Maximal pattern complexity, dual system and pattern recognition

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Abstract

For a family Ω of sets in $\mathbb{R}^2$ and a finite subset S of $\mathbb{R}^2$, let $p_\Omega(S)$ be the number of distinct sets of the form $S \cap \omega$ for all $\omega \in \Omega$. The maximum pattern complexity $p^*_\Omega(k)$ is the maximum of $p_\Omega(S)$ among S with $\#S = k$. The S attaining the maximum is considered as the most effective sampling to distinguish the sets in Ω. We obtain the exact values or at least the order of $p^*_\Omega(k)$ in k for various classes Ω. We also discuss the dual problem in the case that $\#\Omega = \infty$, that is, consider the partition of $\mathbb{R}^2$ generated by a finite family $T \subset \Omega$. The number of elements in the partition is written as $p_{\mathbb{R}^2}(T)$ and $p^*_{\mathbb{R}^2}(k)$ is the maximum of $p_{\mathbb{R}^2}(T)$ among T with $\#T = k$. Here, $p^*_\Omega(k) = p^*_{\mathbb{R}^2}(k)$ does not hold in general.

For the general setting that Ω is an infinite subset of $\mathbb{A}^\Sigma$, where $\mathbb{A}$ is a finite alphabet, Σ is an arbitrary infinite set, and $p^*_\Omega(k) = \max_{\#S=k} \#\Omega|_S$, it is known that the entropy

$$h(\Omega) := \lim_{k\to\infty} \log p^*_\Omega(k)/k$$

exists and takes value in $\{\log 1,\ \log 2, \cdots, \log \#\mathbb{A}\}$. In this paper, we prove that the entropy $h(\Sigma)$ of the dual system coincides with $h(\Omega)$.

1 Introduction

Let Σ be an arbitrary infinite set. For $k = 0, 1, 2, \cdots$, denote

$$\mathcal{F}_k(\Sigma) = \{S;\ S \subset \Sigma \text{ with } \#S = k\},\ \mathcal{F}(\Sigma) = \cup_{k=0}^{\infty}\mathcal{F}_k(\Sigma),$$

where $\#$ denotes the number of elements in a set. For a nonempty set $\Omega \subset \mathbb{A}^\Sigma$, where $\mathbb{A}$ is a finite set with $\#\mathbb{A} \geq 2$, define the *complexity* p_Ω

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which is a function of $S \in \mathcal{F}(\Sigma)$ by

$$p_\Omega(S) = \#\Omega|_S.$$

Here, $\Omega|_S$ is the restriction of Ω to S. That is, $\Omega := \{\omega|_S;\ \omega \in \Omega\}$ and $\omega|_S$ is the restriction of the mapping $\omega : \Sigma \to \mathbb{A}$ to $S \subset \Sigma$. It is customary to assume that Ω is closed in the weak product topology, but we will not do so in this paper. Note that Ω and its closure have the same complexity.

The complexity function as a function of $S \in \mathcal{F}(\Sigma)$ is a naive way to measure how large the set Ω is. But the dependence on S is too detailed and complicated. We simplify it in 2 ways as follows:

(1) $p^*_\Omega(k) := \sup_{S\in\mathcal{F}_k(\Sigma)} p_\Omega(S)$ and (2) $p_{*\Omega}(k) := \inf_{S\in\mathcal{F}_k(\Sigma)} p_\Omega(S)$.
Since $p^*_\Omega(0) = p_{*\Omega}(0) = 1$ hold always, we consider them as functions of $k = 1, 2, \cdots$ and called them the *maximal pattern complexity* and the *minimal pattern complexity* of Ω, respectively. If $p^*_\Omega(k) = p_{*\Omega}(k)$ $(k = 1, 2, \cdots)$ holds, then we call Ω a *uniform set.* In this case, $p_\Omega(S)$ depends only on $\#S$ and define $p_\Omega(k) := p_\Omega(S)$ with $\#S = k$ as a function of $k = 1, 2, \cdots$, which is called the *uniform complexity* of Ω.

These complexity functions are discussed in [15] for $\Sigma = \mathbb{N} := \{0, 1, 2, \cdots\}$, together with the *block complexity* $p^{BL}_\Omega(k) := p_\Omega(\{0, 1, \cdots, k-1\})$. Here, we consider general infinite set Σ, even with continuum cardinality some times. We refer Theorem 1 for general Σ, which is proved in [15] for $\mathbb{N}$, since the restriction to the case $\Sigma = \mathbb{N}$ loses no generality.

The study of maximal pattern complexity started more than 10 years ago by L. Zamboni and the authors [23], [6], [7]. It was introduced for the first time for an infinite word $\omega \in \mathbb{A}^\mathbb{N}$ as

$$p^*_\omega(k) := \sup_{\tau\subset\mathbb{N},\ \#\tau=k} \#\{\omega[n+\tau];\ n \in \mathbb{N}\} \quad (k = 1, 2, \cdots),$$

where $\omega[n + \tau] = \omega(n + \tau_0)\omega(n + \tau_1)\cdots\omega(n + \tau_{k-1}) \in \mathbb{A}^k$ with $\tau = \{\tau_0 < \tau_1 < \cdots < \tau_{k-1}\} \subset \mathbb{N}$. Let $\Omega = \{T^n\omega;\ n \in \mathbb{N}\}$, where T is the shift. Then we have $p^*_\omega(k) = p^*_\Omega(k)$ $(k = 1, 2, \cdots)$.

