Uniform sets and super-stationary sets over general alphabets

published on line in Ergodic Theory and Dynamical Systems

doi:10.1017/S014338571000074X

Teturo KAMAE
Satakedai 5-9-6, 565-0855 Japan
kamae@apost.plala.or.jp

Abstract

Uniform sets and super-stationary sets over the binary alphabet have been extensively studied. In this paper, they are generalized to general alphabets. We generalize the fact that any uniform set contains a super-stationary set so that any uniform complexity is realized by a super-stationary set. This gives a formula to calculate the uniform complexity functions. We also give characterizations of the class of super-stationary sets in general settings in two somewhat different ways than in the binary case. Super-stationary sets are considered as phenomena which are independent of the time scale, but sensitive only to the direction of time, or dependent just on the order of events in time series. Hence, characterizations of super-stationary sets give insights into what is time, what looks like a history without description of time duration, or what remain meaningful after we lose quantitative sense of time.

1 Introduction

Uniform sets and super-stationary sets over a binary alphabet have been extensively studied. In the present paper we investigate these notions in the context of arbitrary finite alphabets. In some cases, earlier results in the binary case extend naturally to this more general setting and our proofs in the present paper, while more general, are simpler than those previously used in the binary case. For instance, as a direct application of the infinite Ramsey Theorem, we prove that every uniform set contains a super-stationary set. This implies that every uniform complexity may be realized by a super-stationary set, and in turn provides a formula for computing uniform complexity functions. In other cases, results in the binary case do not extend to larger alphabets (see, for instance, Theorem 2). Nevertheless we obtain two different characterizations of super-stationary sets (see Theorem 5 and 6) one of which is in the same spirit of Theorem 2.

Let $\mathbb{A}$ be a nonempty finite set called an *alphabet*. Elements of $\mathbb{A}$ are called *letters*. We always assume that $\mathbb{A}$ has at least 2 letters if not mentioned otherwise. For a nonempty closed subset Ω of $\mathbb{A}^{\mathbb{N}}$, where $\mathbb{N} = \{0, 1, 2, \cdots\}$, let $p_\Omega(S) := \#\Omega[S]$ be the *complexity* function depending on the finite sets $S \subset \mathbb{N}$, where $\#$ denotes the number of elements in a set and

$$\Omega[S] := \{\omega(s_1)\omega(s_2)\cdots\omega(s_k) \in \mathbb{A}^k;\ \omega \in \Omega\},$$

with $S = \{s_1 < s_2 < \cdots < s_k\}$. (Note that $p_\Omega(\emptyset) = 1$ if $\Omega \neq \emptyset$.) The *maximal pattern complexity* function $p^*_\Omega(k)$ defined on $k = 0, 1, 2, \cdots$ is the maximum of $p_\Omega(S)$ among the sets $S \subset \mathbb{N}$ with $\#S = k$.

We call Ω a *uniform set* if $p_\Omega(S)$ depends only on $\#S = k$ and the complexity function $p_\Omega(k) := p_\Omega(S)$ as a function of $k = 0, 1, 2, \cdots$ is called the *uniform complexity* function of Ω. In this case, $p_\Omega(k) = p^*_\Omega(k)$ holds for $k = 0, 1, 2, \cdots$.

Note that the definitions of complexity, maximal pattern complexity, uniform set and uniform complexity can be applicable for a general infinite index set Σ in place of $\mathbb{N}$, replacing $\Omega[S]$ by the restriction $\Omega|_S$ of $\Omega \subset \mathbb{A}^\Sigma$ to a finite set $S \subset \Sigma$. The class of uniform complexity functions remains unchanged by this generalization.

For an infinite set $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$, $\omega \in \mathbb{A}^{\mathbb{N}}$ and $\Omega \subset \mathbb{A}^{\mathbb{N}}$, define $\omega[\mathcal{N}] \in \mathbb{A}^{\mathbb{N}}$ and $\Omega[\mathcal{N}] \subset \mathbb{A}^{\mathbb{N}}$ by

$$\omega[\mathcal{N}](n) := \omega(N_n)\ (n \in \mathbb{N})$$

and

$$\Omega[\mathcal{N}] := \{\omega[\mathcal{N}] \in \mathbb{A}^{\mathbb{N}};\ \omega \in \Omega\}.$$

We call Ω a *super-stationary set* if $\Omega[\mathcal{N}] = \Omega$ holds for any infinite subset $\mathcal{N}$ of $\mathbb{N}$. It is clear that a super-stationary set is a uniform set.

Uniform sets and super-stationary sets over the binary alphabet were introduced and studied in connection with problems in symbolic dynamics [2-7, 10, 11]. Uniform sets are related to strategies to maximize the number of partitions obtained by piling a fixed number of congruent sets in a space. If it is attained by taking these sets arbitraray from a family (say, Σ) of congruent sets, this family is called an optimal position. In this case, the partition generated by the sets in Σ constitutes a uniform set by collecting the names in $\{0, 1\}^\Sigma$ of its elements. In particular, the orbit closures of recurrent pattern Sturmian words have optimal positions, and hence, uniform sets are related to them. So far, two different types of recurrent pattern Sturmian words are known; rotation words and Toeplitz words. The uniform sets related to them have different primitive factors, that is, the isomorphic classes of the super-stationary sets contained in them are different (Example 3). Thus, primitive factors are used to distinct dynamical systems.

An element $\omega \in \mathbb{A}^{\mathbb{N}}$ is called an *infinite word*: we regard ω both as a mapping from $\mathbb{N}$ to $\mathbb{A}$ as well as an infinite sequence $\omega(0)\omega(1)\omega(2)\cdots$ of letters in $\mathbb{A}$. On the other hand, an element ξ in $\mathbb{A}^* := \cup_{k=0}^{\infty}\mathbb{A}^k$ is called a *finite word* and represented as a finite sequence $\xi_1\xi_2\cdots\xi_k$ of letters in $\mathbb{A}$, where k is such that $\xi \in \mathbb{A}^k$, which is called the *length* of ξ and is denoted by $|\xi|$. We also denote $\mathbb{A}^+ = \cup_{k=1}^{\infty}\mathbb{A}^k = \mathbb{A}^* \setminus \{\epsilon\}$, where ϵ is the empty word.

