Local time of self-affine sets of Brownian motion type – revisited


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Abstract

In [4], we studied self-affine sets of Brownian motion type. We obtained the image measure induced by the Lebesgue measure on the domain space. If it is absolutely continuous, the density function is called the local time. The local times are obtained by solving so called jigsaw puzzle. In this paper, we are specially interested in the self-affine set with 3 pieces with the orientations “up, down, up” (see Figure 1).

1 Introduction

Let \( u, v, w \) be positive numbers such that
\[
\begin{align*}
u^2 + v^2 + w^2 &= 1 \quad \text{and} \quad u - v + w = 1.
\end{align*}
\]
The solution \((u, v, w)\) can be parametrized by \(\beta\) with \(0 < \beta < 1\) so that

\[
\begin{align*}
u &= \frac{\beta}{1 - \beta + \beta^2}, & v &= \frac{\beta - \beta^2}{1 - \beta + \beta^2}, & w &= \frac{1 - \beta}{1 - \beta + \beta^2}.
\end{align*}
\]

(1.2)

Let \(\Omega_\beta \subset [0, 1] \times [0, 1]\) be the solution of the set equation:

\[
\Omega_\beta = \left( \varphi_{0,u^2} \times \varphi_{0,u} \right) (\Omega_\beta) \cup \left( \varphi_{u^2,u^2+v^2} \times \varphi_{u,u-v} \right) (\Omega_\beta) \cup \left( \varphi_{1-w^2,1} \times \varphi_{1-w,1} \right) (\Omega_\beta),
\]

(1.3)

where \(\varphi_{a,b}\) is the linear map on \(\mathbb{R}\) such that \(\varphi_{a,b}(0) = a, \ \varphi_{a,b}(1) = b\).

The linear mappings \(\varphi_{0,u^2} \times \varphi_{0,u}\) and \(\varphi_{1-w^2,1} \times \varphi_{1-w,1}\) map the unit square to small rectangles \([0, u^2] \times [0, u]\) and \([1 - w^2, 1] \times [1 - w, 1]\), respectively, keeping the direction (orientation “up”), while \(\varphi_{u^2,u^2+v^2} \times \varphi_{u,u-v}\) maps the unit square to a small rectangle \([u^2, u^2 + v^2] \times [u - v, u]\) reversing up and down (orientation “down”). Any of the 3 small rectangles satisfy that the vertical sizes are square root of the horizontal sizes. Therefore, the self-affine set \(\Omega_\beta\) is of Brownian motion type defined in [4]. It is proved to be the graph of a continuous function, which we denote \(N_\beta\). Specially, \(N_{1/2}\) is called \(N\)-function and studied in [1][4][5]. The self-affine sets are discussed in [2] generally as scaling dynamics.

In this paper, we study the fractal properties of the function \(N_\beta\) and generalize the results known in the case \(N_{1/2}\) to the general \(\beta\).

2 Preliminary Lemmas

**Lemma 1.** For any \(0 < \beta < 1\), \(\Omega_\beta\) is the graph of a continuous function, which we denote \(N_\beta\) (see Figure 1).

**Proof** For a compact set \(K \subset [0, 1] \times [0, 1]\), let

\[
\psi(K) = \left( \varphi_{0,u^2} \times \varphi_{0,u} \right) (K) \cup \left( \varphi_{u^2,u^2+v^2} \times \varphi_{u,u-v} \right) (K) \cup \left( \varphi_{1-w^2,1} \times \varphi_{1-w,1} \right) (K).
\]

Then, we have

\[
\Omega_\beta = \bigcap_{k=1}^{\infty} \psi^k([0, 1] \times [0, 1]).
\]

Since \(\psi^k([0, 1] \times [0, 1])\) is a union of rectangles each of which has vertical size at most \(u^k \vee w^k\) and horizontal size at least \(v^{2k}\). Moreover, they are connected one by one from left to right. For any \(\varepsilon > 0\), take \(k\) such that \(2(u^k \vee w^k) < \varepsilon\). Let \(\delta = v^{2k}\). Then, for any \((x, y), (x', y') \in \Omega_\beta\) with \(|x - x'| < \delta\), there exist 2 connected rectangles in \(\psi^k([0, 1] \times [0, 1])\) such that the union contains both of \((x, y), (x', y')\). Then, \(|y - y'| \leq 2(u^k \vee w^k) < \varepsilon\). Thus, \(\Omega_\beta\) is the graph of a continuous function. \(\square\)
Lemma 2. The equation (1.1) is satisfied by $u, v, w > 0$ if and only if (1.2) holds for some $0 < \beta < 1$.