One of main motivations for to study the maximal pattern complexity was to find a topological analogy of the *sequence entropy* introduced by A. G. Kushnirenko [18]. It was found out first that $\omega \in \mathbb{A}^\mathbb{N}$ is not eventually periodic if and only if $p^*_\omega(k) \geq 2k$ for any $k = 1, 2, \cdots$. Then, we became interested in the words $\omega \in \mathbb{A}^\mathbb{N}$ satisfying $p^*_\omega(k) = 2k$ for any $k = 1, 2, \cdots$, which we call *pattern Sturmian words.* It is an analogy of Sturmian words ω satisfying $p^{BL}_\omega(k) = k + 1$ for any $k = 1, 2, \cdots$ with the block complexity $p^{BL}_\omega(k)$. There are 2 different types of recurrent pattern Sturmian words, namely, rotation words and Toeplitz words. They are known to have different *super-stationary structures* [8], [17].

The study of maximal pattern complexity was developed by N. Gjini, H. Rao, B. Tan, Z-Y. Wen, Y-H. Qu, P. Salimov, S. Ferenczi, P. Hubert and the authors [4], [9], [10], [19], [13], [3]. Specially, in [9], [19], the high dimensional words ω in $\mathbb{A}^{\mathbb{N}^2}$ or $\mathbb{A}^{\mathbb{Z}^n}$ are considered.

The exponential growth rate of the block complexity function $p_\omega^{BL}(k)$ in k corresponds to the topological entropy of the dynamical system generated by ω. W. Huang and X. Ye [5] used the exponential growth rate of $p_\omega^*(k)$ as a topological invariant of the dynamical systems to discuss low complexity systems.

The authors used the maximal pattern complexity for the problem of maximizing the partitions of $\mathbb{R}^n$ generated by a fixed number of congruent sets [24]. For example, let B be the unit n-dimensional ball centered at the origin in $\mathbb{R}^n$ with $n \geq 2$. Take k-number of its translates $B+x_i$ $(i = 1, \cdots, k)$ with $x_i \in \mathbb{R}^n$ and consider the partition $\mathbb{P}(B + x_i;\ i = 1, \cdots, k)$ of $\mathbb{R}^n$ generated by these sets. The problem is what is the maximum value of $\#\mathbb{P}(B + x_i;\ i = 1, \cdots, k)$ for the choices of $x_i \in \mathbb{R}^n$ and how to attain the maximum. We know [24] that

$$\max_{\{x_i;\ i=1,\cdots,k\}\subset\mathbb{R}^n} \#\mathbb{P}(B + x_i;\ i = 1, \cdots, k) = 2\sum_{i=0}^{n} \binom{k-1}{i}$$

and if Θ is an infinite subset of $\mathbb{R}^n$ such that for some δ with $0 < \delta < 1$, $\Theta \subset \{x \in \mathbb{R}^n;\ ||x|| = \delta\}$, and that any subset of Θ with cardinality $n+1$ is not on a $(n-1)$-dimensional plane, where $||x|| = \sqrt{x_1^2 + \cdots + x_n^2}$, then Θ is an *optimal position* for Ω, that is, for any $k = 1, 2, \cdots$, any k-elements set $\{x_i;\ i = 1, \cdots, k\}$ from Θ attains the maximum, that is,

$$\#\mathbb{P}(B + x_i;\ i = 1, \cdots, k) = 2\sum_{i=0}^{n} \binom{k-1}{i}.$$

The notion of optimal position was introduced in [11] for the first time. If $\Omega \subset \mathbb{A}^\Sigma$ has an optimal position, say Θ, then we have a uniform set $\Omega|_\Theta \subset \mathbb{A}^\Theta$ and the uniform complexity $p_{\Omega|_\Theta}(k)$. The uniform complexity has many beautiful properties and is studied in [8], [11], [12], [14], [15].

We recall a result obtained in [15].

Definition 1. For any positive integers r, h and $\Omega \subset \mathbb{A}^\Sigma$, a pair

$$((t_1, t_2, \cdots, t_h), \Theta)$$

is called a *r-tree of size* h contained in Ω, if $(t_1, t_2, \cdots, t_h)$ is a sequence of distinct elements in Σ and Θ is a subset of $\Omega|_{\{t_1,t_2,\cdots,t_h\}}$ such that for any $\eta \in \Theta|_{\{t_1,\cdots,t_i\}}$ with $i = 0, 1, \cdots, h-1$, we have

$$\#\{\eta' \in \Theta|_{\{t_1,\cdots,t_i,t_{i+1}\}};\ \eta'|_{\{t_1,\cdots,t_i\}} = \eta\} = r.$$

Theorem 1. (Huang-Kamae-Ye) [15] *The following limit*

$$h(\Omega) := \lim_{k\to\infty}(1/k)\log p^*_\Omega(k)$$

called the entropy of Ω *exists, and takes value* $\log r$ *with a positive integer* $r \le \#\mathbb{A}$. *Moreover, if* $h(\Omega) = \log r$, *then* Ω *contains a* r*-tree of an arbitrary large size.*

In general, for infinite sets $\Omega,\ \Sigma$ and a finite set $\mathbb{A}$, let $\psi : \Omega \times \Sigma \to \mathbb{A}$ be a mapping. For $\sigma \in \Sigma$ and $\omega \in \Omega$, define mappings $\psi^1_\sigma : \Omega \to \mathbb{A}$ and $\psi^2_\omega : \Sigma \to \mathbb{A}$ by $\psi^1_\sigma(\omega) = \psi(\omega,\sigma)$ and $\psi^2_\omega(\sigma) = \psi(\omega,\sigma)$. Hence, $\{\psi^1_\sigma;\ \sigma \in \Sigma\}$ is a subset of $\mathbb{A}^\Omega$ and $\{\psi^2_\omega;\ \omega \in \Omega\}$ is a subset of $\mathbb{A}^\Sigma$. We call one of them *dual* of the other. The mapping ψ is called the *duality* mapping.