The *concatenation* $\xi\eta$ or $\xi\omega$ of $\xi \in \mathbb{A}^*$ with $\eta \in \mathbb{A}^*$ or $\omega \in \mathbb{A}^{\mathbb{N}}$ is defined as the finite or infinite word $\xi_1\cdots\xi_k\eta_1\cdots\eta_l$ or $\xi_1\cdots\xi_k\omega(0)\omega(1)\omega(2)\cdots$, respectively with $k = |\xi|$ and $l = |\eta|$. In this case, ξ is called a *prefix* of $\xi\eta$ or $\xi\omega$, while η is called a *suffix* of $\xi\eta$.

For $\xi = \xi_1\xi_2\cdots\xi_k \in \mathbb{A}^*$, $\eta = \eta_1\eta_2\cdots\eta_l \in \mathbb{A}^*$ with $0 \le k \le l$ and $\omega \in \mathbb{A}^{\mathbb{N}}$, ξ is called a *super-subword* of η or ω if there exists $S = \{s_1 < s_2 < \cdots < s_k\}$ which is a subset of $\{1, 2, \cdots, l\}$ or $\mathbb{N}$, respectively, such that $\xi = \eta[S] := \eta_{s_1}\eta_{s_2}\cdots\eta_{s_k}$ or $\xi = \omega[S] := \omega(s_1)\omega(s_2)\cdots\omega(s_k)$. We denote

$$\xi \ll \eta \quad \text{or} \quad \xi \ll \omega$$

if ξ is a super-subword of η or ω, respectively.

A set $\Xi \subset \mathbb{A}^*$ is called *noncomparable* if for all pairs of distinct words $\xi, \eta \in \Xi$, one has $\xi \ll \eta$ does not hold, or equivalently, for all words $\xi, \eta \in \Xi$, one has $\xi \ll \eta$ implies $\xi = \eta$. For a set $\Xi \subset \mathbb{A}^*$, we denote by Ξ_{min} the set of all minimal words in Ξ with respect to $\ll$, that is, the set of $\xi \in \Xi$ such that $\eta \underset{\neq}{\ll} \xi$ does not hold for any $\eta \in \Xi$. Then, Ξ_{min} is noncomparable and is a finite set by Lemma 3, which is proved later.

For $\xi \in \mathbb{A}^*$, denote

$$\mathcal{P}(\xi) := \{\omega \in \mathbb{A}^{\mathbb{N}};\ \xi \ll \omega \text{ does not hold}\},$$

and for $\Xi \subset \mathbb{A}^*$, denote

$$\mathcal{P}(\Xi) := \bigcap_{\xi \in \Xi} \mathcal{P}(\xi).$$

Note that $\mathcal{P}(\Xi) = \emptyset$ if $\epsilon \in \Xi$ and $\mathcal{P}(\emptyset) = \mathbb{A}^{\mathbb{N}}$. Also, $\mathcal{P}(\Xi) = \mathcal{P}(\Xi_{min})$.

For $\zeta = \zeta_1\zeta_2\cdots\zeta_{l-1}\zeta_l \in \mathbb{A}^*$ and $a \in \mathbb{A}$, denote

$$a^{-1}\zeta = \begin{cases} \zeta_2\cdots\zeta_l & (\text{if } \zeta_1 = a) \\ \zeta_1\zeta_2\cdots\zeta_l & (\text{if } \zeta_1 \neq a) \end{cases},\quad \zeta a^{-1} = \begin{cases} \zeta_1\cdots\zeta_{l-1} & (\text{if } \zeta_l = a) \\ \zeta_1\cdots\zeta_{l-1}\zeta_l & (\text{if } \zeta_l \neq a) \end{cases}.$$

For $\zeta \in \mathbb{A}^*$ and $\xi = \xi_1\cdots\xi_k \in \mathbb{A}^*$, denote

$$\xi^{-1}\zeta = \xi_k^{-1}\cdots\xi_1^{-1}\zeta\ ,\ \zeta\xi^{-1} = \zeta\xi_k^{-1}\cdots\xi_1^{-1}.$$

For $\Xi \subset \mathbb{A}^*$ and $\xi \in \mathbb{A}^*$, we denote

$$\xi^{-1}\Xi = \{\xi^{-1}\zeta;\ \zeta \in \Xi\}\ ,\ \Xi\xi^{-1} = \{\zeta\xi^{-1};\ \zeta \in \Xi\}.$$

We define the condition (#) for $\Xi \subset \mathbb{A}^*$ as follows.

(#) *There are no words* $\xi, \eta \in \mathbb{A}^*$ *such that* $(\xi^{-1}\Xi\eta^{-1})_{min} = \mathbb{A}$*, where each letter in* $\mathbb{A}$ *here is considered as a word with length* 1*.*

We recall some fundamental results concerning uniform sets and super-stationary sets over the binary alphabet $\{0, 1\}$.

