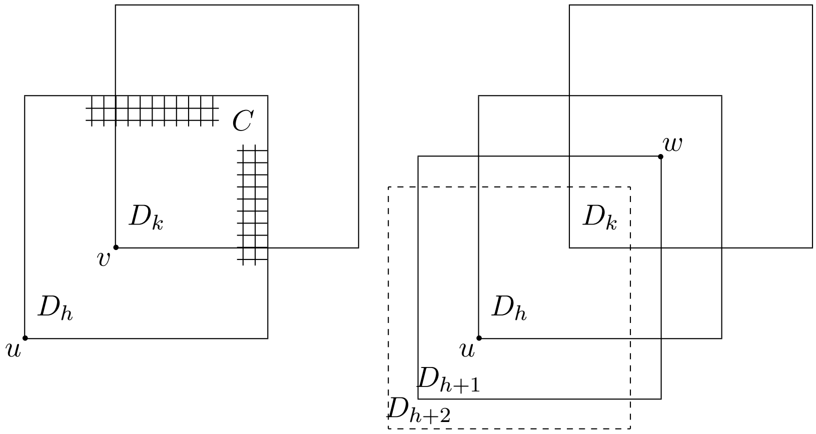

Figure 2: The region C and the configurations.

2 edges of D_k. Then, there exists D_h in $\{D_i;\ i = 1, 2, \cdots, k-1\}$ which is located most right and most up at the same time, since otherwise, there is no C as above. Hence, we have the situation in the left of Figure 2.

If $k = h+1$, then u is a vertex of $\cap_{i=1}^{k-1} D_i$. In this case, we do not put D_k as in the left of Figure 2 in our construction, and have a contradiction. If $k \geq h+2$, then for any l with $h < l < k$, D_l should contain v by our construction. On the other hand, since D_h is more right and more up than D_l by the assumption on D_h, the left-down vertex of D_l should be more left and more down than u, which implies that the right-up vertex of D_l is in the interior of $D_h \cap D_k$. But in this case, we have the configuration in the right of Figure 2, which contradicts our choice of D_{h+2} in our construction. Hence, we have $k = h+2$ and the configuration should be in the right of Figure 2. In this case, $\cap_{i=1}^{k-1} D_i$ has 2 diagonally situated vertices in V_{k-1}, u and w, while we should not put D_k like this in our construction.

In any case, we have a contradiction coming from the supposition that $C \cap \partial D_k$ is not connected. Hence, it is connected. Thus, each $C \in \mathbb{P}(\mathcal{A})$ is connected, which completes the proof. $\square$

Lemma 18. *The above triple (X, G, D) does not admit an optimal position.*

Proof Take any infinite subset Σ of G.

If there exist 3 distinct elements (u_1, v_1), (u_2, v_2), (u_3, v_3) in Σ such that
(1) either $u_1 \leq u_2 \leq u_3$ or $u_1 \geq u_2 \geq u_3$, and
(2) either $v_1 \leq v_2 \leq v_3$ or $v_1 \geq v_2 \geq v_3$,
then, we have $A_1 \cap A_2^c \cap A_3 = \emptyset$ with $A_i = D + (u_i, v_i)$ $(i = 1, 2, 3)$. Hence, $\#\mathbb{P}(A_1, A_2, A_3) \leq 7$, and $\{A_1, A_2, A_3\}$ dose not attain $p^*_{X,G,D}(3) = 8$.

We prove that Σ contains 3 distinct elements as above. Take any 5 distinct elements in Σ, say (u_i, v_i) $(i = 1, 2, 3, 4, 5)$ with $u_1 \le u_2 \le u_3 \le u_4 \le u_5$. Let $v_h = \min_i v_i$ and $v_k = \max_i v_i$ for some $h, k = 1, 2, 3, 4, 5$. If $h+1 < k$ or $k+1 < h$, then $(u_h, v_h),\ (u_l, v_l),\ (u_k, v_k)$ satisfies the above condition with $l = h+1 < k$ or $l = k+1 < h$. Otherwise, then either $\{1, 2\} \cap \{h, k\} = \emptyset$ or $\{4, 5\} \cap \{h, k\} = \emptyset$ holds. Assume the former without loss of generality. If $v_1 \le v_2$, then $(u_1, v_1),\ (u_2, v_2),\ (u_k, v_k)$ satisfies the above condition. If $v_1 \ge v_2$, then $(u_1, v_1),\ (u_2, v_2),\ (u_h, v_h)$ satisfies the above condition.