In our case that Ω is an infinite subset of $\mathbb{A}^\Sigma$, we always define the duality mapping $\psi : \Omega \times \Sigma \to \mathbb{A}$ by $\psi(\omega,\sigma) = \omega(\sigma)$. For $S \in \mathcal{F}(\Sigma)$, note that $p_\Omega(S) = \#\{\psi^2_\omega|_S;\ \omega \in \Omega\}$. For $T \in \mathcal{F}(\Omega)$, define $p_\Sigma(T) = \#\{\psi^1_\sigma|_T;\ \sigma \in \Sigma\}$ and

$$p^*_\Sigma(k) = \sup_{T\in\mathcal{F}_k(\Omega)} p_\Sigma(T) \quad (k = 1,2,\cdots).$$

By the symmetry, the existence of the entropy

$$h(\Sigma) := \lim_{k\to\infty}(1/k)\log p^*_\Sigma(k)$$

follows from Theorem 1. We prove that it coincides with $h(\Omega)$.

Theorem 2. *It holds that* $h(\Sigma) = h(\Omega)$.

The following Theorem 3 is proved in an unpublished paper [16] of one of the authors. The key lemma (Lemma 4) is essentially due to N. Sauer [20]. To be self-contained, we give a proof of it in Section 2.

Theorem 3. *Let* $\Omega \subset \mathbb{A}^\Sigma$ *with* $\#\mathbb{A} = d$. *Assume that* $p^*_\Omega(n) < d^n$ *holds for some* $n = 1,2,\cdots$. *Then, we have*

$$p^*_\Omega(k) \le \sum_{i=0}^{n-1}\binom{k}{i}(d-1)^{k-i} \quad (k = 1,2,\cdots),$$

where we define $\binom{k}{i} = 0$ *if* $0 \le i \le k$ *does not hold.*

We apply the maximal pattern complexity for a problem of pattern recognitions. For example, consider a set of pictures of, say typical human faces as computer graphics. They are represented as configurations of digital data (colors, brightness, etc) at points in $\Sigma = \mathbb{R}^2$. The set of digital data at a point is a finite set, say $\mathbb{A}$, so that a picture is an element in $\mathbb{A}^{\mathbb{R}^2}$. Thus, the set of pictures can be identified with a set $\Omega \subset \mathbb{A}^{\mathbb{R}^2}$.

Choose a subset S (*sampling set*) of $\mathbb{R}^2$ of a fixed size k to identify a human face $\omega \in \Omega$ by scanning and checking whether $\omega|_S$ coincides with the registered one or not. Thus, we can use the information of $\omega|_S$ as like a password. The best candidate for S is those which distinguish the faces in Ω as many as possible. In another word, the best S is those satisfying that $\#\Omega|_S = p^*_\Omega(k)$. We are interested in how to choose the sampling set and how many different sets are distinguished by the observation, related to the pattern recognition problem.

In this paper, other than Section 2 and 3, we assme that $\mathbb{A} = \{0, 1\}$ and $\Sigma = \mathbb{R}^2$. By identifying $\omega \in \Omega$ with the set $\{\sigma;\ \omega(\sigma) = 1\}$, Ω is considered as a family of sets in $\mathbb{R}^2$, that is, a family of monochromatic pictures. With this identification, $p_\Omega(S)$ is considered as the number of distinct sets of the form $\omega \cap S$ for all $\omega \in \Omega$, that is

$$p_\Omega(S) = \#\{\omega \cap S;\ \omega \in \Omega\} \quad (S \in \mathcal{F}(\mathbb{R}^2)),$$

and $p^*_\Omega(k)$ is the maximum of this number among $S \in \mathcal{F}_k(\mathbb{R}^2)$.

As a corollary of Theorem 3, we have

Corollary 1. *If $\mathbb{A} = \{0, 1\}$ and $p^*_\Omega(n) < 2^n$ for some $n = 1, 2, \cdots$, then $p^*_\Omega(k) = O(k^{n-1})$.*

In the case $\mathbb{A} = \{0, 1\}$, the maximum n such that $p^*_\Omega(n) = 2^n$ is called *Vapnik-Chervonenkis dimension* (*VC-dimension*, for short) of the class of subsets Ω of Σ. It is a basic notion in the problems of pattern recognition and machine learning, and has been studied by many authors, for example, V. N. Vapnik, A. Ya. Chervonenkis [21], R. S. Wenocur, R. M. Dudley [22], Peter Bras [2], and A. Blumer, A. Ehrenfeucht, D. Haussler, M. K. Warmuth [1].

We obtain the order of $p^*_\Omega(k)$ in k for various Ω listed below. Sometimes, we get the exact value of $p^*_\Omega(k)$ as well as an optimal position for Ω. We introduce the ortho-normal coordinate system in $\mathbb{R}^2$ with x- and y- axes. We assume that all the sets belonging to the classes below are closed, just for simplicity.

$\mathcal{L}$ = the class of straight lines in $\mathbb{R}^2$

$\mathcal{H}$ = the class of half planes bounded by straight lines in $\mathbb{R}^2$

$\mathcal{D}_1$ = the class of unit discs in $\mathbb{R}^2$

$\mathcal{D}$ = the class of discs in $\mathbb{R}^2$

$\mathcal{Q}_1$ = the class of unit squares in $\mathbb{R}^2$ with edges parallel to x- or y- axis

$\mathcal{Q}$ = the class of squares in $\mathbb{R}^2$ with edges parallel to x- or y- axis

$\mathcal{R}$ = the class of rectangles in $\mathbb{R}^2$ with edges parallel to x- or y- axis

$\mathcal{C}_n$ = the class of convex n-polygons in $\mathbb{R}^2$, $(n = 3, 4, \cdots)$

$\mathcal{C}_\infty$ = the class of convex n-polygons with arbitrary $n = 3, 4, \cdots$ in $\mathbb{R}^2$

