Super-stationary set, Subword problem and the Complexity

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Teturo KAMAE, Hui RAO, Bo TAN and Yu-Mei XUE

Abstract

Let $\Omega \subset \{0,1\}^{\mathbb{N}}$ be a nonempty closed set with $\mathbb{N} = \{0,1,2,\cdots\}$. For $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$ and $\omega \in \{0,1\}^{\mathbb{N}}$, define $\omega[\mathcal{N}] \in \{0,1\}^{\mathbb{N}}$ by $\omega[\mathcal{N}](n) := \omega(N_n)$ $(n \in \mathbb{N})$ and

$$\Omega[\mathcal{N}] := \{\omega[\mathcal{N}] \in \{0,1\}^{\mathbb{N}};\ \omega \in \Omega\}.$$

We call Ω a super-stationary set if $\Omega[\mathcal{N}] = \Omega$ holds for any infinite subset $\mathcal{N}$ of $\mathbb{N}$.

Denoting Ω' the derived set (i.e. the set of accumulating points) of Ω and $\deg\Omega = \inf\{d;\ \Omega^{(d+1)} = \emptyset\}$ with $\Omega^{(1)} = \Omega'$, $\Omega^{(2)} = (\Omega')'$, $\cdots$, it is known [K] that for any nonempty closed subset Ω of $\{0,1\}^{\mathbb{N}}$ such that $\deg(\Omega \circ \rho) < \infty$ for some injection $\rho : \mathbb{N} \to \mathbb{N}$, there exists an infinite subset $\mathcal{M}$ such that $\Omega[\mathcal{M}]$ is a super-stationary set. Moreover, if $\deg(\Omega \circ \rho) = \infty$ for any injection $\rho : \mathbb{N} \to \mathbb{N}$, then $p^*_\Omega(k) = 2^k$ $(k = 1, 2, \cdots)$ holds.

We call $\xi \in \{0,1\}^*$ a super-subword of $\omega \in \{0,1\}^{\mathbb{N}}$ if there exists $S = \{s_1 < s_2 < \cdots < s_k\}$ with $k = |\xi|$ such that $\xi = \omega[S] := \omega(s_1)\omega(s_2)\cdots\omega(s_k)$. Let $\mathcal{P}(\xi)$ be the set of $\omega \in \{0,1\}^{\mathbb{N}}$ having no super-subword ξ. Denote

$$\mathcal{Q}(\Xi) = \cup_{\xi\in\Xi}\mathcal{P}(\xi) \text{ and } \mathcal{P}(\Xi) = \cap_{\xi\in\Xi}\mathcal{P}(\xi),$$

where $\Xi \subset \{0,1\}^*$.

In this paper, we prove that the class of super-stationary sets other than $\{0,1\}^{\mathbb{N}}$ coincides with the class of $\mathcal{Q}(\Xi)$ for nonempty finite sets $\Xi \subset \{0,1\}^+$. Moreover, it also coincides with the class of $\mathcal{P}(\mathrm{L}(\Xi))$ for nonempty finite sets $\Xi \subset \{0,1\}^+$, where $\mathrm{L}(\Xi)$ is the set of minimal covers of Ξ. Using these expressions, we can calculate the complexity of super-stationary sets and prove that the complexity function of a super-stationary set in k is either 2^k or a polynomial function of k for large k.

We also discuss the word problems related to the super-subwords.

1 Introduction

An element $\omega \in \{0,1\}^{\mathbb{N}}$ is called an *infinite* 0-1-*word* which is a mapping from $\mathbb{N}$ to $\{0,1\}$, while it is also considered as an infinite sequence $\omega(0)\omega(1)\omega(2)\cdots$ of 0 and 1. On the other hand, an element u in $\{0,1\}^* := \cup_{k=0}^{\infty}\{0,1\}^k$ is called a *finite* 0-1-*word* and represented as a finite sequence $u_1u_2\cdots u_k$ of 0 and 1, where k is such that $u \in \{0,1\}^k$, which is called the *length* of u and is denoted by $|u|$. We also denote $\{0,1\}^+ = \cup_{k=1}^{\infty}\{0,1\}^k = \{0,1\}^* \setminus \{\epsilon\}$, where ϵ is the empty word.

The *concatenation* $u\omega$ of $u \in \{0,1\}^*$ and $\omega \in \{0,1\}^* \cup \{0,1\}^{\mathbb{N}}$ is defined as the finite or infinite word $u_0u_1\cdots u_{k-1}\omega(0)\omega(1)\omega(2)\cdots$. In this case, u is called a *prefix* of $u\omega$. For $u \in \{0,1\}^*$, the *cylinder set* $[u]$ determined by u is defined by

$$[u] = \{\omega \in \{0,1\}^{\mathbb{N}};\ u \text{ is the prefix of } \omega\}.$$

Let $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\}$ be an infinite subset of $\mathbb{N}$. For $\omega \in \{0,1\}^{\mathbb{N}}$ and $\Omega \subset \{0,1\}^{\mathbb{N}}$, define $\omega[\mathcal{N}] \in \{0,1\}^{\mathbb{N}}$ and $\Omega[\mathcal{N}] \subset \{0,1\}^{\mathbb{N}}$ by

$$\omega[\mathcal{N}](n) := \omega(N_n) \quad (n \in \mathbb{N})$$
$$\Omega[\mathcal{N}] := \{\omega[\mathcal{N}] \in \{0,1\}^{\mathbb{N}};\ \omega \in \Omega\}.$$

Definition 1. A nonempty closed set $\Omega \subset \{0,1\}^{\mathbb{N}}$ is called a *super-stationary* set if $\Omega[\mathcal{N}] = \Omega$ holds for any infinite subset $\mathcal{N}$ of $\mathbb{N}$. Note that if $\mathcal{N} = \{1,2,\cdots\}$, then $\Omega[\mathcal{N}] = T\Omega$, where $T : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is the shift. Hence, if Ω is super-stationary, it is *stationary* in the sense that $T\Omega = \Omega$.

For $\xi \in \{0,1\}^k$ with $k \geq 0$, $\eta = \eta_1\eta_2\cdots\eta_l \in \{0,1\}^l$ with $l \geq k$ and $\omega \in \{0,1\}^{\mathbb{N}}$, ξ is called a *super-subword* of η or ω if there exists $S = \{s_1 < s_2 < \cdots < s_k\}$ which is a subset of $\{1,2,\cdots,l\}$ or $\mathbb{N}$, such that $\xi = \eta[S] := \eta_{s_1}\eta_{s_2}\cdots\eta_{s_k}$ or $\omega[S] := \omega(s_1)\omega(s_2)\cdots\omega(s_k)$, respectively. We also denote $\Omega[S] := \{\omega[S];\ \omega \in \Omega\}$ for $\Omega \subset \{0,1\}^{\mathbb{N}}$. We denote $\xi \ll \eta$ or $\xi \ll \omega$ if ξ is a super-subword of η or ω, respectively.

For $\xi \in \{0,1\}^*$, let

$$\mathcal{P}(\xi) := \{\omega \in \{0,1\}^{\mathbb{N}};\ \xi \ll \omega \text{ does not hold}\}$$

and

$$\mathcal{Q}(\Xi) := \bigcup_{\xi\in\Xi} \mathcal{P}(\xi)\ ,\quad \mathcal{P}(\Xi) := \bigcap_{\xi\in\Xi} \mathcal{P}(\xi)$$

for $\Xi \subset \{0,1\}^*$. Note that $\mathcal{P}(\epsilon) = \emptyset$ and $\mathcal{P}(\emptyset) = \{0,1\}^{\mathbb{N}}$.

Definition 2. Let Ξ be a nonempty finite subset of $\{0,1\}^*$.
(1) We call Ξ *noncomparable* if for any $\xi, \eta \in \Xi$ with $\xi \neq \eta$, $\xi \ll \eta$ does not hold.