Theorem 1. (T. Kamae [8]) *Let* $\Omega \subset \{0,1\}^{\mathbb{N}}$ *be a uniform set. Then, there exists an infinite subset* $\mathcal{N} \subset \mathbb{N}$ *such that* $\Omega[\mathcal{N}]$ *is a super-stationary set. Hence, all the uniform complexity functions are realized by super-stationary sets.*

Theorem 2. (T. Kamae, H. Rao, B. Tan, Y.-M. Xue [9]) *The class of super-stationary sets over the alphabet* $\{0,1\}$*other than* $\{0,1\}^{\mathbb{N}}$ *coincides with the class of sets* $\bigcup_{\xi\in\Xi} \mathcal{P}(\xi)$ *with nonempty finite sets* $\Xi \subset \{0,1\}^+$*.*

Theorem 3. [9] *The complexity function* $p_\Omega(k)$ *of a super-stationary set* Ω *over the alphabet* $\{0,1\}$ *other than* $\{0,1\}^{\mathbb{N}}$ *coincides with a polynomial function of* k *for large* k*.*

In this paper, we generalize Theorem 1 to a general alphabet. The proof for the generalization is somewhat simpler than the original proof of Theorem 1 in [8] and relies on the infinite Ramsey Theorem.

Theorem 4. *Let* $\Omega \subset \mathbb{A}^{\mathbb{N}}$ *be a uniform set. Then, there exists an infinite subset* $\mathcal{N} \subset \mathbb{N}$ *such that* $\Omega[\mathcal{N}]$ *is a super-stationary set. Hence, all the uniform complexity functions are realized by super-stationary sets.*

For a nonempty closed set $\Omega \subset \mathbb{A}^N$, if $\Omega[\mathcal{N}]$ is a superstationary set such that $p_{\Omega[\mathcal{N}]}(k) = p^*_\Omega(k)$ $(k = 1, 2, \cdots)$, then the isomorphic class $((\Omega[\mathcal{N}]))$ containing $\Omega[\mathcal{N}]$ is called a *primitive factor* of Ω, where closed subsets U and V of $\mathbb{A}^{\mathbb{N}}$ are said to be *isomorphic* if there is an isometric bijection between them (see [8]). Hence, Theorem 4 has the following corollary.

Corollary 1. *Each uniform set* $\Omega \subset \mathbb{A}^{\mathbb{N}}$ *has a primitive factor.*

Theorem 2 does not hold for $\#\mathbb{A} \geq 3$ (Example 1). Instead of a union of $\mathcal{P}(\xi)$, we have a characterization of a super-stationary set as an intersection of $\mathcal{P}(\xi)$ satisfying the condition (#). In [9], the relation between the representations of a super-stationary set as a union and as an intersection of the sets $\mathcal{P}(\xi)$ is discussed. (There is an error in the proof of Theorem 2 in [9], the corrected version is available at the author's home page: `http://www14.plala.or.jp/kamae`)

Theorem 5. *The class of super-stationary sets over* $\mathbb{A}$ *coincides with the class of sets* $\mathcal{P}(\Xi)$ *with* $\Xi \subset \mathbb{A}^+$ *satisfying* (#)*. Moreover,* Ξ *can be taken as a noncomparable (hence, finite) set.*

Theorem 3 is generalized as Theorem 8 in Section 7. We have another characterization (Theorem 6) of the super-stationary sets.

Definition 1. (1) The *concatenation* UV of subsets U and V of $\mathbb{A}^* \cup \mathbb{A}^{\mathbb{N}}$ is defined as

$$UV = (U \cap \mathbb{A}^{\mathbb{N}}) \cup \{uv;\ u \in U \cap \mathbb{A}^*,\ v \in V\},$$

which is a subset of $\mathbb{A}^* \cup \mathbb{A}^{\mathbb{N}}$.

(2) The *succession* $U \diamond V$ of subsets U and V of $\mathbb{A}^{\mathbb{N}}$ is defined as

$$U \diamond V = U \cup V \cup \{\xi v;\ \xi \text{ is a prefix of some } u \in U,\ v \in V\},$$

which is a subset of $\mathbb{A}^{\mathbb{N}}$.

(3) The *a-succession* $U \diamond_a V$ of subsets U, V of $\mathbb{A}^{\mathbb{N}}$ and $a \in \mathbb{A}$ is defined as

$$\begin{aligned} & U \diamond_a V \\ = \ & U \cup V \cup \{\xi x v;\ \xi \text{ is a prefix of some } u \in U,\ x \in \{a, \epsilon\},\ v \in V\}, \end{aligned}$$

which is a subset of $\mathbb{A}^{\mathbb{N}}$.

Lemma 1. *The class of super-stationary sets is closed under taking the union, the succession and the a-succession (for any $a \in \mathbb{A}$) between them.*

For $\emptyset \neq B \subset \mathbb{A}$, we denote $I_B = B^* \cup B^{\mathbb{N}}$. For $a \in \mathbb{A}$, we denote $\delta_a = \{a, \epsilon\}$. Denote

$$I(\mathbb{A}) = \{I_B;\ \emptyset \neq B \subset \mathbb{A}\} \text{ and } \delta(\mathbb{A}) = \{\delta_a;\ a \in \mathbb{A}\},$$

which are considered as alphabets (i.e. sets of letters) as well as the families of sets of words in the above sense. Let us denote by $\Lambda(\mathbb{A})$ the set of nonempty finite words λ over the alphabet $I(\mathbb{A}) \cup \delta(\mathbb{A})$ satisfying that

(1) the first and the last letters of λ belong to $I(\mathbb{A})$,

(2) there are no neighboring letters of λ both of which belong to $\delta(\mathbb{A})$,

(3) if I_B and $I_{B'}$ are neighboring in λ, then neither $B \subset B'$ nor $B' \subset B$ hold, and

(4) if I_B and δ_a are neighboring in λ, then $a \notin B$.

The above $\lambda = \lambda_1 \lambda_2 \cdots \lambda_k \in \Lambda(\mathbb{A})$ can be considered as a subset of $\mathbb{A}^{\mathbb{N}}$ defined by the concatenations among the sets $\lambda_1, \lambda_2, \cdots, \lambda_k$ of words in the sense of Definition 1 and collecting all the infinite words.