Thus, there exist 3 distinct elements as above, which completes the proof.
□

5 Primitive factor

Let (X, G, D) be the triple in Corollary 16 with $n \ge 1$ and Σ satisfy the conditions in Corollary 16. Let $\Omega \subset \{0, 1\}^\Sigma$ be the name set with respect to the optimal position Σ of the triple (X, G, D). That is, Ω is the closure of $\{\omega_q;\ q \in \mathbb{R}^n\}$, where $\omega_q \in \{0, 1\}^\Sigma$ is such that

$$\omega_q(v) = \begin{cases} 1 & (q \in B_n(v, 1)) \\ 0 & (q \notin B_n(v, 1)) \end{cases} \quad (v \in \Sigma). \tag{4}$$

Theorem 19. *The above uniform set Ω has the unique primitive factor $[\Xi]$, where $\Xi = \mathcal{P}((01)^l 0) \cup \mathcal{P}((10)^l 1)$ if $n = 2l$ is even, and $\Xi = \mathcal{P}((01)^{l+1}) \cup \mathcal{P}((10)^{l+1})$ if $n = 2l+1$ is odd.*

Proof We sometimes identify $x \in \mathbb{R}^n$ with the vector $\vec{Ox}$, where O is the origin. Let $\mathcal{S}_1 := \{x \in \mathbb{R}^n;\ ||x|| = 1\}$.

Our theorem was essentially proved in [3] for the case $n = 1$.

Assume that $n \ge 2$. It is sufficient to prove that for any infinite subset Σ' of Σ, there exists an injection $\psi : \mathbb{N} \to \Sigma'$ satisfying that $\Omega' \circ \psi = \Xi$, where Ω' is the name set with respect to Σ'. For simplicity, we denote this Σ' by Σ and Ω' by Ω.

There exist accumulation points of Σ. Take one of them and call it r_0. For $v \in \mathcal{S}_1$ and ϵ with $0 < \epsilon < 1/4$, let

$$\Theta(v, \epsilon) = \{w \in \mathbb{R}^n;\ (w, v) > (1 - \epsilon)||w||\}.$$

Take $\epsilon_0,\ \epsilon_1,\ \cdots,\ \epsilon_{n-1}$ with $0 < \epsilon_0, \epsilon_1, \cdots, \epsilon_{n-1} < 1/4$ arbitrary.

Since r_0 is an accumulation point of Σ, $(\Sigma - r_0) \cap \{x \in \mathbb{R}^n;\ ||x|| < \epsilon\}$ is an infinite set for any $\epsilon > 0$. Let $\Sigma_0(\epsilon_0) = (\Sigma - r_0) \cap \{x \in \mathbb{R}^n;\ ||x|| < \epsilon_0\}$.

There exists $v_1 \in \mathcal{S}_1$ such that $\Sigma_0(\epsilon) \cap \Theta(v_1, \eta)$ is an infinite set for any $\epsilon > 0$ and $\eta > 0$. Let $\Sigma_1(\epsilon_0, \epsilon_1) = \pi_1(\Sigma_0(\epsilon_0) \cap \Theta(v_1, \epsilon_1))$, where $\pi_1(w) = w - (w, v_1)v_1$ is the projection defined on $\Sigma_0(\epsilon_0) \cap \Theta(v_1, \epsilon_1)$.

Since there is no subset of Σ of cardinality $n+1$ which is on a hyper-plane, $\pi_1(w) = 0$ holds at most for n elements in $\Sigma_0(\epsilon_0) \cap \Theta(v_1, \epsilon_1)$. On the other hand, since $w \in \Sigma_0(\epsilon_0) \cap \Theta(v_1, \epsilon)$ implies that

$$||w - (w, v_1)v_1||^2 < 2\epsilon||w||^2,$$

there exist infinitely many $w \in \Sigma_0(\epsilon_0) \cap \Theta(v_1, \epsilon)$ such that

$$0 < ||\pi_1(w)||^2 < 2\epsilon ||w||^2$$

for any ϵ with $0 < \epsilon \le \epsilon_1$. This implies that there exist infinitely many distinct elements in $\Sigma_1(\epsilon_0, \epsilon_1)$.