(2) We call $\eta \in \{0,1\}^* \cup \{0,1\}^{\mathbb{N}}$ a *cover* of Ξ if $\xi \ll \eta$ for any $\xi \in \Xi$. A cover η of Ξ is called a *minimal cover* of Ξ if any ζ with $\zeta \underset{\neq}{\ll} \eta$ is not a cover of Ξ. The *least common multiple* of Ξ is, by definition, the set of all minimal covers of Ξ, which is denoted by $\mathrm{L}(\Xi)$. Note that $\mathrm{L}(\Xi)$ is a finite subset of $\{0,1\}^*$.
(3) We call $\eta \in \{0,1\}^*$ a *core* of Ξ, if $\eta \ll \xi$ holds for any $\xi \in \Xi$. A core η of Ξ is called a *maximal core* of Ξ if any ζ with $\eta \underset{\neq}{\ll} \zeta$ is not a core of Ξ. The set of maximal cores of Ξ is called the *greatest common factor* of Ξ and is denoted by $\mathrm{G}(\Xi)$. Clearly, $\mathrm{G}(\Xi)$ is a nonempty finite subset of $\{0,1\}^*$ (possibly, $\{\epsilon\}$).

Definition 3. For a nonempty closed set $\Omega \subset \{0,1\}^{\mathbb{N}}$, let Ω' be the set of accumulating points of Ω, that is,

$$\Omega' = \{\omega \in \Omega;\ \#([\omega|_k] \cap \Omega) = \infty \text{ for any } k \in \mathbb{N}\},$$

where $\omega|_k := \omega(0)\omega(1)\cdots\omega(k-1) \in \{0,1\}^k$. We call Ω' the *derived set* of Ω. Clearly, Ω' is a closed set (possibly, the empty set). We denote $\Omega^{(0)} = \Omega$ and $\Omega^{(i)} = (\Omega^{(i-1)})'$ for $i = 1, 2, \cdots$. For completeness, we define $\emptyset' = \emptyset$. The *degree* of Ω is defined to be the minimum d, if exists, such that $\Omega^{(d+1)} = \emptyset$, which is denoted by $\deg \Omega$. If such d does not exist, then we define $\deg \Omega = \infty$.

A super-stationary set Ω is a *uniform set*, that is, a nonempty closed set such that for any nonempty finite set $S \subset \mathbb{N}$, $\#\Omega[S]$ depends only on $\#S$. For a uniform set Ω, the function $p_\Omega(k) := \#\Omega[S]$ of $k = 1, 2, \cdots$, where $S \subset \mathbb{N}$ satisfies that $\#S = k$, is called the *uniform complexity* function of Ω. Define $p_\Omega(0) = 1$ if necessary.

Theorem 1. (T. Kamae [K]) *Let Ω be a nonempty closed subset of $\{0,1\}^{\mathbb{N}}$.*
(1) If there exists an infinite subset $\mathcal{N}$ of $\mathbb{N}$ such that $\deg \Omega[\mathcal{N}] < \infty$, then there exists an an infinite subset $\mathcal{M}$ of $\mathbb{N}$ such that $\Omega[\mathcal{M}]$ is a super-stationary set.
(2) If $\deg \Omega[\mathcal{N}] = \infty$ for any infinite subset $\mathcal{N}$ of $\mathbb{N}$, then $p^*_\Omega(k) = 2^k$ $(k = 1, 2, \cdots)$.
Hence, all uniform complexity functions are realized by super-stationary sets.

Definition 4. For $\Xi \subset \{0,1\}^*$, define

$$\begin{aligned} Q(\Xi)(k) &:= \#\{\eta \in \{0,1\}^k;\ \xi \ll \eta \text{ does not hold for some } \xi \in \Xi\} \\ P(\Xi)(k) &:= \#\{\eta \in \{0,1\}^k;\ \xi \ll \eta \text{ does not hold for any } \xi \in \Xi\}. \end{aligned}$$

Let Ξ be a nonempty finite subset of $\{0,1\}^*$ and $a \in \{0,1\}$. Denote

$$\Xi_a = \{\xi \in \{0,1\}^*;\ \xi a \in \Xi\}.$$

Denote Ξ_{max} (Ξ_{min}) the set of maximal (minimal, respectively) elements in Ξ with respect to the partial order $\ll$.

We prove the following Main Theorem.

Theorem 2. (Main Theorem)
(1) *The class of super-stationary sets other than* $\{0,1\}^{\mathbb{N}}$ *coincides with the class of sets* $\mathcal{Q}(\Xi)$ *for nonempty finite subsets* Ξ *of* $\{0,1\}^+$.
(2) *For any nonempty finite set* $\Xi \subset \{0,1\}^+$, $\mathcal{Q}(\Xi) = \mathcal{P}(\mathrm{L}(\Xi))$ *holds.*
(3) *For any super-stationary set* Ω *other than* $\{0,1\}^{\mathbb{N}}$, *take a nonempty finite subset* Ξ *of* $\{0,1\}^+$ *such that* $\Omega = \mathcal{Q}(\Xi)$. *Then, we have*

$$p_\Omega(k) = Q(\Xi)(k) = P(\mathrm{L}(\Xi))(k) \ (k = 1, 2, \cdots).$$

(4) *Let* $\Xi \subset \{0,1\}^+$ *be a nonempty finite set. Then, we have*

$$Q(\Xi) = Q(\Xi_{max}) \text{ and } P(\Xi) = P(\Xi_{min}).$$

Denoting $\Xi = \Xi_0 0 \cup \Xi_1 1$, *we have*

$$\begin{aligned} Q(\Xi)(k) &= Q(\Xi_0 \cup \Xi_1 1)(k-1) + Q(\Xi_0 0 \cup \Xi_1)(k-1) \\ P(\Xi)(k) &= P(\Xi_0 \cup \Xi_1 1)(k-1) + P(\Xi_0 0 \cup \Xi_1)(k-1) \\ &\quad (k = 1, 2, \cdots). \end{aligned}$$

In particular, if $\Xi = \Xi_0 0$, *then*

$$Q(\Xi)(k) = 1 + \sum_{i=0}^{k-1} Q(\Xi_0)(i) \ , \ P(\Xi)(k) = 1 + \sum_{i=0}^{k-1} P(\Xi_0)(i) \quad (k = 0, 1, 2, \cdots).$$

(*The same result holds for if* $\Xi = \Xi_1 1$.)

(5) *For a uniform set* Ω, *there exists a super-stationary set having the same uniform complexity function as* Ω. *Hence,* $p_\Omega(k)$ *is either* 2^k $(k = 1, 2, \cdots)$ *or a polynomial function of* k *with rational coefficient for large* k.
(6) *For any super-stationary set* Ω *other than* $\{0,1\}^{\mathbb{N}}$, $\deg \Omega$ *coincides with the degree of the polynomial* $p_\Omega(k)$ *of* k.

2 Proof of the Main Theorem

For $\xi = \xi_1 \xi_2 \cdots \xi_d \in \{0,1\}^+$ and $\omega \in \{0,1\}^{\mathbb{N}}$, define the *$\xi$-position* in ω to be the sequence $0 \le M_1 < M_2 < \cdots < M_\tau$ inductively, as follows:
(1) $M_0 = -1$.

Assume that $0 \le l \le d$ and $M_0 < M_1 < \cdots < M_l$ are already defined.
(2) If either $l = d$ or $l < d$ and $\{n > M_l;\ \omega(n) = \xi_{l+1}\} = \emptyset$, then let $\tau = l$ and the induction process is completed. Otherwise let $M_{l+1} = \min\{n > M_l;\ \omega(n) = \xi_{l+1}\}$ and repeat (2) with $l+1$ in place of l.

This τ is called the *ξ-length* in ω and denoted by $\tau(\xi, \omega)$.

In the same way, we define the *ξ-position* in η and the *ξ-length* in η for $\eta = \eta_1 \eta_2 \cdots \eta_k \in \{0,1\}^*$ to be the sequence $1 \le M_1 < M_2 < \cdots < M_\tau \le k$ with $M_0 = 0$ as above.