Theorem 4. (1) *We have* $p^*_{\mathcal{L}}(k) = \frac{1}{2}k^2 + \frac{1}{2}k + 1$ $(k = 1, 2, \cdots)$. *Moreover,* $\Theta \subset \mathbb{R}^2$ *with* $\#\Theta = \infty$ *is an optimal position for* $\mathcal{L}$ *if and only if any* 3 *points in* Θ *are not on a line.*
(2) *We have* $p^*_{\mathcal{D}_1}(k) = k^2 - k + 2$ $(k = 1, 2, \cdots)$. *Moreover,* $\Theta \subset \mathbb{R}^2$ *with* $\#\Theta = \infty$ *is an optimal position for* $\mathcal{D}_1$ *if* Θ *is a subset of a circle with radius* δ *such that* $0 < \delta < 1$.
(3) *We have* $p^*_{\mathcal{Q}_1}(k) = k^2 - k + 2$ $(k = 1, 2, \cdots)$. *An optimal position for* $\mathcal{Q}_1$ *does not exist.*
(4) *We have* $p^*_{\mathcal{H}}(k) = k^2 - k + 2$ $(k = 1, 2, \cdots)$. *Moreover,* $\Theta \subset \mathbb{R}^2$ *with* $\#\Theta = \infty$ *is an optimal position for* $\mathcal{H}$ *if* Θ *is a subset of a circle with radius* δ *such that* $\delta > 0$.
(5) *We have* $p^*_{\mathcal{C}_\infty}(k) = 2^k$ $(k = 1, 2, \cdots)$. *Moreover,* $\Theta \subset \mathbb{R}^2$ *with* $\#\Theta = \infty$ *is an optimal position for* $\mathcal{C}_\infty$ *if and only if* Θ *is a subset of the boundary of a strictly convex set.*

Theorem 5. (1) *We have* $p^*_{\mathcal{D}}(k) \asymp k^3$ *as* $k \to \infty$ *in the sense that*

$$0 < \liminf_{k\to\infty} p^*_{\mathcal{D}}(k)/k^3 \le \limsup_{k\to\infty} p^*_{\mathcal{D}}(k)/k^3 < \infty.$$

(2) *We have* $p^*_{\mathcal{Q}}(k) \asymp k^3$ *as* $k \to \infty$.
(3) *We have* $p^*_{\mathcal{R}}(k) \asymp k^4$ *as* $k \to \infty$.
(4) *We have* $k^{2n} \prec p^*_{\mathcal{C}_n}(k) \prec k^{2n+1}$ *as* $k \to \infty$ *for any* $n = 3, 4, \cdots$ *in the sense that*

$$\liminf_{k\to\infty} p^*_{\mathcal{C}_n}(k)/k^{2n} > 0 \text{ and } \limsup_{k\to\infty} p^*_{\mathcal{C}_n}(k)/k^{2n+1} < \infty.$$

Remark 1. Some results in the above theorems (e.g. for $\mathcal{L}$, $\mathcal{H}$, $\mathcal{Q}$, $\mathcal{R}$) are well known in term of VC-dimension. But some exact values and the notions of duality and optimal position have not been discussed in this connection.

Organization of the paper

In Section 2, we give 3 examples of duality and prove that the entropies of dual systems are the same (Theorem 2).

In Section 3, we prove Sauer's Lemma (Lemma 3) and Theorem 3 based on it.

Section 4 is devoted to the proof of Theorem 4 obtaining exact estimates of the maximal pattern complexities of $\mathcal{L}$, $\mathcal{D}_1$, $\mathcal{Q}_1$, $\mathcal{H}$, $\mathcal{C}_\infty$. We also discuss about the optimal positions.

In Section 5, we prove a lemma for upper estimates of the VC-dimensions of $\mathcal{Q}$ and $\mathcal{R}$. Using it, we obtain the orders of the maximal pattern complexities of $\mathcal{D}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{C}_n$.

2 Duality

For an infinite subset $\Omega \subset \mathbb{A}^\Sigma$, let $\psi : \Omega \times \Sigma \to \mathbb{A}$ be the duality mapping defined as $\psi(\omega, \sigma) = \omega(\sigma)$.

Example 1. Let $\Omega = \mathcal{D}_1$ and $\Sigma = \mathbb{R}^2$. For $x \in \mathbb{R}^2$, let ω_x be the unit disc centered at x. Then, we have the symmetry that

$$\psi(\omega_x, y) = \psi(\omega_y, x) = 1_{|x-y|\leq 1}$$

for any $x, y \in \mathbb{R}^2$. Therefore, $p^*_\Omega(k) = p^*_\Sigma(k)$ follows. Since the boundaries of 2 distinct discs intersect at most at 2 points, the number of partition increased by adding one disc to k discs is at most $2k$. Hence, we have $p^*_\Sigma(k+1) \leq p^*_\Sigma(k) + 2k$ $(k = 1, 2, \cdots)$. In fact, we know [24] that $p^*_\Sigma(k) = k^2 - k + 2$ $(k = 1, 2, \cdots)$. Thus, we have $p^*_\Omega(k) = p^*_\Sigma(k) = k^2 - k + 2$. The same result holds for $\mathcal{Q}_1$.

Example 2. Let $\Omega = \mathcal{D}$ (or $\mathcal{Q}$) and $\Sigma = \mathbb{R}^2$. By the same reason as Example 1, we have $p^*_\Sigma(k) = k^2 - k + 2$. Let

$$S = \{(-i, 0);\ i = 1, \cdots, k\} \cup \{(i, 0);\ i = 1, \cdots, k\} \cup \{(0, i/k);\ i = 1, \cdots, k\}.$$

Then, since there is a circle (square whose edges are parallel to x- or y-axis, respectively) passing $(-i_1, 0)$, $(i_2, 0)$ and $(0, i_3/k)$ for any $i_1, i_2, i_3 \in \{1, \cdots, k\}$, we have

$$p^*_\Omega(3k) \geq \#\{S \cap \omega;\ \omega \in \mathcal{D}\} \geq k^3.$$

Thus, $p^*_\Omega(k) \asymp k^3$ while $p^*_\Sigma(k) \asymp k^2$.