Now, there exists $v_2 \in \mathcal{S}_1$ with $(v_1, v_2) = 0$ such that $\Sigma_1(\epsilon, \eta) \cap \Theta(v_2, \zeta)$ is an infinite set for any $\epsilon > 0$, $\eta > 0$ and $\zeta > 0$. Let

$$\Sigma_2(\epsilon_0, \epsilon_1, \epsilon_2) = \pi_2(\Sigma_1(\epsilon_0, \epsilon_1) \cap \Theta(v_2, \epsilon_2)),$$

where $\pi_2(w) = w - (w, v_2)v_2$ is the projection defined on $\Sigma_1(\epsilon_0, \epsilon_1) \cap \Theta(v_2, \epsilon_2)$.

If $n = 2$, then $\Sigma_2 = \{O\}$. If $n \ge 3$, then $\pi_2(w) = 0$ holds at most for n elements in $\Sigma_1(\epsilon_0, \epsilon_1) \cap \Theta(v_2, \epsilon_2)$. Then, by the same argument as above, there exist infinitely many distinct elements in $\Sigma_2(\epsilon_0, \epsilon_1, \epsilon_2)$.

In this way, we continue until we get $\Sigma_{n-1} := \Sigma_{n-1}(\epsilon_0, \epsilon_1, \cdots, \epsilon_{n-1})$ such that $\Sigma_{n-1}(\eta_0, \eta_1, \cdots, \eta_{n-1}) \cap \Theta(v_n, \epsilon)$ is an infinite set for any $\eta_0, \eta_1, \cdots, \eta_{n-1} > 0$ and $\epsilon > 0$, but $\pi_n(\Sigma_{n-1} \cap \Theta(v_n, \epsilon)) = \{O\}$. Let

$$\Theta := \Theta(\epsilon_0, \epsilon_1, \cdots, \epsilon_{n-1}) = \pi_1^{-1} \circ \pi_2^{-1} \circ \cdots \circ \pi_{n-1}^{-1} \Sigma_{n-1} \subset \Sigma - r_0.$$

Note that $v_1, v_2, \cdots, v_n$ are chosen independently of the choice of $\epsilon_0, \epsilon_1, \cdots, \epsilon_{n-1}$.

By the construction, Θ is an infinite set, and for any $w \in \Theta$,

$$w = (w, v_1)v_1 + (w, v_2)v_2 + \cdots + (w, v_n)v_n \ , \quad ||w|| < \epsilon_0$$

holds, and $v_1, v_2, \cdots, v_n$ are an orthonormal basis of $\mathbb{R}^n$. Moreover, we have

$$(w, v_j) > (1 - \epsilon_j)\sqrt{(w, v_j)^2 + (w, v_{j+1})^2 + \cdots + (w, v_n)^2}$$

for any $w \in \Theta$ and $j = 1, 2, \cdots, n$, since $(w, v_j)v_j + \cdots + (w, v_n)v_n \in \Theta(v_j, \epsilon_j)$. It follows that

$$\begin{aligned} &0 < (w, v_1) < \epsilon_0, \ \text{and} \\ &0 < (w, v_{j+1}) < 2\sqrt{\epsilon_j}\ (w, v_j) \quad (j = 1, 2, \cdots, n-1). \end{aligned} \tag{5}$$

We need the following lemma to complete the proof.

Lemma 20. *Let* $A = (a_{ij})_{i,j=0,1,2,\cdots,n}$ *be a matrix with* $n \ge 2$ *and*

$$\begin{aligned} a_{ij} &> 0 && (i, j = 0, 1, 2, \cdots, n), \\ a_{i,0} &= 1 && (i = 0, 1, 2, \cdots, n), \ \text{and} \\ \frac{a_{i'j'}}{a_{i'j}} &< \frac{1}{2(n+1)!}\frac{a_{ij'}}{a_{ij}} && (0 \le i < i' \le n, \ 0 \le j < j' \le n). \end{aligned} \tag{6}$$

Then, we have

$$\det A = (-1)^{n(n+1)/2} a_{0n} a_{1,n-1} \cdots a_{n0}(1 + \xi)$$

with $|\xi| < 1/2$. *Hence,* $\det A$ *has the same sign as* $(-1)^{n(n+1)/2}$.

