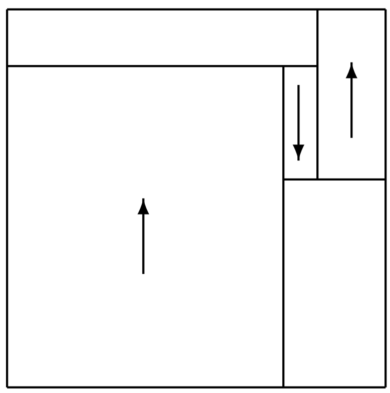
Local time of self-affine sets of Brownian motion type – revisited

Journal of Mathematical Analysis and Applications 437 (2016), pp. 638-644

Yu-Mei XUE* and Teturo KAMAE†

Abstract

In [4], we studied self-affine sets of Brownian motion type. We obtained the image measure induced by the Lebesgue measure on the domain space. If it is absolutely continuous, the density function is called the local time. The local times are obtained by solving so called jigsaw puzzle. In this paper, we are specially interested in the self-affine set with 3 pieces with the orientations "up, down, up" (see Figure 1).



1 Introduction

Let u, v, w be positive numbers such that

$$u^2 + v^2 + w^2 = 1 \text{ and } u - v + w = 1. \tag{1.1}$$

*School of Mathematics and System Sciences, LMIB, BeiHang University, Beijing 100191,PR China (yxue@buaa.edu.cn)

†Advanced Mathematical Institute, Osaka City University, 558-8585 Japan (kamae@apost.plala.or.jp)

The solution (u, v, w) can be parametrized by β with $0 < \beta < 1$ so that

$$u = \frac{\beta}{1-\beta+\beta^2} \ , \ v = \frac{\beta-\beta^2}{1-\beta+\beta^2} \ , \ w = \frac{1-\beta}{1-\beta+\beta^2} \tag{1.2}$$

Let $\Omega_\beta \subset [0,1] \times [0,1]$ be the solution of the set equation:

$$\begin{aligned} \Omega_\beta = &\left(\varphi_{0,u^2} \times \varphi_{0,u}\right)(\Omega_\beta) \\ &\bigcup \left(\varphi_{u^2,u^2+v^2} \times \varphi_{u,u-v}\right)(\Omega_\beta) \bigcup \left(\varphi_{1-w^2,1} \times \varphi_{1-w,1}\right)(\Omega_\beta), \end{aligned} \tag{1.3}$$

where $\varphi_{a,b}$ is the linear map on $\mathbb{R}$ such that $\varphi_{a,b}(0) = a, \ \varphi_{a,b}(1) = b$.

The linear mappings $\varphi_{0,u^2} \times \varphi_{0,u}$ and $\varphi_{1-w^2,1} \times \varphi_{1-w,1}$ map the unit square to small rectangles $[0, u^2] \times [0, u]$ and $[1 - w^2, 1] \times [1 - w, 1]$, respectively, keeping the direction (orientation "up"), while $\varphi_{u^2,u^2+v^2} \times \varphi_{u,u-v}$ maps the unit square to a small rectangle $[u^2, u^2 + v^2] \times [u - v, u]$ reversing up and down (orientation "down"). Any of the 3 small rectangles satisfy that the vertical sizes are square root of the horizontal sizes. Therefore, the self-affine set Ω_β is of Brownian motion type defined in [4]. It is proved to be the graph of a continuous function, which we denote N_β. Specially, $N_{1/2}$ is called N-function and studied in [1][4][5]. The self-affine sets are discussed in [2] generally as scaling dynamics.

In this paper, we study the fractal properties of the function N_β and generalize the results known in the case $N_{1/2}$ to the general β.