Theorem 6. *The class of super-stationary sets over $\mathbb{A}$ coincides with the class of sets which are nonempty finite unions of sets belonging to $\Lambda(\mathbb{A})$.*

Example 1. Let $\mathbb{A} = \{0, 1, 2\}$, $\Xi = \{00, 10\}$ and $\Omega = \mathcal{P}(\Xi)$. Then, Ξ satisfies the condition (#). In fact,

$$\Omega = \{2\}^{\mathbb{N}} \diamond_0 \{1, 2\}^{\mathbb{N}} = I_{\{2\}} \delta_0 I_{\{1,2\}} \in \Lambda(\{0, 1, 2\}),$$

which is super-stationary. Let $\Omega^0 = \mathcal{P}(\Xi)$ be considered over the alphabet $\{0,1\}$. Then, Ξ does not satisfy (#) since $\{00,10\}0^{-1} = \{0,1\}$. In fact, $\Omega^0 = \{01^\infty,\ 1^\infty\}$, and Ω^0 is not super-stationary since

$$\Omega^0[\{1,2,\cdots\}] = \{1^\infty\} \neq \Omega^0.$$

It follows therefore by Theorem 2 that Ω^0 cannot be written as $\bigcup_{\xi\in\Xi}\mathcal{P}(\xi)$ with a finite set $\Xi \subset \{0,1\}^+$. If Ω is written as $\bigcup_{\xi\in\Xi}\mathcal{P}(\xi)$ with a finite set $\Xi \subset \{0,1,2\}^+$, then we have $\Xi \subset \{0,1\}^+$ and $\Omega^0 = \bigcup_{\xi\in\Xi}\mathcal{P}(\xi)$ over $\{0,1\}$, since if ξ contains 2, then $\mathcal{P}(\xi) \supset \{0,1\}^{\mathbb{N}}$, which is a contradiction. Thus, there exists a super-stationary set other than $\mathbb{A}^{\mathbb{N}}$ which cannot be written as $\bigcup_{\xi\in\Xi}\mathcal{P}(\xi)$ with a finite set $\Xi \subset \mathbb{A}^+$ if $\#A \geq 3$.

Example 2. Let $\theta > 0$ be an irrational number. Let $\mathbb{A} = \{0,1,\cdots,d-1\}$ with an integer $d \geq 2$ and $0 = a_0 < a_1 < \cdots < a_{d-1} < a_d = 1$. Let $f : [0,\infty) \to \mathbb{A}$ be such that $f(x) = i$ if $\{x\} \in [a_i, a_{i+1})$ for any $i \in \mathbb{A}$, where $\{x\}$ is the fractional part of $x \in [0,\infty)$. Let $\Omega \subset \mathbb{A}^{\mathbb{N}}$ be the closure of the following set:

$$\{\omega \in \mathbb{A}^{\mathbb{N}};\ \text{there exists } x \in \mathbb{R} \text{ such that } \omega(n) = f(x+n\theta)\ (\forall n \in \mathbb{N})\}.$$

Then, it is known [7] that $p^*_\Omega(k) = dk$ $(k = 1,2,\cdots)$. Moreover, it is easy to see that Ω is a uniform set if $a_{i+1} - a_i = 1/d$ $(\forall i \in \mathbb{A})$. The unique primitive factor of Ω is $((\mathcal{P}(\Xi_+)))$ with

$$\Xi_\pm = \{ij \in \mathbb{A}^2;\ i \neq j \text{ and } i \pm 1 \not\equiv j\ (mod\ d)\}\ (\pm \text{ respectively})$$

if $d \geq 3$, and $\Xi_+ = \Xi_- = \{010, 101\}$ if d=2.

To prove this it is sufficient to prove that for any infinite set $\mathcal{N} \subset \mathbb{N}$, there exists an infinite set $\mathcal{M} \subset \mathcal{N}$ such that either $\Omega[\mathcal{M}] = \mathcal{P}(\Xi_+)$ or $\Omega[\mathcal{M}] = \mathcal{P}(\Xi_-)$ since $((\mathcal{P}(\Xi_+))) = ((\mathcal{P}(\Xi_-)))$. Take any infinite set $\mathcal{N} \subset \mathbb{N}$. Then, there exists $x_\infty \in \mathbb{R}/\mathbb{Z}$ and $\mathcal{M} = \{M_0 < M_1 < \cdots\} \subset \mathcal{N}$ such that either

$$x_\infty - c < M_0\theta < M_1\theta < M_2\theta < \cdots < x_\infty\ (mod\ 1)$$

or

$$x_\infty + c > M_0\theta > M_1\theta > M_2\theta > \cdots > x_\infty\ (mod\ 1),$$

where $c = \min_{i\in\mathbb{A}}(a_{i+1} - a_i)$. In the former case, $\omega[\mathcal{M}](m) = \omega[\mathcal{M}](n)$ or $\omega[\mathcal{M}](m) \equiv \omega[\mathcal{M}](n) + 1\ (mod\ d)$ holds for any $\omega \in \Omega$ and $n, m \in \mathbb{N}$ with $n < m$. Hence, $\Omega[\mathcal{M}] = \mathcal{P}(\Xi_+)$. In the same way, we have $\Omega[\mathcal{M}] = \mathcal{P}(\Xi_-)$ in the latter case.

Example 3. Let $\mathbb{A} = \{0,1\}$. We consider $\{0,1\}^{\mathbb{N}}$ as the additive group $\mathbb{Z}_2$ of 2-adic integers, where $n \in \mathbb{N}$ is identified with $e_0e_1e_2\cdots \in \{0,1\}^{\mathbb{N}}$ such that $n = \sum_{i=0}^{\infty} e_i 2^i$ and $-n-1$ is identified with $(1-e_0)(1-e_1)(1-e_2)\cdots \in$

$\{0,1\}^{\mathbb{N}}$. For $\alpha \in \{0,1\}^{\mathbb{N}}$, define $\rho(\alpha) = \inf\{i \in \mathbb{N};\ \alpha(i) = 1\}$. Note that $\rho(n) = \sup\{k \in \mathbb{N};\ 2^k \text{ divides } n\}$ for any $n \in \mathbb{Z} \setminus \{0\}$ and $\rho(0) = \infty$.