Proof Assume (6). Let τ be a permutation on $\{0, 1, \cdots, n\}$. Assume that there exist u, v with $u < v$ and $\tau(u) < \tau(v)$. Let $\tau' = \tau(u, v)$, where (u, v) is the transposition. Then, we have

$$\frac{\prod_{i=0}^n a_{i\tau(i)}}{\prod_{i=0}^n a_{i\tau'(i)}} = \frac{a_{u\tau(u)}a_{v\tau(v)}}{a_{u\tau'(u)}a_{v\tau'(v)}} = \frac{a_{v\tau(v)}/a_{v\tau(u)}}{a_{u\tau(v)}/a_{u\tau(u)}} < \frac{1}{2(n+1)!} .$$

This implies that

$$0 < \prod_{i=0}^n a_{i\tau(i)} < \frac{1}{2(n+1)!} a_{0n}a_{1,n-1}\cdots a_{n0}$$

for any permutation τ other than $(n, n-1, \cdots, 1, 0)$. Hence, we have

$$\det A = (-1)^{n(n+1)/2} a_{0n}a_{1,n-1}\cdots a_{n0}(1+\xi)$$

with $|\xi| < 1/2$. □

Let us continue the proof of Theorem 19.

Choose ϵ_0^0, ϵ_1^0, $\cdots$, ϵ_{n-1}^0 with $0 < \epsilon_0^0, \epsilon_1^0, \cdots, \epsilon_{n-1}^0 < 1/4$ arbitrary and take $w_0 \in \Theta(\epsilon_0^0, \epsilon_1^0, \cdots, \epsilon_{n-1}^0)$. Let $a_{0j} = (w_0, v_j)$ $(j = 1, 2, \cdots, n)$ and $a_{00} = 1$. Then by (5), $a_{0j} > 0$ $(j = 1, 2, \cdots, n)$.

Choose ϵ_0^1, ϵ_1^1, $\cdots$, ϵ_{n-1}^1 with $0 < \epsilon_0^1, \epsilon_1^1, \cdots, \epsilon_{n-1}^1 < 1/4$ such that

$$\epsilon_0^1 < \frac{a_{01}}{2(n+1)!}\ , \quad 2\sqrt{\epsilon_j^1} < \frac{1}{2(n+1)!}\frac{a_{0,j+1}}{a_{0j}} \quad (j = 1, 2, \cdots, n-1)$$

and take $w_1 \in \Theta(\epsilon_0^1, \epsilon_1^1, \cdots, \epsilon_{n-1}^1)$. Let $a_{1j} = (w_1, v_j)$ $(j = 1, 2, \cdots, n)$ and $a_{10} = 1$. Then by (5), $a_{1j} > 0$ $(j = 1, 2, \cdots, n)$ and

$$\frac{a_{1,j+1}/a_{1j}}{a_{0,j+1}/a_{0j}} < \frac{1}{2(n+1)!} \quad (j = 0, 1, \cdots, n-1).$$

In this way, we can choose $w_0, w_1, w_2, \cdots$ in $\Sigma - r_0$. Let $a_{0i} = 1,\ a_{ij} = (w_i, v_j)$ for $i = 0, 1, 2, \cdots$ and $j = 1, 2, \cdots, n$. Then, we have $a_{ij} > 0$ $(j = 1, 2, \cdots, n)$ and

$$\frac{a_{i,j+1}/a_{ij}}{a_{i-1,j+1}/a_{i-1,j}} < \frac{1}{2(n+1)!} \quad (j = 0, 1, \cdots, n-1)$$

for any $i = 1, 2, \cdots$.

Therefore, for any $l_0, l_1, \cdots, l_n$ with $l_0 < l_1 < \cdots < l_n$, the matrix $A = (a_{l_i j})_{i,j=0,1,\cdots,n}$ satisfies $(-1)^{n(n+1)/2}\det A > 0$ by Lemma 20.