2 Preliminary Lemmas

Lemma 1. *For any $0 < \beta < 1$, Ω_β is the graph of a continuous function, which we denote N_β* (see Figure 1)*.*

Proof For a compact set $K \subset [0,1] \times [0,1]$, let

$$\psi(K) = \left(\varphi_{0,u^2} \times \varphi_{0,u}\right)(K) \cup \left(\varphi_{u^2,u^2+v^2} \times \varphi_{u,u-v}\right)(K) \cup \left(\varphi_{1-w^2,1} \times \varphi_{1-w,1}\right)(K).$$

Then, we have

$$\Omega_\beta = \bigcap_{k=1}^{\infty} \psi^k([0,1] \times [0,1]).$$

Since $\psi^k([0,1] \times [0,1])$ is a union of rectangles each of which has vertical size at most $u^k \vee w^k$ and horizontal size at least v^{2k}. Moreover, they are connected one by one from left to right. For any $\varepsilon > 0$, take k such that $2(u^k \vee w^k) < \varepsilon$. Let $\delta = v^{2k}$. Then, for any $(x, y), (x', y') \in \Omega_\beta$ with $|x - x'| < \delta$, there exist 2 connected rectangles in $\psi^k([0,1] \times [0,1])$ such that the union contains both of $(x, y), (x', y')$. Then, $|y - y'| \le 2(u^k \vee w^k) < \varepsilon$. Thus, Ω_β is the graph of a continuous function. $\square$

Figure 1: The set equation and converging to $\Omega_{1/2}$

Lemma 2. *The equation* (1.1) *is satisfied by* $u, v, w > 0$ *if and only if* (1.2) *holds for some* $0 < \beta < 1$.

Proof It is clear that (1.2) for some $0 < \beta < 1$ implies (1.1) together with $u, v, w > 0$. Conversely, let $u, v, w > 0$ satisfy (1.1). Then, $0 < u < 1$ follows and

$$v^2 + w^2 = 1 - u^2 \text{ and } v^2 w^2 = (u - u^2)^2$$

hold. Hence, v^2, w^2 are the solution of the quadratic equation

$$X^2 - (1 - u^2)X + (u - u^2)^2 = 0 \tag{2.1}$$

in X, the discriminant of which is $(1-u)^3(1+3u)$ and is positive. Since $-v + w = 1 - u > 0$ and $v < w$, this means that v and w are determined uniquely by u. That is,

$$v = \frac{1 - u^2 - \sqrt{(1-u)^3(1+3u)}}{2}, \quad w = \frac{1 - u^2 + \sqrt{(1-u)^3(1+3u)}}{2} \tag{2.2}$$

We can represent u with $0 < u < 1$ uniquely as $u = \beta/(1 - \beta + \beta^2)$ by β with $0 < \beta < 1$. Then, (1.2) follows from (2.2). □

3 Local time and related results

Let λ be the Lebesgue measure on the interval $[0,1]$ and μ_β be the induced measure of λ by N_β. That is, $\mu_\beta = \lambda \circ N_\beta^{-1}$. Let

$$\rho_\beta(x) = \begin{cases} \dfrac{2}{\beta}\ x & (0 \le x \le \beta) \\ \dfrac{2}{1-\beta}\ (1-x) & (\beta \le x \le 1) \end{cases} . \tag{3.1}$$

Theorem 1. *The measure μ_β is absolutely continuous with respect to the Lebesgue measure, the density of which is ρ_β. We call it the local time of Ω_β or N_β.*

Proof By Theorem 3 in [4], it is sufficient to prove that ρ_β has a solution of the jigsaw puzzle for

$$\{\ [0,u],\ [1-w,u],\ [1-w,1]\ ;\ 1,-1,1\ \}.$$

That is,

$$\rho_\beta = J_{[0,u]}\rho_\beta + \tilde{J}_{[1-w,u]}\rho_\beta + J_{[1-w,1]}\rho_\beta,$$

where for an interval $[a,b]$ and a function f defined on $[0,1]$, we define functions $J_{[a,b]}f$ and $\tilde{J}_{[a,b]}f$ defined on $[a,b]$ by

$$J_{[a,b]}f(x) = (b-a)\ f(\varphi_{a,b}^{-1}x) \text{ and } \tilde{J}_{[a,b]}f(x) = (b-a)\ f(\varphi_{b,a}^{-1}x)$$

Let see this. In Figure 2, let

$$A = (0,0),\ B = (\beta, 2),\ C = (1,0),\ D = (1-w, 0),\ E = (u, 0)$$

and determine F,G,H by the condition that △AFE, △DGC and △EHD are similar to △ABC keeping the correspondence of vertices in order. We prove that the x-coordinates of F, G and H are same as those of D, E and B, respectively.