Let $\beta \in \{0,1\}^{\mathbb{N}}$ and $B \subset \mathbb{N} \cup \{\infty\}$ be such that $\#B = \#(\mathbb{N} \setminus B) = \infty$. For $\alpha \in \{0,1\}^{\mathbb{N}}$, define $\omega_\alpha \in \{0,1\}^{\mathbb{N}}$ by

$$\omega_\alpha(n) = 1 \text{ if and only if } \rho(\alpha + n - \beta) \in B \quad (\forall n \in \mathbb{N})$$

and let

$$\Omega = \text{closure of } \{\omega_\alpha;\ \alpha \in \{0,1\}^{\mathbb{N}}\} \subset \{0,1\}^{\mathbb{N}}.$$

Then, it is known [4] that $p^*_\Omega(k) = 2k$ $(k = 1, 2, \cdots)$. Here, we'll give an alternative proof that Ω has the unique primitive factor $((\mathcal{P}(101, 110)))$ [4]. On the other hand, the rotation words over $\{0,1\}$ have the unique primitive facor $((\mathcal{P}(010, 101)))$ as is shown in Example 2, which is different from $((\mathcal{P}(101, 110)))$ [4].

To prove that Ω has the unique primitive factor $((\mathcal{P}(101, 110)))$, it is sufficient to prove that for any $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$, there exists $\mathcal{M} = \{M_0 < M_1 < M_2 < \cdots\} \subset \mathcal{N}$ such that either $\Omega[\mathcal{M}] = \mathcal{P}(101, 110)$ or $\Omega[\mathcal{M}] = \mathcal{P}(010, 001)$ since clearly $((\mathcal{P}(101, 110))) = ((\mathcal{P}(010, 001)))$.

Take any $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$. Then, there exists a subset $\mathcal{M} = \{M_0 < M_1 < M_2 < \cdots\}$ of $\mathcal{N}$ such that M_i converges to an element $\gamma \in \{0,1\}^{\mathbb{N}}$ in the sense of $\mathbb{Z}_2$ as $i \to \infty$. Moreover, by taking a further subsequence if necessary, we may assume that

$$0 < \rho(\gamma - M_0) < \rho(\gamma - M_1) < \rho(\gamma - M_2) < \cdots,$$

and either

$$\rho(\gamma - M_i) \in B \ \text{ and } (\rho(\gamma - M_{i-1}),\ \rho(\gamma - M_i)) \cap (\mathbb{N} \setminus B) \neq \emptyset \quad (\forall i \in \mathbb{N})$$

or

$$\rho(\gamma - M_i) \notin B \ \text{ and } (\rho(\gamma - M_{i-1}),\ \rho(\gamma - M_i)) \cap B \neq \emptyset \quad (\forall i \in \mathbb{N}),$$

where we put $M_{-1} = -1$.

We consider the latter case first. For $\alpha \in \{0,1\}^{\mathbb{N}}$, assume that $\{\rho(\alpha + M_i - \beta);\ i \in \mathbb{N}\}$ does not contain ∞ and is unbounded. Then, for any $n \in \mathbb{N}$, there exists $i > n$ with $\rho(\alpha + M_i - \beta) > \rho(\gamma - M_n)$. Hence, we have

$$\rho(\alpha + M_n - \beta) = \rho((\alpha + M_i - \beta) + (\gamma - M_i) - (\gamma - M_n)) = \rho(\gamma - M_n) \notin B.$$

Therefore, $\omega_\alpha(M_n) = 0$ for any $n \in \mathbb{N}$ and $\omega_\alpha[\mathcal{M}] = 0^\infty$. Assume next that $\rho(\alpha + M_i - \beta) = \infty$ for some $i \in \mathbb{N}$. This holds if and only if $M_i = \beta - \alpha$. Then, for any $n \neq i$, we have

$$\rho(\alpha + M_n - \beta) = \rho((\alpha + M_i - \beta) + (\gamma - M_i) - (\gamma - M_n)) = \rho(\gamma - M_{n \wedge i}) \notin B,$$

and hence, $\omega_\alpha(M_n) = 0$ for any $n \neq i$. Therefore, $\omega_\alpha[\mathcal{M}]$ may have 1 at most at one place. Finally, consider the case that $\{\rho(\alpha + M_i - \beta);\ i \in \mathbb{N}\}$ is bounded. Let $r = \max_{i\in\mathbb{N}} \rho(\alpha + M_i - \beta)$. Let i_0 be the smallest $i \in \mathbb{N}$ such that $\rho(\alpha + M_{i_0} - \beta) = r$. Then, for any $n \in \mathbb{N}$ with $\rho(\alpha + M_n - \beta) < r$, we have

$$\rho(\alpha + M_{i_0} - \beta) = \rho((\alpha + M_n - \beta) + (\gamma - M_n) - (\gamma - M_{i_0})) > \rho(\alpha + M_n - \beta).$$

This is possible only if

$$\rho(\alpha + M_n - \beta) = \rho(\gamma - M_{n\wedge i_0}) \notin B.$$

Hence, $\omega_\alpha(M_n) = 0$ for any $n \in \mathbb{N}$ with $\rho(\alpha + M_n - \beta) < r$. Therefore, if $r \notin B$, then $\omega_\alpha[\mathcal{M}] = 0^\infty$. Hence, consider the case that $r \in B$. If there is another $i_1 > i_0$ such that $\rho(\alpha + M_{i_1} - \beta) = r$, then since

$$\rho(\alpha + M_{i_1} - \beta) = \rho((\alpha + M_{i_0} - \beta) + (M_{i_1} - M_{i_0})),$$

we have $\rho(\gamma - M_{i_0}) = \rho(M_{i_1} - M_{i_0}) > r$. Since $\rho(M_n - M_{i_0}) = \rho(\gamma - M_{i_0}) > r$ for any $n > i_0$, this implies that

$$\rho(\alpha + M_n - \beta) = \rho((\alpha + M_{i_0} - \beta) + (M_n - M_{i_0})) = \rho(\alpha + M_{i_0} - \beta) = r \in B.$$

Thus, $\omega_\alpha[\mathcal{M}] = 0^{i_0}1^\infty$. If there is no other i with $\rho(\alpha + M_i - \beta) = r$, then $\omega_\alpha = 0^{i_0}10^\infty$.