Lemma 1. *For any $\xi = \xi_1\xi_2\cdots\xi_d \in \{0,1\}^+$ and $\omega \in \{0,1\}^{\mathbb{N}}$, if there exists $0 \le m_1 < m_2 < \cdots < m_k$ with $k \le d$ such that $\omega(m_1)\omega(m_2)\cdots\omega(m_k) = \xi_1\xi_2\cdots\xi_k$. Let $0 \le M_1 < M_2 < \cdots < M_\tau$ be the ξ-position in ω with $\tau = \tau(\xi,\omega)$. Then, we have $k \le \tau$ and $M_i \le m_i$ $(i = 1,2,\cdots,k)$.*

Proof It holds that $M_1 = \min\{n;\ \omega(n) = \xi_1\} \le m_1$. Assume that $M_i \le m_i$ for i with $1 \le i \le k-1$. Then, we have

$$M_{i+1} = \min\{n > M_i;\ \omega(n) = \xi_{i+1}\} \le \min\{n > m_i;\ \ \omega(n) = \xi_{i+1}\} \le m_{i+1}.$$

Hence, $k \le \tau$ and $M_i \le m_i$ $(i = 1,2,\cdots,k)$ holds. □

Lemma 2. *It holds that $\mathcal{P}(\xi) = \{\omega \in \{0,1\}^{\mathbb{N}};\ \tau(\xi,\omega) < |\xi|\}$.*

The proof is obvious and omitted.

Lemma 3. *For any $\xi \in \{0,1\}^+$, $\mathcal{P}(\xi)$ is a super-stationary set. Hence, for any nonempty finite set $\Xi \subset \{0,1\}^+$, $\mathcal{Q}(\Xi)$ is a super-stationary set.*

Proof It is clear that $\mathcal{P}(\xi)$ is a nonempty closed set such that $\mathcal{P}(\xi)[\mathcal{N}] \subset \mathcal{P}(\xi)$ for any infinite subset $\mathcal{N}$ of $\mathbb{N}$. Therefore, it suffices to show that $\mathcal{P}(\xi)[\mathcal{N}] \supset \mathcal{P}(\xi)$ for any infinite subset $\mathcal{N}$ of $\mathbb{N}$.

Take an arbitrary $\omega \in \mathcal{P}(\xi)$ and $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$. Let $0 \le M_1 < M_2 < \cdots < M_\tau$ be the ξ-position in ω with $\tau < |\xi|$. Define $\eta \in \{0,1\}^{\mathbb{N}}$ by

$$\eta(n) = \begin{cases} \xi_i & \text{if } n = N_{M_i} \text{ for some } i = 1,2,\cdots,\tau \\ \overline{\xi_i} & \text{if } N_{M_{i-1}} < n < N_{M_i} \text{ for some } i = 1,2,\cdots,\tau+1 \end{cases}$$

for any $n \in \mathbb{N}$, where we put $N_{M_0} = -1$, $N_{M_{\tau+1}} = \infty$ and $\overline{0} = 1$, $\overline{1} = 0$. Then, the ξ-position in η is $N_{M_1}, N_{M_2}, \cdots, N_{M_\tau}$ with $\tau(\xi,\eta) = \tau(\xi,\omega) < |\xi|$. Hence, $\eta \in \mathcal{P}(\xi)$ and $\eta[\mathcal{N}] = \omega$, which implies that $\mathcal{P}(\xi)$ is a super-stationary set. The last statement holds since a finite union of super-stationary sets is super-stationary. □

Lemma 4. *Let Ω be a super-stationary set such that $\Omega \ne \{0,1\}^{\mathbb{N}}$. Then, $\sup\{\#\{n \in \mathbb{N};\ \omega(n) \ne \omega(n+1)\};\ \omega \in \Omega\} < \infty$.*

Proof Suppose that $\sup\{\#\{n \in \mathbb{N};\ \omega(n) \ne \omega(n+1)\};\ \omega \in \Omega\} = \infty$. Then, for any $\xi \in \{0,1\}^*$, there exists $\omega \in \Omega$ such that $\xi \ll \omega$. Since Ω is a super-stationary set, this implies that $\Omega = \{0,1\}^{\mathbb{N}}$, which is a contradiction. □

Proof of (1): Let Ω be a super-stationary set such that $\Omega \ne \{0,1\}^{\mathbb{N}}$. Let

$$K := \sup\{\#\{n \in \mathbb{N};\ \omega(n) \ne \omega(n+1)\};\ \omega \in \Omega\},$$

which is finite by Lemma 4. For $a = 0,1$, let

$$\Omega_a(K) = \{\omega \in \Omega;\ \#\{n \in \mathbb{N};\ \omega(n) \ne \omega(n+1)\} = K \text{ and } \omega(0) = a\}.$$

Then, $\Omega_a(K) \neq \emptyset$ for some $a \in \{0,1\}$, which we assume.

For $\omega \in \Omega_a(K)$, let

$$\omega = a^{n_1}\overline{a}^{n_2}\cdots b^{n_K}\overline{b}^{\infty}$$

be the block decomposition with $n_1 = n_1(\omega), \cdots, n_K = n_K(\omega) \in \{1, 2, \cdots\}$ and $b \equiv a + K - 1 \ (mod\ 2)$.

We call the i-th block (of $\Omega_a(K)$) *isolated*, where $i = 1, 2, \cdots, K$, if $n_i(\omega) = 1$ for any $\omega \in \Omega_a(K)$. Note that the first block is not isolated. To show this, take $\eta \in \Omega$ such that $\eta[\{1, 2, \cdots\}] = \omega$, which exists since $\Omega[\mathcal{N}] = \Omega$ for any infinite subset $\mathcal{N} \subset \mathbb{N}$. If $\eta(0) \neq a$, then

$$\#\{n \in \mathbb{N};\ \eta(n) \neq \eta(n+1)\} = K + 1,$$

contradicting the maximality of K. Hence, $\eta(0) = a$, which implies that $\eta \in \Omega_a(K)$ and $n_1(\eta) = n_1(\omega) + 1 \geq 2$.

We also prove that there do not exist 2 neighboring isolated blocks. Take the i-th block of ω with $1 \leq i < K$. Let it be $\cdots j$, the last j of which is located as $\omega(l)$. Then, $\omega(l) = j$ and $\omega(l+1) = \overline{j}$. Take $\eta \in \Omega$ such that $\eta[\mathbb{N} \setminus \{l+1\}] = \omega$, which is obtained from ω by inserting some $c \in \{0,1\}$ in between the l-th and the $(l+1)$-st blocks of ω, that is, $\cdots jc\overline{j}\cdots$. Therefore, $\eta \in \Omega_a(K)$ and at least one of the i-th or the $(i+1)$-st block is not isolated.

It holds that if i-th block is not isolated for some $i = 1, 2, \cdots, K$, then n_i can take any value in $\{1, 2, \cdots\}$, independently of other n_j's. This is because if i-th block is not isolated, then for some $\omega \in \Omega_a(K)$, $n_i(\omega) \geq 2$ and its i-th block is $jj\cdots$ with the first j located as $\omega(l)$. Take $\eta \in \Omega$ such that $\eta[\mathbb{N} \setminus \{l+1\}] = \omega$. Then, jj in ω is replaced by $jcj\cdots$ with $c \in \{0,1\}$ in η. The maximality of K implies that $c = j$. Hence, $\eta \in \Omega_a(K)$ and $n_i(\eta) = n_i(\omega) + 1$. Thus, n_i can be arbitrarily large.