Let F= (x,y). Then by the similarlity,

$$x = \beta u = \frac{\beta^2}{1-\beta+\beta^2} = 1-w.$$

Let G= (x,y). Then by the similarlity,

$$1-x = (1-\beta)w = \frac{(1-\beta)^2}{1-\beta+\beta^2}.$$

Hence,

$$x = 1 - \frac{(1-\beta)^2}{1-\beta+\beta^2} = \frac{\beta}{1-\beta+\beta^2} = u.$$

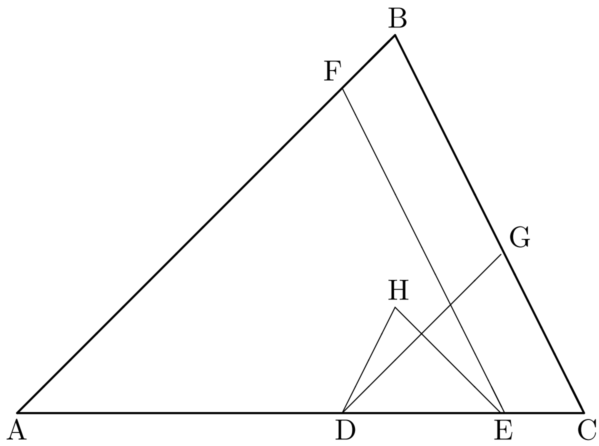


Figure 2: The graph of ρ_β

Finaly, let H$= (x, y)$. Then, since

$$x - (1-w) : u - x = 1 - \beta : \beta,$$

$$x = (1-\beta)u + \beta(1-w) = \frac{(1-\beta)\beta + \beta\beta^2}{1-\beta+\beta^2} = \beta.$$

Thus, the x-coordinates of F, G and H are same as those of D, E and B, respectively.

The graph of our function ρ_β is the broken line ABC which we identify with the function. In this sense, we have to prove that

$$\mathrm{ABC} = \mathrm{AFE} + \mathrm{DHE} + \mathrm{DGC}.$$

For $x \in [0, 1-w]$, this holds since

$$\frac{2}{\beta}\, x = \frac{2}{\beta}\, x + 0 + 0.$$

For $x \in [1-w, \beta]$, this holds since the derivatives of the both sides are

$$\frac{2}{\beta} = -\frac{2}{1-\beta} + \frac{2}{1-\beta} + \frac{2}{\beta}.$$

For $x \in [\beta, u]$, this holds since the derivatives of the both sides are

$$-\frac{2}{1-\beta} = -\frac{2}{1-\beta} - \frac{2}{\beta} + \frac{2}{\beta}.$$

For $x \in [u, 1]$, this holds since

$$\frac{2}{1-\beta}\ (1-x) = 0 + 0 + \frac{2}{1-\beta}\ (1-x).$$

Thus, ρ_β has a solution of the jigsaw puzzle. □

Since Ω_β is of Brownian motion type and has a bounded local time, the following corollary follows from [4].

Corollary 1. *The graph of N_β has a nonzero and finite $(3/2)$-Hausdorff measure. Hence, it has the Hausdorff dimension $3/2$.*

Moreover, since Ω_β is admissible in the sense of Definition 2 in [5], the following corollary follows from [5].

Corollary 2. *For any y with $0 < y < 1$, the level set $\{x \in [0,1];\ N_\beta(x) = y\}$ has the Hausdorff dimension $1/2$.*

Remark 1. There is an error in the proof of Lemma 2 of [5]. The corrected version is available at <http://www14.plala.or.jp/kamae>.

4 Other examples

We give some more examples of self-affine sets of Brownian motion type which have triangles or trapezoids as the local times.

Recall Example 7 in [4], where there is an error, $J_2 = J_3 = [a, a+1/2]$ should be read as $J_2 = J_3 = [a/2, a/2+1/2]$. Denoting β for a, we restate it as follows.