Thus, we proved that $\Omega[\mathcal{M}] \subset \mathcal{P}(101, 110)$.

Conversely, take any $\omega \in \mathcal{P}(101, 110)$. Then, one of the following cases holds:

$$\omega = 0^\infty,\ 0^i1^\infty\ (\exists i \in \mathbb{N})\ \text{ or }\ 0^i10^\infty\ (\exists i \in \mathbb{N}).$$

Note that $\omega_\alpha[\mathcal{M}] = 0^\infty$ for $\alpha = \beta - M_0 + 2^r$ with $r > \rho(\gamma - M_0)$ and $r \notin B$ since $\rho(\alpha + M_0 - \beta) = \rho(2^r) = r \notin B$ and

$$\rho(\alpha+M_n-\beta) = \rho((\alpha+M_0-\beta)+(M_n-M_0)) = \rho(M_n-M_0) = \rho(\gamma-M_0) \notin B$$

for any $n > 0$. Let $\alpha = \beta - M_i + 2^r$ with $\rho(\gamma - M_{i-1}) < r < \rho(\gamma - M_i)$ and $r \in B$. Then, for any $n \geq i$, we have

$$\rho(\alpha + M_n - \beta) = \rho((\alpha + M_i - \beta) + (M_n - M_i)) = \rho(\alpha + M_i - \beta) = r \in B$$

and for any $n < i$, we have

$$\rho(\alpha + M_n - \beta) = \rho((\alpha + M_i - \beta) + (M_n - M_i)) = \rho(\gamma - M_n) \notin B.$$

Hence, $\omega_\alpha[\mathcal{M}] = 0^i1^\infty$. Let $\alpha = \beta - M_i + 2^r$ with $\rho(\gamma - M_i) < r$ and $r \in B$. Then, $\rho(\alpha + M_i - \beta) = r \in B$ and for any $n \neq i$,

$$\rho(\alpha + M_n - \beta) = \rho((\alpha + M_i - \beta) + (M_n - M_i)) = \rho(\gamma - M_{n\wedge i}) \notin B.$$

Therefore, $\omega_\alpha[\mathcal{M}] = 0^i10^\infty$.

Thus, $\Omega[\mathcal{M}] = \mathcal{P}(101, 110)$. In the other case, we have $\Omega[\mathcal{M}] = \mathcal{P}(010, 001)$.

2 Proof of Theorem 4

Let $\Omega \subset \mathbb{A}^{\mathbb{N}}$ be a nonempty closed set. We say Ω is *k-super-stationary* ($k = 1, 2, \cdots$) if for any $S \subset \mathbb{N}$ with $\#S = k$, $\Omega[S] = \Omega[\{0, 1, \cdots, k-1\}]$ holds. For a set W and $n \in \mathbb{N}$, denote by $\mathcal{F}_n(W)$ the family of subsets $S \subset W$ with $\#S = n$. Then, $(\mathbb{N}, \mathcal{F}_n(\mathbb{N}))$ is the complete n-graph defined on the vertex set $\mathbb{N}$. Moreover, we consider it as a colored n-graph in the sense that n-edge $S \in \mathcal{F}_n(\mathbb{N})$ has a color $\Omega[S]$, where the colors are the subsets of $\mathbb{A}^n$.

By the infinite Ramsey Theorem (see [1]), there exists an infinite subset $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\}$ of $\mathbb{N}$ such that $\mathcal{F}_n(\mathcal{N})$ is monochromatic. That is, there exists $\Xi \subset \mathbb{A}^n$ such that $\Omega[S] = \Xi$ for any $S \in \mathcal{F}_n(\mathcal{N})$. This implies that $\Omega^1 := \Omega[\mathcal{N}]$ is n-super-stationary, that is,

$$\Omega^1[\{0, 1, \cdots, n-1\}] = \Omega^1[S]$$

for any $S \subset \mathbb{N}$ with $\#S = n$.

Take $m > n$ and apply the infinite Ramsey Theorem again for Ω^1. Then, there exists an infinite subset $\mathcal{M} = \{M_0 < M_1 < M_2 < \cdots\}$ of $\mathbb{N}$ such that $\Omega^2 := \Omega^1[\mathcal{M}]$ is m-super-stationary. It follows that $\Omega[\mathcal{N} \circ \mathcal{M}] = \Omega^2$, where $\mathcal{N} \circ \mathcal{M} := \{N_{M_0} < N_{M_1} < N_{M_2} < \cdots\}$, and

$$\Omega^1[\{0, 1, \cdots, n-1\}] = \Omega^1[\{M_0, M_1, \cdots, M_{n-1}\}] = \Omega^2[\{0, 1, \cdots, n-1\}].$$

In the same way, for any $l > m$, there exists an infinite subset $\mathcal{L} = \{L_0 < L_1 < L_2 < \cdots\}$ of $\mathbb{N}$ such that $\Omega^3 := \Omega^2[\mathcal{L}]$ is l-super-stationary. It follows that

$$\Omega[\mathcal{N} \circ \mathcal{M} \circ \mathcal{L}] = \Omega^3 \text{ and } \Omega^2[\{0, 1, \cdots, m-1\}] = \Omega^3[\{0, 1, \cdots, m-1\}].$$

In this way, we can define a sequence $\mathcal{N}^1 \supset \mathcal{N}^2 \supset \cdots$ of infinite subsets of $\mathbb{N}$ such that $\Omega_k := \Omega[\mathcal{N}^k]$ ($k = 1, 2, \cdots$) is k-super-stationary. Then, it holds that

(1) $\Omega_k[\{0, 1, \cdots, k-1\}] = \Omega_k[S]$ for any $S \subset \mathbb{N}$ with $\#S = k$ for any $k = 1, 2, \cdots$, and

(2) $\Omega_k[\{0, 1, \cdots, k-1\}] = \Omega_l[\{0, 1, \cdots, k-1\}]$ for any $k, l = 1, 2, \cdots$ with $k < l$.