Example 3. Let $\Sigma = \mathbb{N} = \{0, 1, 2, \cdots\}$ and $\Omega \subset \{0, 1\}^{\mathbb{N}}$ be the set of all 0-1-words with at most one 1-block, that is,

$$\{0^i 1^j 0^\infty;\ i \in \mathbb{N},\ j \in \mathbb{N}\} \cup \{0^i 1^\infty;\ i \in \mathbb{N}\}.$$

Then, we have $p^*_\Omega(k) = 1 + k + \binom{k}{2}$. On the other hand, since $\omega \in \Omega$ gives a partition of $\mathbb{N}$ by the interval $\{i \in \mathbb{N};\ \omega(i) = 1\}$, we have $p^*_\Sigma(k+1) - p^*_\Sigma(k) \leq 2$. In fact, we have $p^*_\Sigma(k) = 2k$ $(k = 1, 2, \cdots)$. Thus, $p^*_\Omega(k) \asymp k^2$ while $p^*_\Sigma(k) \asymp k$.

Proof of Theorem 2

By the symmetry, it is sufficient to prove that $h(\Sigma) \geq h(\Omega)$. If $h(\Omega) = 0$, then this holds clearly. Assume that $h(\Omega) = \log r$ with an integer $r \geq 2$.

Take any $k = 1, 2, \cdots$. By Theorem 1, Ω contains a r-tree, say

$$((t_1, t_2, \cdots, t_H), \Theta)$$

with $H = k + r^k$, where $U := \{t_1, t_2, \cdots, t_H\} \subset \Sigma$ and $\Theta \subset \Omega|_U$. Our aim is to construct $W \in \mathcal{F}_k(\Omega)$ such that

$$\#\Sigma|_W = \#\{(w(t);\ w \in W);\ t \in \Sigma\} \geq r^k.$$

We determine W gradually starting from any $W_0 \subset \Theta|_{\{t_1,\cdots,t_k\}} \subset \mathcal{F}_k(\Omega|_{\{t_1,\cdots,t_k\}})$. For each $h = 0, 1, \cdots, r^k$, we construct $W_h \subset \Theta|_{U_h}$, where $U_h = \{t_1, t_2, \cdots, t_{k+h}\}$, inductively as follows:
(1) W_0 is any set with $\# W_0 = k$,
(2) $\# W_{h+1} = k$ and $W_{h+1}|_{U_h} = W_h$,
(3) if $\#\{(w(t);\ w \in W_h);\ t \in U_h\} < r^k$ and $h < r^k$, then

$$\#\{(w(t);\ w \in W_{h+1});\ t \in U_{h+1}\} = \#\{(w(t);\ w \in W_h);\ t \in U_h\} + 1.$$

(3) is possible, since if $h < r^k$, then any $w \in W_h$ has r distinct extensions to U_{h+1}. Therefore, there are r^k distinct extensions from W_h to W_{h+1}. If

$$\#\{(w(t);\ w \in W_h);\ t \in U_h\} = \#\{(w(t);\ w \in W_{h+1});\ t \in U_h\} < r^k,$$

then we can find W_{h+1} such that $(w(t_{k+h+1});\ w \in W_{h+1})$ is not in $\{(w(t);\ w \in W_h);\ t \in U_h\}$. Thus, we get $\#\Sigma|_{W_{r^k}} = r^k$.

Define $W \in \mathcal{F}_k(\Omega)$ so that $W|_U = W_{r^k}$. Then, we have

$$p^*_\Sigma(k) \geq p_\Sigma(W) \geq r^k \ (k = 1, 2, \cdots).$$

Thus, $h(\Sigma) \geq h(\Omega)$. □

3 Proof of Theorem 3

Let $\Omega \subset \mathbb{A}^\Sigma$ with $\#\mathbb{A} = d$. Take $n = 1, 2, \cdots$ and $k \geq n$ since otherwise, our statement is clear. Since $p^*_\Omega(n) < d^n$, for each $U \in \mathcal{F}_n(\Sigma)$, there is $\xi_U \in \mathbb{A}^U$ such that $\xi_U \notin \Omega|_U$. Choose one of $\xi_U \in \mathbb{A}^U$ as this for any $U \in \mathcal{F}_n(\Sigma)$. Take any $S \in \mathcal{F}_k(\Sigma)$. Let $\mathbb{F} = \{\xi_U;\ U \in \mathcal{F}_n(S)\}$ and

$$\mathcal{P}(\mathbb{F}, S) := \{\eta \in \mathbb{A}^S;\ \eta|_U \notin \mathbb{F} \text{ for any } U \in \mathcal{F}_n(S)\}.$$

Then, we have $\Omega|_S \subset \mathcal{P}(\mathbb{F}, S)$. Hence, our theorem follows from the following lemma. □

Let $F \subset \cup_{U \in \mathcal{F}_n(S)} \mathbb{A}^U$. We call F a *simple complete list of words* on S of size n if $\#(F \cap \mathbb{A}^U) = 1$ for any $U \in \mathcal{F}_n(S)$. Then, $\mathbb{F}$ is a simple complete list of words on S of size n.

Lemma 4. *Let $n \leq k$ and S be a set with $\# S = k$. Let F be a simple complete list of words on S of size n. Then, we have $\#\mathcal{P}(F, S) \leq \sum_{i=0}^{n-1} \binom{k}{i} (d-1)^{k-i}$.*

Proof For $n = 1$ our statement is clear since $\#\mathcal{P}(F, S) = (d-1)^k$. We use the induction on n. Assume that $n \geq 2$ and our statement holds for $n - 1$.