Proof It is clear that (1.2) for some $0 < \beta < 1$ implies (1.1) together with $u, v, w > 0$. Conversely, let $u, v, w > 0$ satisfy (1.1). Then, $0 < u < 1$ follows and

$$v^2 + w^2 = 1 - u^2 \text{ and } v^2w^2 = (u - u^2)^2$$

hold. Hence, $v^2, w^2$ are the solution of the quadratic equation

$$X^2 - (1 - u^2)X + (u - u^2)^2 = 0 \quad (2.1)$$

in $X$, the discriminant of which is $(1 - u)^3(1 + 3u)$ and is positive. Since $-v + w = 1 - u > 0$ and $v < w$, this means that $v$ and $w$ are determined uniquely by $u$. That is,

$$v = \frac{1 - u^2 - \sqrt{(1 - u)^3(1 + 3u)}}{2}, \quad w = \frac{1 - u^2 + \sqrt{(1 - u)^3(1 + 3u)}}{2} \quad (2.2)$$

We can represent $u$ with $0 < u < 1$ uniquely as $u = \beta/(1 - \beta + \beta^2)$ by $\beta$ with $0 < \beta < 1$. Then, (1.2) follows from (2.2).
3 Local time and related results

Let $\lambda$ be the Lebesgue measure on the interval $[0, 1]$ and $\mu_\beta$ be the induced measure of $\lambda$ by $N_\beta$. That is, $\mu_\beta = \lambda \circ N_\beta^{-1}$. Let

$$\rho_\beta(x) = \begin{cases} 
\frac{2}{\beta} x & (0 \leq x \leq \beta) \\
\frac{2}{1-\beta} (1-x) & (\beta \leq x \leq 1)
\end{cases}. \tag{3.1}$$

Theorem 1. The measure $\mu_\beta$ is absolutely continuous with respect to the Lebesgue measure, the density of which is $\rho_\beta$. We call it the local time of $\Omega_\beta$ or $N_\beta$.

Proof By Theorem 3 in [4], it is sufficient to prove that $\rho_\beta$ has a solution of the jigsaw puzzle for

$$\{ [0,u], [1-w,u], [1-w,1]; 1, -1, 1 \}.$$ 

That is,

$$\rho_\beta = J_{[0,u]}\rho_\beta + J_{[1-w,u]}\rho_\beta + J_{[1-w,1]}\rho_\beta,$$

where for an interval $[a,b]$ and a function $f$ defined on $[0,1]$, we define functions $J_{[a,b]}f$ and $\tilde{J}_{[a,b]}f$ defined on $[a,b]$ by

$$J_{[a,b]}f(x) = (b-a) f(\varphi_{a,b}^{-1}x) \text{ and } \tilde{J}_{[a,b]}f(x) = (b-a) f(\varphi_{b,a}^{-1}x).$$

Let see this. In Figure 2, let

$$A = (0,0), \ B = (\beta, 2), \ C = (1,0), \ D = (1-w,0), \ E = (u,0)$$

and determine $F, G, H$ by the condition that $\triangle AFE, \triangle DGC$ and $\triangle EHD$ are similar to $\triangle ABC$ keeping the correspondence of vertices in order. We prove that the $x$-coordinates of $F, G$ and $H$ are same as those of $D, E$ and $B$, respectively.

Let $F = (x,y)$. Then by the similarity,

$$x = \beta u = \frac{\beta^2}{1 - \beta + \beta^2} = 1 - w.$$ 

Let $G = (x,y)$. Then by the similarity,

$$1 - x = (1-\beta)w = \frac{(1-\beta)^2}{1 - \beta + \beta^2}.$$ 

Hence,

$$x = 1 - \frac{(1-\beta)^2}{1 - \beta + \beta^2} = \frac{\beta}{1 - \beta + \beta^2} = u.$$
Finally, let $H = (x, y)$. Then, since
\[ x - (1 - w) : u - x = 1 - \beta : \beta, \]
\[ x = (1 - \beta)u + \beta(1 - w) = \frac{(1 - \beta)\beta + \beta\beta^2}{1 - \beta + \beta^2} = \beta. \]
Thus, the $x$-coordinates of $F$, $G$ and $H$ are same as those of $D$, $E$ and $B$, respectively.