By taking subsequence along an $\mathcal{N}$, the following Lemma holds:

Lemma 5. *Any element of the form $\omega = a^{n_1}\overline{a}^{n_2}\cdots b^{n_K}\overline{b}^{\infty}$ with the condition that $n_i \in \mathbb{N}$ if the i-th block is not isolated and $n_i \in \{0,1\}$ if i-th block is isolated belongs to Ω.*

Definition 5. Corresponding to the above set of words $a^{n_1}\overline{a}^{n_2}\cdots b^{n_K}\overline{b}^{\infty}$, we define the sequence $T := t_1t_2\cdots t_Kt_{K+1}$ of symbols I and δ so that $t_i = I$ if the i-th block of $\Omega_a(K)$ is not isolated and $t_i = \delta$ if the i-th block of $\Omega_a(K)$ is isolated. Then, δ-symbol is not at the first place, nor at the last place. Moreover, there are no cosecutive δ-symbols.

Since all δ's in T are followed by I, we can replace δI by one symbol. Let $S = s_1s_2\cdots s_d$ be the sequence of symbols I and J obtained from T by replacing all δI by J. Using this S, we define a sequence $S(a) = s_1(a_1)s_2(a_2)\cdots s_d(a_d)$ by

$$a_1 = a \ \text{ and } \ a_{i+1} = \begin{cases} \overline{a_i} & (s_i = I) \\ a_i & (s_i = J) \end{cases} \quad (i = 1, 2, \cdots, d-1).$$

We call S or $S(a)$ the *type* of $\Omega_a(K)$.

Let $S(a) = s_1(a_1)s_2(a_2)\cdots s_d(a_d)$ be the type of $\Omega_a(K)$. Consider the set of $\omega \in \{0,1\}^{\mathbb{N}}$ with the property that there exist positive integers $m_1, m_2, \cdots, m_{d-1}$ such that

$$\omega = s_1(a_1, m_1)s_2(a_2, m_2)\cdots s_d(a_{d-1}, m_{d-1})s_d(a_d, \infty), \tag{1}$$

$$\text{where} \quad s_i(a_i, m_i) := \begin{cases} a_i^{m_i} & (s_i = I) \\ a_i\overline{a_i}^{m_i-1} & (s_i = J) \end{cases} \quad (i = 1, 2, \cdots, d).$$

Define $\xi = \xi_1\xi_2\cdots\xi_d \in \{0,1\}^d$ by $\xi_i = a_{i+1}$ $(i = 1, 2, \cdots, d-1)$ and

$$\xi_d = \begin{cases} \overline{a_d} & (s_d = I) \\ a_d & (s_d = J) \end{cases} . \tag{2}$$

Lemma 6. *Every $\omega \in \Omega_a(K)$ is written as (1) with positive m_i's together with $m_i \geq 2$ if $s_i = J$. On the other hand, any $\omega \in \{0,1\}^{\mathbb{N}}$ which is written as (1) just with positive m_i's belongs to Ω. Moreover, for any $\omega \in \{0,1\}^{\mathbb{N}}$ written as (1) with positive m_i's, let $M_1, M_2, \cdots, M_\tau$ be the ξ-position in ω. Then, it holds that $\tau(\xi, \omega) = d-1$ and*

$$M_i = m_1 + m_2 + \cdots + m_i \quad (i = 1, 2, \cdots, d-1).$$

Hence, $\omega \in \mathcal{P}(\xi)$. In particular, $\Omega_a(K) \subset \mathcal{P}(\xi)$.

Proof The first 2 statements are clear from the definition of $S(a)$ and Lemma 5. Let $\omega \in \{0,1\}^{\mathbb{N}}$ be written as (1) with positive m_i's. Since $\xi_1 = a_2$ and $m_1 = \min\{n;\ \omega(n) = a_2\}$, $m_1 = M_1$ holds. Since $\xi_2 = a_3$ and $m_1 + m_2 = \min\{n > m_1;\ \omega(n) = a_3\}$, $M_2 = m_1 + m_2$. The poof proceeds in this way arriving at $M_{d-1} = m_1 + m_2 + \cdots + m_{d-1}$. Since $\{n > M_{d-1};\ \omega(n) = \xi_d\} = \emptyset$, we have $\tau(\xi, \omega) = d-1$ and $\omega \in \mathcal{P}(\xi)$. □

Lemma 7. *$\mathcal{P}(\xi) \subset \Omega$ holds.*

Proof Take any $\omega \in \mathcal{P}(\xi)$. Let $M_1, M_2, \cdots, M_\tau$ be the ξ-position in ω with $\tau \leq d-1$. Then, we have

$$\omega(n) = \begin{cases} \xi_i & \text{if } n = M_i \text{ for some } i = 1, 2, \cdots, \tau \\ \overline{\xi_i} & \text{if } M_{i-1} < n < M_i \text{ for some } i = 1, 2, \cdots, \tau+1 \end{cases}$$

for any $n \in \mathbb{N}$, where we put $M_0 = -1$ and $M_{\tau+1} = \infty$.

Let $m_1 = M_1$ and $m_i = M_i - M_{i-1}$ $(i = 2, 3, \cdots, \tau)$. Then it is easy to see that

$$\omega = s_1(a_1, m_1)s_2(a_2, m_2)\cdots s_\tau(a_\tau, m_\tau)s_{\tau+1}(a_{\tau+1}, \infty).$$

Hence, we have

$$\omega = \lim_{m\to\infty} s_1(a_1, m_1)s_2(a_2, m_2)\cdots s_\tau(a_\tau, m_\tau)s_{\tau+1}(a_{\tau+1}, m) \\ s_{\tau+2}(a_{\tau+2}, 1)\cdots s_d(a_{d-1}, 1)s_d(a_d, \infty)$$

Since the term under the "lim" belongs to Ω by Lemma 6 and Ω is a closed set, we have $\omega \in \Omega$, which completes the proof. $\square$

We have proved $\Omega_a(K) \subset \mathcal{P}(\xi) \subset \Omega$. In the same way, if $\Omega_{\overline{a}}(K) \neq \emptyset$, then there exists $\zeta \in \{0,1\}^+$ such that $\Omega_{\overline{a}}(K) \subset \mathcal{P}(\zeta) \subset \Omega$. Any case, there exists a nonempty finite set $\Xi_1 \subset \{0,1\}^+$ such that $\Omega(K) \subset \mathcal{Q}(\Xi_1) \subset \Omega$, where we put $\Omega(K) = \Omega_0(K) \cup \Omega_1(K)$.

If $\mathcal{Q}(\Xi_1) \subsetneq \Omega$, then put $\Omega^1 = \Omega \setminus \mathcal{Q}(\Xi_1)$. Let

$$L := \sup\{\#\{n \in \mathbb{N};\ \omega(n) \neq \omega(n+1)\};\ \omega \in \Omega^1\}.$$

Then, $L < K$ and there exists $a \in \{0,1\}$ such that $\Omega^1_a(L) \neq \emptyset$, where

$$\Omega^1_a(L) := \{\omega \in \Omega^1;\ \#\{n \in \mathbb{N};\ \omega(n) \neq \omega(n+1)\} = L \text{ and } \omega(0) = a\}.$$

For $\omega \in \Omega^1_a(L)$, let

$$\omega = a^{n_1}\overline{a}^{n_2} \cdots b^{n_L}\overline{b}^{\infty}$$

be the block decomposition with $n_1 = n_1(\omega), \cdots, n_L = n_L(\omega) \in \{1,2,\cdots\}$, where $b \equiv a + L - 1\ (mod\ 2)$.

Likewise, the i-th block is *isolated*, where $i = 1, 2, \cdots, L$, if $n_i(\omega) = 1$ for any $\omega \in \Omega^1_a(L)$. Note that the first block is not isolated. To show this, take $\eta \in \Omega$ such that $\eta[\{1,2,\cdots\}] = \omega$. If $\eta \in \mathcal{Q}(\Xi_1)$, then we have a contradiction that $\omega \in \mathcal{Q}(\Xi_1)$ since $\mathcal{Q}(\Xi_1)$ is a super-stationary set by Lemma 3. Hence, $\eta \in \Omega^1$. If $\eta(0) \neq a$, then

$$\#\{n \in \mathbb{N};\ \eta(n) \neq \eta(n+1)\} = L + 1,$$

contradicting the maximality of L. Hence, $\eta(0) = a$, which implies that $\eta \in \Omega_a(L)$ and $n_1(\eta) = n_1(\omega) + 1 \geq 2$.