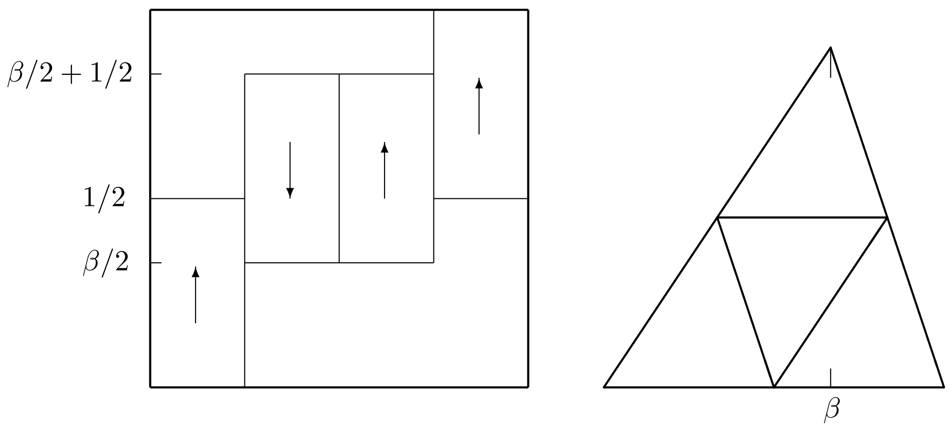


Figure 3: Jigsaw puzzles

Example 1. ([4]) The self-affine set of Brownian motion type satisfying the set equation given in the left side of Figure 3, that is, consider the self-affine set Ω satisfying that

$$\Omega = \left(\varphi_{0,1/4} \times \varphi_{0,1/2}\right)(\Omega) \bigcup \left(\varphi_{1/4,1/2} \times \varphi_{\beta/2+1/2,\beta/2}\right)(\Omega) \bigcup \left(\varphi_{1/2,3/4} \times \varphi_{\beta/2,\beta/2+1/2}\right)(\Omega) \bigcup \left(\varphi_{3/4,1} \times \varphi_{1/2,1}\right)(\Omega)$$

has the same triangular function as in Figure 2 as the local time, since the jigsaw puzzle problem is solved as in the right side of Figure 3.

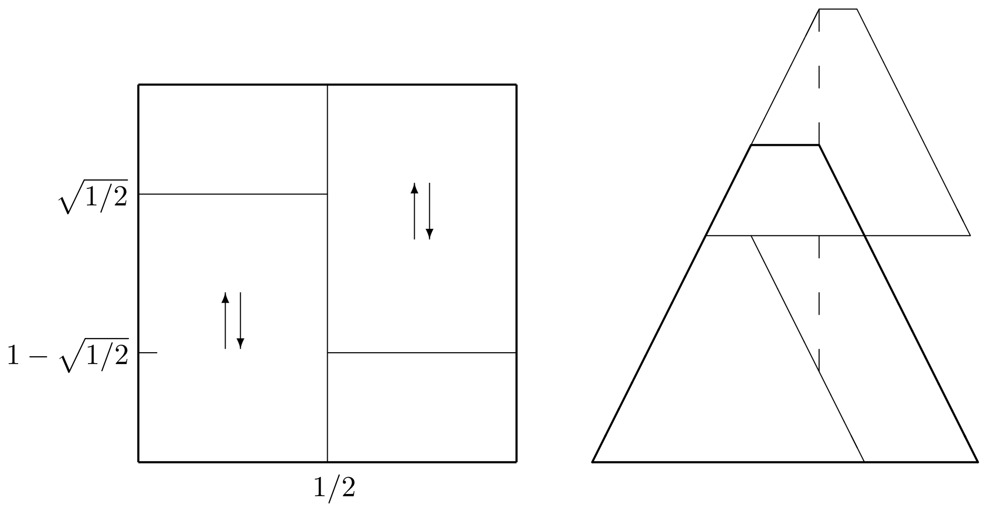


Figure 4: Set equation and jigsaw puzzle

Recall the following example in [2].

Example 2. (Dai, Feng, and Wang [2]) Consider the self-affine set of Brownian motion type defined by the set equation in the left side of Figure 4, where the orienttions can be either upward or downward independently. Let $J_1 = [0, \sqrt{2}/2]$, $J_2 = [(2-\sqrt{2})/2, 1]$ and $\tau_1, \tau_2 \in \{-1, 1\}$. Then, the following isosceles trapezoid is the solution of the jigsaw puzzle for $\{J_1, J_2;\ \pm 1, \pm 1\}$ as shown in the right side of Figure 4:

$$\rho(x) = \begin{cases} \dfrac{4+3\sqrt{2}}{2}\, x & 0 \le x \le \sqrt{2}-1 \\ \dfrac{2+\sqrt{2}}{2} & \sqrt{2}-1 < x \le 2-\sqrt{2} \\ \dfrac{4+3\sqrt{2}}{2}\,(1-x) & 2-\sqrt{2} < x \le 1 \end{cases}$$

Hence, it is the local time of the self-affine set.