Define an injection $\psi : \mathbb{N} \to \Sigma$ by $\psi(i) = r_0 + w_i$ $(i \in \mathbb{N})$. To complete the proof, we have to prove that $\Omega \circ \psi = \Xi$, where Ω is the name set with respect to Σ.

For $q \in \mathbb{R}^n$, let ω_q be as in (4). Then, it holds that $\omega_q \circ \psi(i) = \omega_q(r_0 + w_i) = 1$ if and only if $r_0 + w_i \in B_n(q, 1)$. To prove that $\omega_q \circ \psi \in \Xi$, it is sufficient to prove that $\omega_q \circ \psi$ has at most n number of i's such that

$$\omega_q \circ \psi(i) \neq \omega_q \circ \psi(i+1). \tag{7}$$

If $||q|| \leq 1-\delta$, then $\omega_q \circ \psi(i)$ can take 0 for at most one i, while if $||q|| \geq 1+\delta$, then $\omega_q \circ \psi(i)$ can take 1 for at most one i. Therefore, $\omega_q \circ \psi$ has at most 2 i's as in (7).

Assume that $1 - \delta < ||q|| < 1 + \delta$. Let

$$T_q = \{x \in \mathbb{R}^n;\ (x, q) = \frac{||q||^2 + \delta^2 - 1}{2}\}$$

be the hyper-plane in $\mathbb{R}^n$. Then, $\omega_q \circ \psi(i) = 1$ if and only if q and $r_0 + w_i$ are not separated by T_q since $||r_0 + w_i|| = \delta$, where by definition, $x, y \in \mathbb{R}^n$ are separated by T_q if x and y are not on T_q, but in the opposite sides of $\mathbb{R}^n$ separated by T_q.

Let T be a hyper-plane in $\mathbb{R}^n$ and

$$M(T) := \{i \in \mathbb{N};\ q \text{ and } r_0 + w_i \text{ are not separated by } T\}.$$

We can find a continuously moving family of hyper-planes $T(t)$ with $t \in [0, 1]$ such that $T(0) = T_q$, $T(1)$ contains n elements in $\{r_0 + w_i;\ i \in \mathbb{N}\}$ and $M(T(t)) = M(T_q)$ for any $t \in [0, 1)$. Let $T(1) \cap \{r_0 + w_i;\ i \in \mathbb{N}\} = \{r_0 + w_{l_1}, r_0 + w_{l_2}, \cdots, r_0 + w_{l_n}\}$ with $l_1 < l_2 < \cdots < l_n$. Then, $T(1)$ is the plane defined by the equation

$$F(x_1, x_2, \cdots, x_n) = \begin{vmatrix} 1 & x_1 & x_2 & \cdots & x_n \\ 1 & a_{l_1 1} & a_{l_1 2} & \cdots & a_{l_1 n} \\ 1 & a_{l_2 1} & a_{l_2 2} & \cdots & a_{l_2 n} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 1 & a_{l_n 1} & a_{l_n 2} & \cdots & a_{l_n n} \end{vmatrix} = 0,$$

where the coordinate system $(x_1, x_2, \cdots, x_n)$ is defined with respect to the orthonormal basis $v_1, v_2, \cdots, v_n$ and r_0 as its origin.

Then, for any $x, y \in \mathbb{R}^n$, x and y are separated by $T(1)$ if $F(x)$ and $F(y)$ have different signs, while x and y are not separated by $T(1)$ if $F(x)$ and $F(y)$ have the same sign. Hence, $\omega_q \circ \psi(i) \neq \omega_q \circ \psi(j)$ if $F(w_i)$ and $F(w_j)$ have different signs, while $\omega_q \circ \psi(i) = \omega_q \circ \psi(j)$ if $F(w_i)$ and $F(w_j)$ have the same sign.