In this way, we can prove the same things for the block decomposition for $\Omega^1_a(L)$ as for $\Omega_a(K)$. Hence, there exists $\lambda \in \{0,1\}^+$ such that $\Omega^1_a(L) \subset \mathcal{P}(\lambda) \subset \Omega$.

Let $\Omega^1(L) := \Omega^1_0(L) \cup \Omega^1_1(L)$. Then, there exists a nonempty finite set $\Xi_2 \subset \{0,1\}^+$ such that $\Omega^1(L) \subset \mathcal{Q}(\Xi_2) \subset \Omega$. Hence,

$$\Omega(K) \cup \Omega^1(L) \subset \mathcal{Q}(\Xi_1 \cup \Xi_2) \subset \Omega.$$

If $\Omega \setminus \mathcal{Q}(\Xi_1 \cup \Xi_2) \neq \emptyset$, then put $\Omega^2 = \Omega \setminus \mathcal{Q}(\Xi_1 \cup \Xi_2)$ and let

$$M := \sup\{\#\{n \in \mathbb{N};\ \omega(n) \neq \omega(n+1)\};\ \omega \in \Omega^2\}.$$

Then, $M < L$. In this way, we continue until $\Omega = \mathcal{Q}(\Xi)$ for some nonempty finite set $\Xi \subset \{0,1\}^+$. Since $K > L > M > \cdots \geq 0$, it finish within $K+1$ steps.

Thus, any super-stationary set other than $\{0,1\}^{\mathbb{N}}$ can be written as $\mathcal{Q}(\Xi)$ for some nonempty finite set $\Xi \subset \{0,1\}^+$, which completes the proof of (1) of the Main Theorem. $\square$

Proof of (2): For any minimal cover η of Ξ, it holds that $1 \le |\eta| \le \sum_{\xi\in\Xi} |\xi|$. Therefore, $\mathrm{L}(\Xi)$ is a finite subset of $\{0,1\}^+$.

Assume that $\omega \notin \mathcal{Q}(\Xi)$. Then, $\omega \notin \mathcal{P}(\xi)$ for any $\xi \in \Xi$. That is, $\xi \ll \omega$ for any $\xi \in \Xi$, and hence, ω is a cover of Ξ. Therefore, there exists a minimal cover of Ξ, say $\eta \in \{0,1\}^+$, such that $\eta \ll \omega$. Thus, $\omega \notin \mathcal{P}(\mathrm{L}(\Xi))$.

Conversely, let $\omega \notin \mathcal{P}(\mathrm{L}(\Xi))$. Then, there exists a minimal cover η of Ξ such that $\omega \notin \mathcal{P}(\eta)$. That is, $\eta \ll \omega$. Since η is a minimal cover of Ξ, this implies that $\xi \ll \omega$ for any $\xi \in \Xi$. Hence, $\omega \neq \mathcal{P}(\xi)$ for any $\xi \in \Xi$. Thus $\omega \neq \mathcal{Q}(\Xi)$. □

Proof of (3): Since $\Omega = \mathcal{Q}(\Xi) = \mathcal{P}(\mathrm{L}(\Xi))$ is a super-stationary set by (1) and (2) of the Main Theorem,

$$\begin{aligned} p_\Omega(k) &= \#\Omega[\{0,1,\cdots,k-1\}] \\ &= \#\mathcal{Q}(\Xi)[\{0,1,\cdots,k-1\}] \\ &= \#\mathcal{P}(\mathrm{L}(\Xi))[\{0,1,\cdots,k-1\}] \end{aligned}$$

holds for $k = 1, 2, \cdots$.

It is clear from the definition that

$$\begin{aligned} & \mathcal{Q}(\Xi))[\{0,1,\cdots,k-1\}] \\ \subset\ & \{\eta \in \{0,1\}^k;\ \zeta \ll \eta \text{ does not hold for some } \zeta \in \mathrm{L}(\Xi)\} \\ & \mathcal{P}(\mathrm{L}(\Xi))[\{0,1,\cdots,k-1\}] \\ \subset\ & \{\eta \in \{0,1\}^k;\ \zeta \ll \eta \text{ does not hold for any } \zeta \in \mathrm{L}(\Xi)\} \end{aligned}$$

for any $k = 1, 2, \cdots$.

Conversely, if $\eta \in \{0,1\}^k$ satisfies that $\xi \ll \eta$ does not hold for some $\xi \in \Xi$. Let this ξ be $\xi_1\xi_2\cdots\xi_d$. Then, $\omega := \eta\overline{\xi_d}^\infty \in \mathcal{P}(\xi)$ since $\xi \ll \omega$ does not hold. Hence, $\omega \in \mathcal{Q}(\Xi)$ and $\omega[\{0,1,\cdots,k-1\}] = \eta$. Therefore, $\eta \in \mathcal{Q}(\Xi)[\{0,1,\cdots,k-1\}]$ and

$$\begin{aligned} & \mathcal{Q}(\Xi)[\{0,1,\cdots,k-1\}] \\ =\ & \{\zeta \in \{0,1\}^k;\ \zeta \ll \eta \text{ does not hold for some } \zeta \in \mathrm{L}(\Xi)\}. \end{aligned}$$

On the other hand, if $\eta \in \{0,1\}^k$ satisfies that $\zeta \ll \eta$ does not hold for any $\zeta \in \mathrm{L}(\Xi)$, then $\xi \ll \eta$ does not hold for some $\xi \in \Xi$. Let this ξ be $\xi_1\xi_2\cdots\xi_d$. Then, $\omega := \eta\overline{\xi_d}^\infty \in \mathcal{P}(\xi)$ since $\xi \ll \omega$ does not hold. Hence, $\omega \in \mathcal{Q}(\Xi) = \mathcal{P}(\mathrm{L}(\Xi))$ and $\omega[\{0,1,\cdots,k-1\}] = \eta$. Therefore, $\eta \in \mathcal{P}(\mathrm{L}(\Xi))[\{0,1,\cdots,k-1\}]$ and

$$\begin{aligned} & \mathcal{P}(\mathrm{L}(\Xi))[\{0,1,\cdots,k-1\}] \\ =\ & \{\zeta \in \{0,1\}^k;\ \zeta \ll \eta \text{ does not hold for any } \zeta \in \mathrm{L}(\Xi)\}. \end{aligned}$$

Thus,

$$\begin{aligned} \#\mathcal{Q}(\Xi)[\{0,1,\cdots,k-1\}] &= Q(\Xi)(k) \\ \#\mathcal{P}(\mathrm{L}(\Xi))[\{0,1,\cdots,k-1\}] &= P(\mathrm{L}(\Xi))(k) \end{aligned}$$

for any $k = 1, 2, \cdots$, which completes the proof. □

Proof of (4): Let $\Xi \subset \{0,1\}^+$. Then, it is clear that

$$Q(\Xi) = Q(\Xi_{max}) \ , \ P(\Xi) = P(\Xi_{min}).$$

Let $\Xi = \Xi_0 0 \cup \Xi_1 1$. Then,

$$\begin{aligned}
&\{\eta \in \{0,1\}^k;\ \xi \ll \eta \text{ does not hold for some (any) } \xi \in \Xi\} = \\
&\{\eta' 0;\ \eta' \in \{0,1\}^{k-1},\ \xi \ll \eta' \text{ does not hold for some (any) } \xi \in \Xi_0 \cup \Xi_1 1\} \\
&\qquad \bigcup \ \{\eta' 1;\ \eta' \in \{0,1\}^{k-1},\ \xi \ll \eta' \text{ does not hold} \\
&\qquad\qquad\qquad \text{for some (any, respectively })\xi \in \Xi_0 0 \cup \Xi_1\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
Q(\Xi)(k) &= Q(\Xi_0 0 \cup \Xi_1)(k-1) + Q(\Xi_0 \cup \Xi_1 1)(k-1) \\
P(\Xi)(k) &= P(\Xi_0 0 \cup \Xi_1)(k-1) + P(\Xi_0 \cup \Xi_1 1)(k-1)
\end{aligned}$$

for any $k = 1, 2, \cdots$.