Let

$$\Theta = \bigcap_{l=1}^{\infty} \overline{\bigcup_{k=l}^{\infty} \Omega_k}.$$

Then, for any infinite set $\mathcal{N} = \{N_0 < N_1 < \cdots\} \subset \mathbb{N}$ and $n, k = 1, 2, \cdots$ with $N_{n-1} < k$, we have

$$\begin{aligned} & \Theta[\{N_0, N_1, \cdots, N_{n-1}\}] \\ = \; & \Omega_k[\{N_0, N_1, \cdots, N_{n-1}\}] \\ = \; & \Omega_k[\{0, 1, \cdots n-1\}] \\ = \; & \Theta[\{0, 1, \cdots n-1\}]. \end{aligned}$$

Since Θ is a compact set, it follows that $\Theta[\mathcal{N}] = \Theta$. Thus, Θ is super-stationary. Moreover, if Ω is a uniform set, then Ω and Θ have the same complexity function since $p_\Omega(k) = p_{\Omega[\mathcal{N}^k]}(k) = p_\Theta(k)$ $(k = 1, 2, \cdots)$, Thus, we have

Lemma 2. *Let $\Omega \subset \mathbb{A}^\mathbb{N}$ be a nonempty closed set. Then, there exists a sequence $\mathcal{N}^1 \supset \mathcal{N}^2 \supset \cdots$ of infinite subsets of $\mathbb{N}$ such that $\Omega[\mathcal{N}^k]$ is k-super-stationary for any $k = 1, 2, \cdots$. Moreover,*

$$\Theta := \bigcap_{l=1}^{\infty} \overline{\bigcup_{k=l}^{\infty} \Omega[\mathcal{N}^k]}$$

is a super-stationary set. If Ω is a uniform set, then Ω and Θ have the same complexity function.

Proof of Theorem 4:

Let $\Omega \subset \mathbb{A}^\mathbb{N}$ be a uniform set. Let $\mathcal{N}^1 \supset \mathcal{N}^2 \supset \cdots$ and Θ be as in Lemma 2 for this Ω. By Theorem 5 which will be proved later, there exists a finite set $\Xi \subset \mathbb{A}^+$ such that $\Theta = \mathcal{P}(\Xi)$. Let $k = \max_{\xi \in \Xi} |\xi|$. Since $\Omega[\mathcal{N}^k]$ is k-super-stationary and $\Omega[\mathcal{N}^k][\{0, 1, \cdots, k-1\}] = \Theta[\{0, 1, \cdots, k-1\}]$, we have $\Omega[\mathcal{N}^k] \subset \mathcal{P}(\Xi) = \Theta$. Since $\Omega[\mathcal{N}^k]$ and Θ have the same complexity function, this implies that $\Omega[\mathcal{N}^k] = \Theta$. Thus, $\Omega[\mathcal{N}^k]$ is a super-stationary set. □

3 Proof of Lemma 1

Let Ω^1 and Ω^2 be super-stationary subsets of $\mathbb{A}^\mathbb{N}$ and $a \in \mathbb{A}$. Let $\mathcal{N} = \{N_0 < N_1 < \cdots\}$ be any infinite subset of $\mathbb{N}$.

Then, it holds that

$$(\Omega^1 \cup \Omega^2)[\mathcal{N}] = \Omega^1[\mathcal{N}] \cup \Omega^2[\mathcal{N}] = \Omega^1 \cup \Omega^2.$$

Thus, $\Omega^1 \cup \Omega^2$ is super-stationary, and the class of super-stationary sets is closed under the union.

Let $\omega \in \Omega^1 \diamond \Omega^2$. If $\omega \in \Omega^1$, then $\omega[\mathcal{N}] \in \Omega^1 \subset \Omega^1 \diamond \Omega^2$. Otherwise, there exist $\omega^1 \in \Omega^1$, $\omega^2 \in \Omega^2$ and $k \in \mathbb{N}$ such that $\omega = \omega^1(0) \cdots \omega^1(k-1)\omega^2$. Let $N_{n-1} \le k - 1 < N_n$ (we put $N_{-1} = -1$). Then, since Ω^1 and Ω^2 are super-stationary, we have

$$\omega[\mathcal{N}] = \omega^1[\{N_0, N_1, \cdots, N_{n-1}\}]\omega^2[\{N_n - k, N_{n+1} - k, \cdots\}] \in \Omega^1 \diamond \Omega^2.$$

Thus, $(\Omega^1 \diamond \Omega^2)[\mathcal{N}] \subset \Omega^1 \diamond \Omega^2$.

Let us prove the opposite inclusion. Let $\omega \in \Omega^1 \diamond \Omega^2$. If $\omega \in \Omega^1$, then there exists $\theta \in \Omega^1$ such that $\theta[\mathcal{N}] = \omega$. Hence, $\omega \in (\Omega^1 \diamond \Omega^2)[\mathcal{N}]$. Otherwise, there exist $\omega^1 \in \Omega^1$, $\omega^2 \in \Omega^2$ and $k \in \mathbb{N}$ such that $\omega = \omega^1(0) \cdots \omega^1(k-1)\omega^2$.