If $k = n$, then our statement is clear since $\#\mathcal{P}(F, S) = d^n - 1$. We use the induction on k. Assume that $k \geq n + 1$ and our statement holds for $k - 1$.

Take $s_0 \in S$ and $S' := S \setminus \{s_0\}$. Let $F' := \{\xi_U \in F;\ s_0 \notin U\}$ and $F'' := \{(\xi_U)|_{U\setminus\{s_0\}};\ \xi_U \in F,\ s_0 \in U\}$. Then, F' is a simple complete list of words of size n on S', while F'' is a simple complete list of words of size $n-1$ on S'. Since $\#S' = k-1$, by the induction hypothesis, we have $\#\mathcal{P}(F', S') \le \sum_{i=0}^{n-1} \binom{k-1}{i} (d-1)^{k-1-i}$. Also, since $\#F'' = n-1$, by the induction hypothesis, we have $\#\mathcal{P}(F'', S') \le \sum_{i=0}^{n-2} \binom{k-1}{i} (d-1)^{k-1-i}$.

For any $\eta \in \mathcal{P}(F, S)$, we have $\eta|_{S'} \in \mathcal{P}(F', S')$. Define the mapping $\pi : \mathcal{P}(F, S) \to \mathcal{P}(F', S')$ by $\pi(\eta) = \eta|_{S'}$. If $\eta' \in \mathcal{P}(F', S') \setminus \mathcal{P}(F'', S')$, then there exists $U' \in \mathcal{F}_{n-1}(S')$ such that $\eta'|_{U'} \in F''$. This implies that there exists $a \in \mathbb{A}$ such that the extension $\eta \in \mathbb{A}^S$ of η' with $\eta(s_0) = a$ satisfies that $\eta|_U \in F$, where $U = U' \cup \{s_0\}$. Therefore, $\eta \notin \mathcal{P}(F, S)$. Thus,

$$\#\pi^{-1}(\eta') \le \begin{cases} d-1 & \text{if } \eta' \in \mathcal{P}(F', S') \setminus \mathcal{P}(F'', S') \\ d & \text{if } \eta' \in \mathcal{P}(F', S') \cap \mathcal{P}(F'', S') \end{cases},$$

which implies that $\#\mathcal{P}(F, S) \le (d-1)\#\mathcal{P}(F', S') + \#\mathcal{P}(F'', S')$. Hence,

$$\begin{aligned} & \#\mathcal{P}(F, S) \\ \le\ & (d-1)\#\mathcal{P}(F', S') + \#\mathcal{P}(F'', S') \\ \le\ & (d-1)\sum_{i=0}^{n-1} \binom{k-1}{i} (d-1)^{k-1-i} + \sum_{i=0}^{n-2} \binom{k-1}{i} (d-1)^{k-1-i} \\ =\ & \sum_{i=0}^{n-1} \left(\binom{k-1}{i} + \binom{k-1}{i-1} \right) (d-1)^{k-i} = \sum_{i=0}^{n-1} \binom{k}{i} (d-1)^{k-i}, \end{aligned}$$

which completes the proof. □

4 Proof of Theorem 4

(1) If $S \subset \mathbb{R}^2$ with $\#S = k$ satisfies that any 3 points in S are not on a line, then it is clear that $p_{\mathcal{L}}(S) = 1 + k + \binom{k}{2}$. Moreover, if some 3 points, say $u, v, w \in S$ are on a line, then $\{u, v\},\ \{u, w\},\ \{v, w\}$ are not in $\{\omega \cap S;\ \omega \in \mathcal{L}\}$, while a unique set containing $\{u, v, w\}$ is in it. Therefore, $p_{\mathcal{L}}(S)$ decreases at least 2 from $1 + k + \binom{k}{2}$. Thus, we have the required statements.

(2) Let $\psi : \mathcal{D}_1 \times \mathbb{R}^2 \to \mathbb{A}$ be the duality map. By Example 1, we have the symmetry that $\psi(\omega_x, y) = \psi(\omega_y, x)$ and hence, $p_{\mathcal{D}_1}(S) = p_{\mathbb{R}^2}(\{\omega_x;\ x \in S\})$ for any $S \in \mathcal{F}(\mathbb{R}^2)$, where ω_x is the unit disc centered at $x \in \mathbb{R}^2$. We know ([24]) that $p^*_{\mathbb{R}^2}(k) = k^2 - k + 2$ $(k = 1, 2, \cdots)$ and an infinite set $\{\omega_x;\ x \in \Theta\}$, where $\Theta \subset \mathbb{R}^2$ is contained in a circle with radius δ such that

$0 < \delta < 1$, is an optimal position for $p^*_{\mathbb{R}^2}$. Hence by the symmetry, we have $p^*_{\mathcal{D}_1}(k) = p^*_{\mathbb{R}^2}(k) = k^2 - k + 2$ $(k = 1, 2, \cdots)$. Moreover, the above Θ is an optimal position for $p^*_{\mathcal{D}_1}(k)$.

(3) Let $\psi : \mathcal{Q}_1 \times \mathbb{R}^2 \to \mathbb{A}$ be the duality mapping. Similar to the above (2), we have $\psi(\theta_x, y) = \psi(\theta_y, x)$, where θ_x is the unit square centered at $x \in \mathbb{R}^2$. Moreover, we know ([24]) that $p^*_{\mathbb{R}^2}(k) = k^2 - k + 2$ $(k = 1, 2, \cdots)$ and that there is no optimal position for $p^*_{\mathbb{R}^2}(k)$. Hence, $p^*_{\mathcal{Q}_1}(k) = k^2 - k + 2$ $(k = 1, 2, \cdots)$ and there is no optimal position for $p^*_{\mathcal{Q}_1}(k)$.