Assume that $\Xi_1 = \emptyset$. Then by the above equality, we have

$$Q(\Xi)(i) = Q(\Xi)(i-1) + Q(\Xi_0)(i-1) \quad (i = 1, 2, \cdots).$$

Adding this equality for $i = 1, 2, \cdots, k$, we have

$$Q(\Xi)(k) = 1 + \sum_{j=0}^{k-1} Q(\Xi_0)(j) \quad (k = 1, 2, \cdots)$$

since $Q(\Xi)(0) = 1$. Note that this equality also holds for $k = 0$ since the both sides are 1. The same equality holds for P insread of Q. □

Proof of (5): Since $Q(\{0\})(k) = Q(\{1\})(k) = 1$ for any $k = 1, 2, \cdots$, we can prove that $Q(\Xi)(k)$ is a polynomial function of k with rational coefficient for sufficiently large k by (4) using the induction on $\sum_{\xi \in \Xi} |\xi|$. □

For $\xi = \xi_1 \xi_2 \cdots \xi_d \in \{0,1\}^+$, let $\xi' := \xi_1 \xi_2 \cdots \xi_{d-1}$ and $\Xi' := \{\xi';\ \xi \in \Xi\}$.

Lemma 8. *$\mathcal{Q}(\Xi)' = \mathcal{Q}(\Xi')$ holds. Hence, the class of super-stationary sets added the empty set is closed under the operations of taking derived set.*

Proof Since

$$\mathcal{Q}(\Xi)' = (\cup_{\xi \in \Xi} \mathcal{P}(\xi))' = \cup_{\xi \in \Xi} \mathcal{P}(\xi)',$$

it is sufficient to prove that $\mathcal{P}(\xi)' = \mathcal{P}(\xi')$ for any $\xi \in \{0,1\}^+$. Let $\xi = \xi_1 \xi_2 \cdots \xi_d \in \{0,1\}^+$.

Assume that $\omega \notin \mathcal{P}(\xi')$. Let $M_1, M_2, \cdots, M_{d-1}$ be the ξ'-position in ω. Let $S = \{0, 1, \cdots, M_{d-1}\}$. Then, any $\eta \in \mathcal{P}(\xi)$ such that $\eta[S] = \omega[S]$ must satisfy that $\omega(n) = \overline{\xi_d}$ for any $n > M_{d-1}$ since otherwise, $\xi \ll \eta$. Hence, $\omega \notin \mathcal{P}(\xi)'$.

Conversely, assume that $\omega \in \mathcal{P}(\xi')$. Let $M_1, M_2, \cdots, M_\tau$ be the ξ-position in ω with $\tau \leq d-2$. Then, for any $N \geq M_\tau$ and $n \in \mathbb{N}$,

$$\eta = \omega(0)\omega(1)\cdots\omega(N)\overline{\xi_{d-1}}^n\overline{\xi_d}^\infty \in \mathcal{P}(\xi).$$

Thus, we have $\omega \in \mathcal{P}(\xi)'$, which completes the proof. □

Lemma 9. *$p_{\mathcal{P}(\xi)}(k) = \sum_{i=0}^{d-1}\binom{k}{i}$ $(k = 1, 2, \cdots)$ holds for any $\xi \in \{0,1\}^+$ with $|\xi| = d$. Hence, $p_{\mathcal{P}(\xi)}(k)$ is a polynomial of k of degree $|\xi| - 1$.*

Proof We use the inducdion on $|\xi|$. If $|\xi| = 1$, then $\mathcal{P}(\xi)$ consists of one element $\overline{\xi_1}^\infty$. Hence, our Lemma holds. Assume that our Lemma holds for $|\xi| = 1, 2, \cdots, d$. Let $|\xi| = d + 1 \geq 2$. By (4-2) of the Main Theorem and the induction hypothesis, we have

$$p_{\mathcal{P}(\xi)}(k) = P(\xi)(k) = 1 + \sum_{i=0}^{k-1} P(\xi')(i) = 1 + \sum_{i=0}^{k-1}\sum_{j=0}^{d-1}\binom{i}{j}$$
$$= 1 + \sum_{j=0}^{d-1}\sum_{i=0}^{k-1}\left(\binom{i+1}{j+1} - \binom{i}{j+1}\right) = 1 + \sum_{j=0}^{d-1}\binom{k}{j+1} = \sum_{j=0}^{d}\binom{k}{j},$$

which completes the proof. □

Proof of (6): Let Ω be a super-stationary set other than $\{0,1\}^{\mathbb{N}}$. Then by (1) of the Main Theorem, there exists a nonempty finite set $\Xi \subset \{0,1\}^+$ such that $\Omega = \mathcal{Q}(\Xi)$. Then by Lemmas 8 and 9, we have

$$\deg\Omega = \max\{|\xi|;\ \xi \in \Xi\} - 1 = \text{the degree of the polynomial } p_\Omega(k) \text{ of } k,$$

which completes the proof. □

3 Word problems related to super-subwords

Let Ξ be a nonempty finite subset of $\{0,1\}^*$ and $a \in \{0,1\}$. If there exists $\xi \in \Xi$ such that $\xi = \overline{a}^n$ for some $n \in \mathbb{N}$, then let $\Xi_{(a)} = \emptyset$. Otherwise, let

$$\Xi_{(a)} = \{\xi \in \{0,1\}^*;\ \xi a\overline{a}^n \in \Xi \text{ for some } n \in \mathbb{N}\}.$$

Theorem 3. *For a nonempty finite set $\Xi \subset \{0,1\}^*$, the following statements hold.*
(1) $\mathrm{L}(\Xi) = \mathrm{L}(\Xi_{max})$
(2) $\mathrm{G}(\Xi) = \mathrm{G}(\Xi_{min})$
(3) $\mathrm{L}(\Xi) \subset \mathrm{L}(\Xi_0 \cup \Xi_1 1)0 \cup \mathrm{L}(\Xi_0 0 \cup \Xi_1)1$
(4) $\mathrm{G}(\Xi) \subset \mathrm{G}(\Xi_{(0)})0 \cup \mathrm{G}(\Xi_{(1)})1$

Proof (1)(2) are clear. To prove (3), take an arbitrary $\eta \in \mathrm{L}(\Xi)$ and $\xi \in \Xi$. If the last letters of η and ξ coincide, then $\xi \ll \eta$ implies $\xi' \ll \eta'$. Otherwise, $\xi \ll \eta$ implies $\xi \ll \eta'$. Hence, $\eta \in \mathrm{L}(\Xi_0 \cup \Xi_1 1)0 \cup \mathrm{L}(\Xi_0 0 \cup \Xi_1)1$.

To prove (4), take an arbitrary $\eta \in \mathrm{G}(\Xi)$. Let the last letter of η be $a \in \{0,1\}$. Since η is a core of Ξ, a appears in every element in Ξ and $\eta' \ll \xi$ for any $\xi \in \Xi_{(a)}$. Hence, $\eta \in \mathrm{G}(\Xi_{(a)})a$, which completes the proof. □

Theorem 4. (1) *For any noncomparable nonempty finite set* $\Theta \subset \{0,1\}^+$*,* $G(L(\Theta)) = \Theta$ *holds.*
(2) *For any noncomparable nonempty finite set* $\Theta \subset \{0,1\}^+$*,* $\Theta = L(\Xi)$ *holds for some nonempty finite set* $\Xi \subset \{0,1\}^+$ *if and only if* $L(G(\Theta)) = \Theta$ *holds.*

Proof To prove (1), let Θ be a noncomparable nonempty finite subset of $\{0,1\}^+$. Let $\xi = \xi_1\xi_2\cdots\xi_d \in \Theta$. Since ξ is a core of $L(\Theta)$, there exists $\eta \in G(L(\Theta))$ such that $\xi \ll \eta$. Suppose that $\xi \neq \eta$. Then, there exists k with $0 \le k \le d$ and $a \in \{0,1\}$ such that $\xi \ll \xi_1\cdots\xi_k a \xi_{k+1}\cdots\xi_d \ll \eta$.