The graph of our function $\rho_\beta$ is the broken line $ABC$ which we identify with the function. In this sense, we have to prove that
\[ ABC = AFE + DHE + DGC. \]
For $x \in [0, 1 - w]$, this holds since
\[ \frac{2}{\beta} x = \frac{2}{\beta} x + 0 + 0. \]
For $x \in [1 - w, \beta]$, this holds since the derivatives of the both sides are
\[ \frac{2}{\beta} = -\frac{2}{1 - \beta} + \frac{2}{1 - \beta} + \frac{2}{\beta}. \]
For $x \in [\beta, u]$, this holds since the derivatives of the both sides are
\[ -\frac{2}{1 - \beta} = -\frac{2}{1 - \beta} - \frac{2}{\beta} + \frac{2}{\beta}. \]
For $x \in [u, 1]$, this holds since
\[
\frac{2}{1 - \beta} (1 - x) = 0 + 0 + \frac{2}{1 - \beta} (1 - x).
\]
Thus, $\rho_\beta$ has a solution of the jigsaw puzzle. \hfill \Box

Since $\Omega_\beta$ is of Brownian motion type and has a bounded local time, the following corollary follows from [4].

**Corollary 1.** The graph of $N_\beta$ has a nonzero and finite $(3/2)$-Hausdorff measure. Hence, it has the Hausdorff dimension $3/2$.

Moreover, since $\Omega_\beta$ is admissible in the sense of Definition 2 in [5], the following corollary follows from [5].

**Corollary 2.** For any $y$ with $0 < y < 1$, the level set $\{x \in [0, 1]; N_\beta(x) = y\}$ has the Hausdorff dimension $1/2$.

**Remark 1.** There is an error in the proof of Lemma 2 of [5]. The corrected version is available at <http://www14.plala.or.jp/kamae>.

### 4 Other examples

We give some more examples of self-affine sets of Brownian motion type which have triangles or trapezoids as the local times.

Recall Example 7 in [4], where there is an error, $J_2 = J_3 = [a, a + 1/2]$ should be read as $J_2 = J_3 = [a/2, a/2 + 1/2]$. Denoting $\beta$ for $a$, we restate it as follows.

**Example 1.** ([4]) The self-affine set of Brownian motion type satisfying the set equation given in the left side of Figure 3, that is, consider the self-affine
set $\Omega$ satisfying that

$$
\Omega = (\varphi_{0,1/4} \times \varphi_{0,1/2}) (\Omega) \cup (\varphi_{1/4,1/2} \times \varphi_{\beta/2+1/2,\beta/2}) (\Omega) \\
\cup (\varphi_{1/2,3/4} \times \varphi_{\beta/2,2+1/2}) (\Omega) \cup (\varphi_{3/4,1} \times \varphi_{1/2,1}) (\Omega)
$$

has the same triangular function as in Figure 2 as the local time, since the jigsaw puzzle problem is solved as in the right side of Figure 3.

![Figure 4: Set equation and jigsaw puzzle](image)

Recall the following example in [2].

**Example 2.** (Dai, Feng, and Wang [2]) Consider the self-affine set of Brownian motion type defined by the set equation in the left side of Figure 4, where the orientations can be either upward or downward independently. Let $J_1 = [0, \sqrt{2}/2]$, $J_2 = [(2-\sqrt{2})/2, 1]$ and $\tau_1, \tau_2 \in \{-1, 1\}$. Then, the following isosceles trapezoid is the solution of the jigsaw puzzle for $\{J_1, J_2; \pm 1, \pm 1\}$ as shown in the right side of Figure 4:

$$
\rho(x) = \begin{cases} 
\frac{4 + 3\sqrt{2}}{2} x & 0 \leq x \leq \sqrt{2} - 1 \\
\frac{2 + \sqrt{2}}{2} & \sqrt{2} - 1 < x \leq 2 - \sqrt{2} \\
\frac{4 + 3\sqrt{2}}{2} (1 - x) & 2 - \sqrt{2} < x \leq 1
\end{cases}
$$

Hence, it is the local time of the self-affine set.
Example 3. Let $1 < a < b < 1$. Assume that $b - a$ is a solution of the equation $(2x + 1)^4 = x(x + 2)^4$, that is,

$$b - a = \frac{7 + \sqrt{117}}{4} - \frac{1}{2} \sqrt{\left(\frac{7 + \sqrt{117}}{2}\right)^2 - 4} = 0.1137 \cdots .$$

Let

$$\rho(x) = \begin{cases} 
\frac{2}{a(1 - a + b)} x & 0 \leq x \leq a \\
\frac{1 - a + b}{2} & a \leq x \leq b \\
\frac{(1 - b)(1 - a + b)}{2} (1 - x) & b \leq x \leq 1 
\end{cases}$$

Then, by Figure 5, $\rho$ is the local time of the self-affine set of Brownian motion type satisfying the set equation suggested by the figure. In this way, the ratio between the upper and lower sides can not be arbitrary if a trapezoid is the local time of a self-affine set of Brownian motion type.

![Figure 5: Jigsaw puzzle](image)

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References