(4) Let $\mathcal{D}_r$ be the set of discs in $\mathbb{R}^2$ with radius r and Θ be an infinite subset of a circle with radius δ such that $0 < \delta < r$. By above (2), we have $p^*_{\mathcal{D}_r}(k) = k^2 - k + 2$ $(k = 1, 2, \cdots)$ and Θ is an optimal position for $\mathcal{D}_r$. Letting $r \to \infty$, we get (4).

(5) Let $\Theta \subset \mathbb{R}^2$ with $\#\Theta = \infty$ be a subset of the boundary of a strictly convex set. Let $S \in \mathcal{F}_k(\Theta)$ with an arbitrary $k = 1, 2, \cdots$. Take any subset U of S. Let $\hat{U}$ be the convex hull of U. Then, it is clear that $\hat{U} \in \mathcal{C}_\infty$ and $\hat{U} \cap S = U$. Thus, $p_{\mathcal{C}_\infty}(S) = 2^k$.

Assume that $\Theta \subset \mathbb{R}^2$ with $\#\Theta = \infty$ is not a subset of the boundary of a strictly convex set. Then, there exist $S \in \mathcal{F}(\Theta)$ and $s \in S$ such that $s \in \widehat{S \setminus \{s\}}$. This implies that $S \setminus \{s\}$ is not in the class $\{\omega \cap S;\ \omega \in \mathcal{C}\}$. Thus, Θ is not an optimal position. □

5 Proof of Theorm 5

Lemma 5. (1) *For any $S \in \mathcal{F}_5(\mathbb{R}^2)$, there exist 3 points, say $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ in S such that either $x_1 \le x_2 \le x_3$ and $y_1 \le y_2 \le y_3$, or $x_1 \le x_2 \le x_3$ and $y_1 \ge y_2 \ge y_3$.*
(2) *For any $S \in \mathcal{F}_4(\mathbb{R}^2)$, either there exist 3 points in S satisfying the condition in (1), or S coincides with $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\}$ such that $x_1 \le x_2 \le x_3 \le x_4$ and $\{y_1, y_4\} \subset (\min\{y_2, y_3\}, \max\{y_2, y_3\})$.*

Proof (1) is proved as Lemma 18 in [24].
(2) Let $S = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\}$ with $x_1 \le x_2 \le x_3 \le x_4$. Assume that

$$\{y_1, y_4\} \cap \{\min\{y_1, y_2, y_3, y_4\}, \max\{y_1, y_2, y_3, y_4\}\} \ne \emptyset.$$

Without loss of generality, we may assume that $y_1 = \min\{y_1, y_2, y_3, y_4\}$. If any one of the relations $y_2 \le y_3$, $y_2 \le y_4$ or $y_3 \le y_4$ holds, then we have 3 points in S satisfying the condition in (1) including (x_1, y_1). If else, then we have $y_2 > y_3 > y_4$ and the 3 points $(x_2, y_2), (x_3, y_3), (x_4, y_4)$ satisfy the condition in (1).

Therefore, if (1) does not hold, then we have

$$\{y_1, y_4\} \cap \{\min\{y_1, y_2, y_3, y_4\}, \max\{y_1, y_2, y_3, y_4\}\} = \emptyset,$$

which implies that $\{y_1, y_4\} \subset (\min\{y_2, y_3\}, \max\{y_2, y_3\})$. $\square$

Proof of Theorem 5

(1) Take any $S \subset \mathbb{R}^2$ with $\#S = 4$. If there exists $s \in S$ such that $s \in \widehat{S \setminus \{s\}}$, then $S \setminus \{s\}$ cannot be $S \cap D$ with $D \in \mathcal{D}$. Hence, $p_{\mathcal{D}}(S) < 2^4$.

Now assume that S coincides with $\{s_1, s_2, s_3, s_4\}$ such that $\square s_1 s_2 s_3 s_4$ is a convex quadrilateral. Since $\angle s_1 + \angle s_2 + \angle s_3 + \angle s_4 = 2\pi$, either $\angle s_1 + \angle s_3 \leq \pi$ or $\angle s_2 + \angle s_4 \leq \pi$. Assume without loss of generality that $\angle s_1 + \angle s_3 \leq \pi$. Then, $S \cap D$ cannot be $\{s_1, s_3\}$ for any $D \in \mathcal{D}$. These arguments imply that $p_{\mathcal{D}}(S) < 2^4$ for any $S \in \mathcal{F}_4(\mathbb{R}^2)$. Therefore, $p^*_{\mathcal{D}}(4) < 2^4$ and hence, $p^*_{\mathcal{D}}(k) = O(k^3)$ holds by Corollary 1.

Conversely with S in Example 2, we have

$$p^*_{\Omega}(3k) \geq \#\{S \cap \omega;\ \omega \in \mathcal{D}\} \geq k^3\ (k = 1, 2, \cdots).$$

Thus $\liminf_{k\to\infty} p^*_{\mathcal{D}}(k)/k^3 \geq 1/3^3 > 0$, which completes the proof.

(2) Take an arbitrary $S \in \mathcal{F}_4(\mathbb{R}^2)$. If there are 3 points s_1, s_2, s_3 in S satisfying the condition in (1) of Lemma 5, then $S \cap Q$ cannot be $\{s_1, s_3\}$ for any $Q \in \mathcal{Q}$. Hence, $p_{\mathcal{Q}}(S) < 2^4$.