Let $\zeta = \zeta^1\zeta^2$ with

$$\begin{aligned} \zeta^1 &= \overline{\xi_1}^{n_1}\xi_1\overline{\xi_2}^{n_2}\xi_2\cdots\overline{\xi_k}^{n_k}\xi_k, \text{ and} \\ \zeta^2 &= \xi_{k+1}\overline{\xi_{k+1}}^{n_{k+1}}\xi_{k+2}\overline{\xi_{k+2}}^{n_{k+2}}\cdots\xi_d\overline{\xi_d}^{n_d} \end{aligned}$$

with sufficiently large $n_1, n_2, \cdots, n_d$. Then, $\xi \ll \zeta$ holds, but $\eta \ll \zeta$ does not hold since $\xi_1\cdots\xi_k a\xi_{k+1}\cdots\xi_d \ll \zeta$ does not hold. We'll prove that $\theta \ll \zeta$ holds for any $\theta \in \Theta$.

To prove this, let $\theta = \theta_1\theta_2\cdots\theta_l \in \Theta\backslash\{\xi\}$ and consider the $\xi_1\cdots\xi_k$-position $M_1, \cdots, M_\tau$ and the $\xi_1\cdots\xi_k$-length τ in θ.

If $\tau < k$, then $\theta = \overline{\xi_1}^{m_1}\xi_1\overline{\xi_2}^{m_2}\xi_2\cdots\overline{\xi_\tau}^{m_\tau}\xi_\tau$ holds with some $m_1, \cdots, m_\tau \ge 0$. Hence, $\theta \ll \zeta^1 \ll \zeta$.

If $\tau = k$, then $\xi_1\cdots\xi_k \ll \theta_1\cdots\theta_{M_\tau} \ll \zeta^1$ holds. Hence, $\xi_{k+1}\cdots\xi_d \ll \theta_{M_\tau+1}\cdots\theta_l$ does not hold since otherwise, we have a contradiction $\xi \ll \theta$. Consider the the $\xi_d\cdots\xi_{k+1}$-length σ in $\theta_l\cdots\theta_{M_\tau+1}$. Since $\xi_{k+1}\cdots\xi_d \ll \theta_{M_\tau+1}\cdots\theta_l$ does not hold, $\xi_d\cdots\xi_{k+1} \ll \theta_l\cdots\theta_{M_\tau+1}$ does not hold. By the same argument as above, this implies that $\sigma < d-k$ and $\theta_l\cdots\theta_{M_\tau+1} \ll \overline{\xi_d}^{n_d}\xi_d\cdots\overline{\xi_{k+1}}^{n_{k+1}}\xi_{k+1}$. Hence, $\theta_{M_\tau+1}\cdots\theta_l \ll \zeta^2$. Together with $\theta_1\cdots\theta_{M_\tau} \ll \zeta^1$, we have $\theta \ll \zeta$.

Thus, we proved that ζ is a cover of Ξ such that $\eta \ll \zeta$ does not hold, which contradicts with $\eta \in G(L(\Xi))$. Hence, we should have $\xi = \eta \in G(L(\Xi))$. Therefore, we have $\Xi \subset G(L(\Xi))$. Now, we prove the oposite inclusion.

Take $\eta = \eta_1\eta_2\cdots\eta_d \in \{0,1\}^*$ such that $\eta \notin \Xi$. Since $\Xi \subset G(L(\Xi))$ and $G(L(\Xi))$ is noncomparable, if $\eta \ll \xi$ for some $\xi \in \Xi$, then $\eta \notin G(L(\Xi))$. Therefore, consider the case where $\eta \ll \xi$ does not hold for any $\xi \in \Xi$. Let

$$\zeta = \overline{\eta_1}^{n_1}\eta_1\overline{\eta_2}^{n_2}\eta_2\cdots\eta_{d-1}\overline{\eta_d}^{n_d}$$

with sufficiently large $n_1, n_2, \cdots, n_d$. Then, by the same argument as above, $\xi \ll \zeta$ holds for any $\xi \in \Xi$, but $\eta \ll \zeta$ does not hold. Hence, ζ is a cover of

Ξ such that $\eta \ll \zeta$ does not hold. Therefore, $\eta \notin G(L(\Xi))$. Thus, we proved that $\Xi \supset G(L(\Xi))$, which completes the proof of (1).

Let us prove (2). If $L(G(\Theta) = \Theta$ holds, then $\Theta = L(\Xi)$ holds with $\Xi = G(\Theta)$. Moreover, Ξ is a nonempty finite subset of $\{0,1\}^+$ since Θ is so and $L(\Xi) = \Theta$, $\Xi = G(\Theta)$. Assume that $\Theta = L(\Xi)$ holds with a nonempty finite set $\Xi \subset \{0,1\}^+$. We may assume that Ξ is noncomparable since otherwise, we can take a noncomparable subset Ξ_0 of Ξ such that $L(\Xi_0) = L(\Xi)$. Then, by (3), $L(G(\Theta)) = L(G(L(\Xi))) = L(Xi) = \Theta$, which completes the proof. $\square$

Example 1. Let $\Xi = \{110, 101, 100, 010\}$. Then by Thorem 3,

$$\begin{aligned}
\mathrm{L}(\Xi) &\subset \mathrm{L}(11,101,10,01)0 \cup \mathrm{L}(110,10,100,010)1 \\
&\subset \mathrm{L}(11,101)0 \cup \mathrm{L}(110,100,010)1 \\
&\subset \mathrm{L}(1,10)10 \cup \mathrm{L}(11,10,01)01 \\
&\subset \mathrm{L}(10)10 \cup \mathrm{L}(11,1,01)001 \cup \mathrm{L}(1,10,0)101 \\
&\subset \{1010\} \cup \mathrm{L}(11,01)001 \cup \mathrm{L}(10)101 \\
&\subset \{1010\} \cup \mathrm{L}(1,0)1001 \cup \{10101\} \\
&\subset \{1010\} \cup \mathrm{L}(1)01001 \cup \mathrm{L}(0)11001 \cup \{10101\} \\
&\subset \{1010\} \cup \{101001\} \cup \{011001\} \cup \{10101\} \\
&= \{1010, 101001, 011001, 10101\}.
\end{aligned}$$

Discarding non-minimal covers from the above, we have $\mathrm{L}(\Xi) = \{1010, 011001\}$.

Conversely,

$$\begin{aligned}
\mathrm{G}(1010, 011001) &\subset \mathrm{G}(101,0110)0 \cup \mathrm{G}(10,01100)1 \\
&\subset \mathrm{G}(1,011)00 \cup \mathrm{G}(10,01)10 \cup \mathrm{G}(10)1 \\
&\subset \mathrm{G}(1)00 \cup \mathrm{G}(1,\epsilon)010 \cup \mathrm{G}(\epsilon,0)110 \cup \{101\} \\
&= \{100, 010, 110, 101\}.
\end{aligned}$$

Since $\{100, 010, 110, 101\}$ is noncomparable, we have $\mathrm{G}(1010, 011001) = \{100, 010, 110, 101\}$. Hence, $\mathrm{G}(\mathrm{L}(\Xi)) = \Xi$.