Example 3. Let $1 < a < b < 1$. Assume that $b - a$ is a solution of the equation $(2x+1)^4 = x(x+2)^4$, that is,

$$b - a = \frac{7+\sqrt{117}}{4} - \frac{1}{2}\sqrt{\left(\frac{7+\sqrt{117}}{2}\right)^2 - 4} = 0.1137\cdots.$$

Let

$$\rho(x) = \begin{cases} \dfrac{2}{a(1-a+b)}\, x & 0 \le x \le a \\ \dfrac{2}{1-a+b} & a \le x \le b \\ \dfrac{2}{(1-b)(1-a+b)}\,(1-x) & b \le x \le 1 \end{cases}$$

Then, by Figure 5, ρ is the local time of the self-affine set of Brownian motion type satisfying the set equation suggested by the figure. In this way, the ratio between the upper and lower sides can not be arbitrary if a trapezoid is the local time of a self-affine set of Brownian motion type.

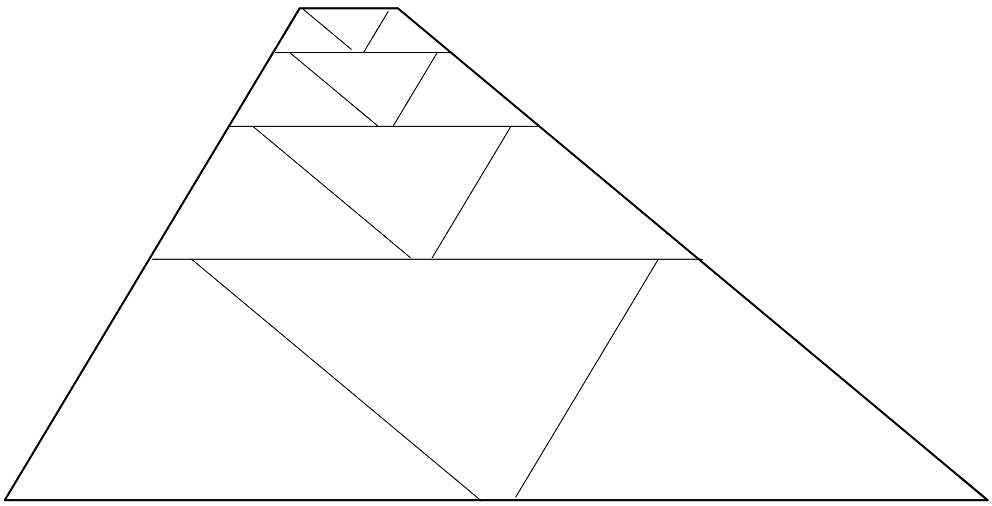


Figure 5: Jigsaw puzzle

Acknowledgement: This work is partially supported by the Fundamental Research Funds for the Central Universities (No. YWF-15-SXXY-005 and YWF-15-SXXY-009) and NSFC grant (No. 11290141 and No.11571030).

References

[1] Teturo Kamae, Stochastic analysis based on deterministic Brownian motion, Israel J. Math. 125 (2001), pp. 317-346

[2] Xin-Rong Dai, De-Jun Feng, and Yang Wang, Classification of refinable splines, Constr. Approx. 24 (2006), pp. 187-200

[3] De-Jun Feng and Huyi Hu, Dimension theory of iterated function systems, Communications on Pure and Applied Mathematics, Vol. LXII (2009), pp. 1435-1500

[4] Yu-Mei Xue and Teturo Kamae, Local time of self-affine sets of Brownian motion type and the jigsaw puzzle problem, J. Math. Anal. Appl. 419 (2014) pp. 79-93

[5] Li Peng and Teturo Kamae, Hausdorff dimension of the level sets of self-affine functions, J. Math. Anal. Appl. 423 (2015) pp. 1400-1409