By Lemma 5, to prove that $p^*_{\mathcal{Q}}(4) < 2^4$, it is sufficient to prove that if $S = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\}$ such that $x_1 \leq x_2 \leq x_3 \leq x_4$ and $\{y_1, y_4\} \subset (\min\{y_2, y_3\}, \max\{y_2, y_3\})$, then $p_{\mathcal{Q}}(S) < 2^4$. Assume that $|x_1 - x_4| \leq |y_2 - y_3|$. Then, $S \cap Q$ cannot be $\{(x_2, y_2), (x_3, y_3)\}$ for any $Q \in \mathcal{Q}$ and hence, $p_{\mathcal{Q}}(S) < 2^4$. Assume next that $|x_1 - x_4| \geq |y_2 - y_3|$. Then, $S \cap Q$ cannot be $\{(x_1, y_1), (x_4, y_4)\}$ for any $Q \in \mathcal{Q}$ and hence, $p_{\mathcal{Q}}(S) < 2^4$. Thus $p^*_{\mathcal{Q}}(4) < 2^4$, and hence, $p^*_{\mathcal{Q}}(k) = O(k^3)$ holds by Corollary 1.

Conversely, with S in Example 2, we have $p^*_{\mathcal{Q}}(3k) \geq p_{\mathcal{Q}}(S) \geq k^3$ $(k = 1, 2, \cdots)$. Thus, $\liminf_{k\to\infty} p^*_{\mathcal{Q}}(k)/k^3 \geq 1/3^3 > 0$, which completes the proof.

(3) Take an arbitrary $S \in \mathcal{F}_5(\mathbb{R}^2)$. By Lemma 5, there are 3 points s_1, s_2, s_3 in S satisfying the condition in (1) of Lemma 5. Then, $S \cap R$ cannot be $\{s_1, s_3\}$ for any $Q \in \mathbb{R}$. Hence, $p^*_{\mathbb{R}}(5) < 2^5$. Therefore by Corollary 1, we have $p^*_{\mathbb{R}}(k) = O(k^4)$.

Conversely, let

$$\begin{aligned} S = &\{(-k, 0), (-k+1, 0), \cdots, (-1, 0)\} \cup \{(1, 0), (2, 0), \cdots, (k, 0)\} \\ &\cup \{(0, -k), (0, -k+1), \cdots, (0, -1)\} \cup \{(0, 1), (0, 2), \cdots, (0, k)\}. \end{aligned}$$

Then, for any $i, j, l, h \in \{1, 2, \cdots, k\}$, there exists $R \in \mathcal{R}$ such that the set

$$\begin{aligned} &\{(-i, 0), (-i+1, 0), \cdots, (-1, 0)\} \cup \{(1, 0), (2, 0), \cdots, (j, 0)\} \\ &\cup \{(0, -l), (0, -l+1), \cdots, (0, -1)\} \cup \{(0, 1), (0, 2), \cdots, (0, h)\} \end{aligned}$$

coincides with $S \cap \mathcal{R}$. Hence, we have $p^*_{\mathcal{R}}(4k) \geq p_{\mathcal{R}}(S) \geq k^4$ $(k = 1, 2, \cdots)$. Thus, $\liminf_{k\to\infty} p^*_{\mathcal{R}}(k)/k^4 \geq 1/4^4 > 0$, which completes the proof.

(4) Take an arbitrary $S \in \mathcal{F}_{2n+2}(\mathbb{R}^2)$. If there exists $s \in S$ such that $s \in \widehat{S \setminus \{s\}}$, then $S\setminus\{s\}$ cannot be of the form $S \cap C$ for any $C \in \mathcal{C}_n$, which implies $p_{\mathcal{C}_n}(S) < 2^{2n+2}$. Hence, assume that S is the set of vertices of a convex $(2n+2)$-polygon, say $s_1 s_2 \cdots s_{2n+2}$. We prove that $U := \{s_2, s_4, \cdots, s_{2n+2}\}$ cannot be of the form $S \cap C$ for any $C \in \mathcal{C}_n$.

Suppose that $U = S \cap C$ holds with some $C \in \mathcal{C}_n$. Since $s_1 \notin U$ while $\{s_{2n+2}, s_2\} \subset U$, one of edges in C should separate s_1 from $\{s_{2n+2}, s_2\}$. In the same way, there is an edge of C which separates s_3 from $\{s_2, s_4\}$. These 2 edges must be different by the convexity of the polygon $s_1 s_2 \cdots s_{2n+2}$. Continuing this argument, we get a conclusion that we need a $(n+1)$-polygon C to have $U = S \cap C$, and hence, a contradiction. Therefore, we have $p_{\mathcal{C}_n}(S) < 2^{2n+2}$, which implies that $p^*_{\mathcal{C}_n}(n) < 2^{2n+2}$. Therefore, we have $p^*_{\mathcal{C}_n}(k) = O(k^{2n+1})$ by Corollary 1. Thus, $p^*_{\mathcal{C}_n}(k) \prec k^{2n+1}$.

Take a sufficiently large k. Let $S = \{s_1, s_2, \cdots, s_k\}$ be such that

$$s_j = (\cos(2\pi j/k),\ \sin(2\pi j/k)) \quad (j = 1, 2, \cdots, k).$$

Take any sequence $1 \le u_1 < u_2 < \cdots < u_{2n} \le k$. Let

$$\begin{aligned} U &= \{s_{u_1}, s_{u_1+1}, \cdots, s_{u_2}\} \cup \{s_{u_3} s_{u_3+1} \cdots s_{u_4}\} \\ &\qquad \cup \cdots \cup \{s_{u_{2n-1}}, s_{u_{2n-1}+1}, \cdots, s_{u_{2n}}\}. \end{aligned}$$

Let C be the n-polygon having edges including $\overline{s_{u_2} s_{u_3}},\ \overline{s_{u_4} s_{u_5}},\ \cdots,\ \overline{s_{u_{2n}} s_{u_1}}$. Then, we have $U = S \cap C$. Since the choice of $u_1, u_2, \cdots, u_{2n}$ determines the set U in one-to-one way, we have

$$p^*_{\mathcal{C}_n}(k) \ge \#\{S \cap C;\ C \in \mathcal{C}_n\} \ge \binom{k}{2n}.$$

Thus, $k^{2n} \prec p^*_{\mathcal{C}_n}(k)$.

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