4 Complexity functions with degree ≤ 2

Example 2. Let $\Omega = \mathcal{Q}(110, 101, 100, 010) = \mathcal{P}(1010, 011001)$. Then, by (4) of the Main Theorem, we have

$$\begin{aligned}
p_\Omega(k) &= p_{\mathcal{Q}(\mathrm{L}(\Xi))}(k) \\
&= Q(110, 101, 100, 010)(k) \\
&= Q(11, 101, 10, 01)(k-1) + Q(110, 10, 100, 010)(k-1) \\
&= Q(101)(k-1) + Q(110, 100, 010)(k-1) \\
&= \sum_{i=0}^{2} \binom{k-1}{i} + 1 + \sum_{i=0}^{k-2} Q(11, 10, 01)(i) \\
&= \frac{1}{2}k^2 - \frac{1}{2}k + 1 + 1 + 1 + \sum_{i=1}^{k-2} (Q(11, 10, 0)(i-1) + Q(1, 10, 0)(i-1)) \\
&= \frac{1}{2}k^2 - \frac{1}{2}k + 3 + \sum_{i=1}^{k-2} (Q(11, 10)(i-1) + Q(10)(i-1)) \\
&= \frac{1}{2}k^2 - \frac{1}{2}k + 3 + 2 + \sum_{i=2}^{k-2} (Q(11, 1)(i-2) + Q(1, 10)(i-2) + i) \\
&= \frac{1}{2}k^2 - \frac{1}{2}k + 5 + \sum_{i=2}^{k-2} (Q(11)(i-2) + Q(10)(i-2) + i) \\
&= \frac{1}{2}k^2 - \frac{1}{2}k + 5 + \sum_{i=2}^{k-2} (i - 1 + i - 1 + i) \\
&= \frac{1}{2}k^2 - \frac{1}{2}k + 5 + \frac{(3k-4)(k-3)}{2} \\
&= 2k^2 - 7k + 11.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
p_\Omega(k) &= p_{\mathcal{P}(\mathrm{L}(\Xi))}(k) \\
&= P(1010, 011001)(k) \\
&= P(101, 011001)(k-1) + P(1010, 01100)(k-1) \\
&= P(101)(k-1) + P(1010, 01100)(k-1) \\
&= \sum_{i=0}^{2} \binom{k-1}{i} + 1 + \sum_{i=0}^{k-2} P(101, 0110)(i) \\
&= \frac{1}{2}k^2 - \frac{1}{2}k + 1 + 1 + 1 + \sum_{i=1}^{k-2} (P(10, 0110)(i-1) + P(101, 011)(i-1)) \\
&= \frac{1}{2}k^2 - \frac{1}{2}k + 3 + \sum_{i=1}^{k-2} \left(P(10)(i-1) + 1 + \sum_{j=0}^{i-2} P(10, 01)(j) \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}k^2 - \frac{1}{2}k + 3 + \sum_{i=1}^{k-2} i + k - 2 + \sum_{i=2}^{k-2} (1 + 2(i-2)) \\
&= \frac{1}{2}k^2 + \frac{1}{2}k + 1 + \frac{(k-1)(k-2)}{2} + (k-3)^2 \\
&= 2k^2 - 7k + 11.
\end{aligned}$$

Consider the complexity functions of the super-stationary sets with degree 0, 1 or 2. They are finite unions of the following sets.

$$\begin{gathered}
\mathcal{P}(0)\ ,\ \mathcal{P}(1) \\
\mathcal{P}(00)\ ,\ \mathcal{P}(01)\ ,\ \mathcal{P}(10)\ ,\ \mathcal{P}(11) \\
\mathcal{P}(000)\ ,\ \mathcal{P}(001)\ ,\ \mathcal{P}(010)\ ,\ \mathcal{P}(011) \\
\mathcal{P}(100)\ ,\ \mathcal{P}(101)\ ,\ \mathcal{P}(110)\ ,\ \mathcal{P}(111)
\end{gathered}$$

All the complexity functions are listed below:

$$\begin{array}{cccc}
 & 1 & 2 & \\
k+1 & k+2 & 2k & 2k+2 \\
3k-2 & 3k-1 & 4k-4 &
\end{array}$$

$$\begin{array}{llll}
\frac{1}{2}k^2 + \frac{1}{2}k + 1 & \frac{1}{2}k^2 + \frac{1}{2}k + 2 & \frac{1}{2}k^2 + \frac{3}{2}k - 1 & \frac{1}{2}k^2 + \frac{3}{2}k + 2 \\
\frac{1}{2}k^2 + \frac{5}{2}k - 4 & \frac{1}{2}k^2 + \frac{5}{2}k - 2 & \frac{1}{2}k^2 + \frac{7}{2}k - 6 &
\end{array}$$

$$\begin{array}{llll}
k^2 - k + 2 & k^2 - 1 & k^2 & k^2 + k - 5 \\
k^2 + k - 4 & k^2 + k + 2 & k^2 + 2k - 8 & k^2 + 2k - 3 \\
k^2 + 3k - 8 & & &
\end{array}$$

$$\begin{array}{llll}
\frac{3}{2}k^2 - \frac{7}{2}k + 5 & \frac{3}{2}k^2 - \frac{5}{2}k + 2 & \frac{3}{2}k^2 - \frac{3}{2}k - 2 & \frac{3}{2}k^2 - \frac{1}{2}k - 6 \\
\frac{3}{2}k^2 - \frac{1}{2}k - 3 & \frac{3}{2}k^2 + \frac{1}{2}k - 11 & \frac{3}{2}k^2 + \frac{1}{2}k - 8 &
\end{array}$$

$$
\begin{array}{llll}
2k^2-7k+11 & 2k^2-6k+8 & 2k^2-5k+4 & 2k^2-4k \\
2k^2-4k+2 & 2k^2-3k-5 & 2k^2-3k-3 & 2k^2-2k-9 \\
2k^2-2k-8 & & &
\end{array}
$$

$$
\begin{array}{llll}
\frac{5}{2}k^2-\frac{19}{2}k+14 & \frac{5}{2}k^2-\frac{17}{2}k+10 & \frac{5}{2}k^2-\frac{15}{2}k+6 & \frac{5}{2}k^2-\frac{15}{2}k+7 \\
\frac{5}{2}k^2-\frac{13}{2}k+1 & \frac{5}{2}k^2-\frac{13}{2}k+2 & \frac{5}{2}k^2-\frac{11}{2}k-3 &
\end{array}
$$

$$
\begin{array}{llll}
3k^2-13k+20 & 3k^2-12k+16 & 3k^2-12k+17 & 3k^2-11k+11 \\
3k^2-11k+12 & 3k^2-10k+7 & 3k^2-9k+2 &
\end{array}
$$

$$
\frac{7}{2}k^2-\frac{33}{2}k+26 \quad \frac{7}{2}k^2-\frac{31}{2}k+22 \quad \frac{7}{2}k^2-\frac{29}{2}k+17
$$

$$
4k^2-20k+32,
$$

where the value for $p_\Omega(k)$ in the above is valid for $k \geq 6$.

5 Open problem

Let Ω and Λ be super-stationary sets with finite degree. It is interesting to ask when they are isomorphic to each other in the sense of [K]. If they are isomorphic, then

$$
\begin{aligned}
& d := \deg \Omega = \deg \Lambda \\
& p_{\Omega^{(i)}}(k) = p_{\Lambda^{(i)}}(k) \quad (k = 1, 2, \cdots;\ i = 0, 1, \cdots, d)
\end{aligned}
$$

holds.

Problem: Is this condition sufficient for them to be isomorphic?

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(Kamae's papers are downloadable from http://www14.plala.or.jp/kamae.)

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Teturo KAMAE (kamae@apost.plala.or.jp)
Matsuyama University, 790-8578 Japan

Hui RAO (hrao@math.tsinghua.edu.cn)
Department of Mathematics, Tsinghua University, Beijing 100084, P.R. China

Bo TAN (bo_tan@163.com)
Department of Mathematics, Huazhong University of Science and Technology, Wuhan 430074, P.R. China

Yu-Mei XUE (yxue@buaa.edu.cn)
Department of Mathematics and LMIB, Beijing University of Aeronautics and Astronautics, Beijing 10083, P.R. China