Therefore, if $l_k < i < l_{k+1}$, then by Lemma 20, $F(w_i)$ has the same sign as $(-1)^k(-1)^{n(n+1)/2}$, and hence, $\omega_q \circ \psi(i)$ is constant for $l_k < i < l_{k+1}$. Since $F(\omega_i)$ and $F(\omega_j)$ with $l_{k-1} < i < l_k$ and $l_{h-1} < j < l_h$ have the same sign if $h - k$ is even and have different signs if $h - k$ is odd, the 0-1-sequence

$\omega_q \circ \psi(i), \omega_q \circ \psi(i+1), \cdots, \omega_q \circ \psi(j)$ with $l_{k-1} < i < l_k$, $l_{h-1} < j < l_h$ and $i+1 = l_k, i+2 = l_{k+1}, \cdots, j-1 = l_{k+j-i-2}$ changed values at most $j-i-1$ times, where $j-i-1$ is the number of l_m's in $\{i, i+1, \cdots, j\}$. Therefore, the 0-1-sequence $\omega_q \circ \psi(0), \omega_q \circ \psi(1), \omega_q \circ \psi(2), \cdots$ changed values at most n times. Hence, $\omega_q \circ \psi \in \Xi$.

Conversely, for any choice of $\{l_1 < l_2 < \cdots < l_n\} \subset \mathbb{N}$, the hyper-plane T containing $\{r_0 + w_{l_1}, r_0 + w_{l_2}, \cdots, r_0 + w_{l_n}\}$ is determined. By the assumption for Σ, no other point of $\{r_0 + w_i;\ i \in \mathbb{N}\}$ is on T.

Then, an $(n-2)$-dimensional plane $T \cap \{x \in \mathbb{R}^n;\ ||x|| = \delta\}$ is determined with center, say p, and radius, say ρ. Let

$$q = \left(1 \pm \frac{\sqrt{1-\rho^2}}{\sqrt{\delta^2 - \rho^2}}\right) p\,.$$

Then, $\omega_q \circ \psi \in \Xi$. By moving q slightly, we can give the values at $\omega_q \circ \psi(l_j)$ $(j = 1, 2, \cdots, n)$ (where $F(w_{l_j}) = 0$) arbitrary. Hence, any element in $\{0,1\}^{\mathbb{N}}$ which changes values at most n times can be realized by an element in $\Omega \circ \psi$, since by taking $l_{j+1} = l_j + 1$ and choosing the values at $\omega_q \circ \psi(l_j)$ and $\omega_q \circ \psi(l_{j+1})$, we can cancel some changes of the values. Hence, $\Xi \subset \Omega \circ \psi$

Thus, $\Omega \circ \psi = \Xi$, which completes the proof. □

Acknowledgement: The authors thank Prof. Shigeru Tanaka (Tsuda Women's College) for his useful suggestion to us. The authors thank also Beijing University of Aeronautics and Astronautics for inviting one of the authors to give an oportunity of the collaboration.

References

[1] T. Kamae, Uniform set and complexity, *Discrete Math.* **309** (2009), pp. 3738-3747.

[2] T. Kamae, H. Rao, B. Tan, Y.-M. Xue, Super-stationary set, subword problem and the complexity *Discrete Math.* **309** (2009), pp. 4417-4427.

[3] T. Kamae, H. Rao, B. Tan, Y.-M. Xue, Language structure of pattern sturmian word, *Discrete Math.* **306** (2006), pp. 1651-1668.

[4] Y.-M. Xue, Transformations with discrete spectrum and sequence entropy, *Master Thesis at Osaka City University*, 2000 (in Japanese).

[5] N. Gjini, T. Kamae, B. Tan, Y.-M. Xue, Maximal pattern complexity for Toeplitz words, *Ergod. Th. & Dynam. Sys.* **26** (2006), pp. 1-14.

[6] T. Kamae, H. Rao, Maximal pattern complexity over ℓ letters, *European J. Combin.* **27** (2006), pp. 125-137.

[7] T. Kamae, H. Rao, Y.-M. Xue, Maximal pattern complexity for 2-dimensional words, *Theoret. Comput. Sci.* **359** (2006), pp. 15-27.

[8] T. Kamae and L. Zamboni, Sequence entropy and the maximal pattern complexity of infinite words, *Ergod. Th. & Dynam. Sys.* **22** (2002), pp. 1191-1199.

[9] T. Kamae and L. Zamboni, Maximal pattern complexity for discrete systems, *Ergod. Th. & Dynam. Sys.* **22** (2002), pp. 1201-1214.

(All the above papers except for [4] can be downloaded from the site:
http://www14.plala.or.jp/kamae)