Then, since Ω^1 and Ω^2 are super-stationary, there exist $\omega^3 \in \Omega^1$ and $\omega^4 \in \Omega^2$ such that

$$\omega^3[\{N_0, N_1, \cdots, N_{k-1}\}] = \omega^1[\{0, 1, \cdots, k-1\}]$$

and

$$\omega^4[\{0, N_{k+1} - N_k, N_{k+2} - N_k, \cdots\}] = \omega^2.$$

Since $\omega = (\omega^3(0)\omega^3(1)\cdots\omega^3(N_k - 1)\omega^4)[\mathcal{N}]$, we have $\omega \in (\Omega^1 \diamond \Omega^2)[\mathcal{N}]$.

Thus, $(\Omega^1 \diamond \Omega^2)[\mathcal{N}] = \Omega^1 \diamond \Omega^2$, and $\Omega^1 \diamond \Omega^2$ is super-stationary.

The proof of the closedness under the a-succession is almost same as above and is omitted.

4 Preliminary lemmas

Lemma 3. *For any infinite set $\Xi \subset \mathbb{A}^+$, there exists an infinite increasing sequence $\xi^1 \underset{\neq}{\ll} \xi^2 \underset{\neq}{\ll} \cdots$ of elements in Ξ. Hence, if $\Xi \subset \mathbb{A}^+$ is noncomparable, then Ξ is a finite set.*

Proof We use the induction on $\#\mathbb{A}$. The statement is clear if $\#\mathbb{A} = 1$. Assume that $\#\mathbb{A} \geq 2$ and the statement holds for any alphabet $\mathbb{B}$ with $\mathbb{B} \subsetneq \mathbb{A}$.

Take any $\xi^1 \in \Xi$. Take any $\xi^2 \in \Xi$ with $\xi^1 \underset{\neq}{\ll} \xi^2$ if exists. Take any $\xi^3 \in \Xi$ with $\xi^2 \underset{\neq}{\ll} \xi^3$ if exists. In this way, if we can continue this process forever, we complete the proof.

If otherwise, then there exists $\xi \in \Xi$ such that $\xi \underset{\neq}{\ll} \eta$ doesn't hold for any $\eta \in \Xi$. Let $\xi = \xi_1\xi_2\cdots\xi_k$. Then, this implies that

$$\eta = \eta^1\xi_1\eta^2\xi_2\cdots\eta^{i-1}\xi_{i-1}\eta^i \tag{4.1}$$

with $i \leq k$ and $\eta^j \in (\mathbb{A} \setminus \{\xi_j\})^*$ $(j = 1, 2, \cdots, i)$.

There exists i such that there exist infinitely many $\eta \in \Xi$ decomposed as (4.1) with this i. Let Ξ^0 be the set of all η as this. There exists $k = 1, 2, \cdots, i$ such that the set of η^k's obtained from $\eta \in \Xi^0$ in (4.1) is an infinite set. Without loss of generality, we assume that $k = 1$. Since $\eta^1 \in (\mathbb{A} \setminus \{\xi_1\})^*$, we can use the induction hypothesis. Hence, there exists an infinite subset Ξ^1 of Ξ^0 such that all η^1's coming from Ξ^1 are distinct and linearly ordered with respect to $\ll$.

Consider the set of η^2's in (4.1) coming from $\eta \in \Xi^1$. Again, we can use the induction hypothesis and take an infinite subset $\Xi' \subset \Xi^1$ such that η^2's coming from Ξ' are linearly ordered. It is easy to find an infinite subset Ξ^2 of Ξ' such that all the pair (η^1, η^2)'s coming from Ξ^2 are distinct and linearly ordered with respect to the product $\ll \times \ll$. In this way, we can find an infinite subset Γ of Ξ^0 such that all $(\eta^1, \eta^2, \cdots \eta^i)$'s coming from Γ are distinct and linearly ordered with respect to $\ll \times \ll \times \cdots \times \ll$. This implies that Γ itself is an infinite linearly ordered set with respect to $\ll$, which completes the proof. □

Lemma 4. *Let $\Omega \subset \mathbb{A}^{\mathbb{N}}$ be a super-stationary set. Then, there exists a noncomparable set $\Xi \subset \mathbb{A}^+$ such that $\Omega = \mathcal{P}(\Xi)$.*

Proof Since Ω is a nonempty closed set, there exists $\Gamma \subset \mathbb{A}^+$ such that $\Omega = \mathbb{A}^{\mathbb{N}} \setminus \bigcup_{\xi\in\Gamma}(\xi)$, where

$$(\xi) := \{\omega \in \mathbb{A}^{\mathbb{N}};\ \xi \text{ is a prefix of } \omega\}$$

is the cylinder set.

Since Ω is super-stationary, if $\omega \in \Omega$, then $\xi \ll \omega$ does not hold for any $\xi \in \Gamma$. Hence, we have

$$\Omega = \mathbb{A}^{\mathbb{N}} \setminus \bigcup_{\xi\in\Gamma}(\xi) = \mathcal{P}(\Gamma).$$

Let $\Xi = \Gamma_{min}$. Then, Ξ is noncomparable and $\Omega = \mathcal{P}(\Gamma) = \mathcal{P}(\Xi)$ holds, since if $\xi \ll \eta$, then $\xi \ll \omega$ follows from $\eta \ll \omega$, and hence, $\mathbb{P}(\xi) \subset \mathbb{P}(\eta)$. □

Lemma 5. *Let $\Omega \subset \mathbb{A}^{\mathbb{N}}$ be a super-stationary set. For $a \in \mathbb{A}$, let $\Omega_a = \{\omega;\ a\omega \in \Omega\}$. Then, Ω_a is super-stationary if it is not empty and it holds that*

$$\Omega = \bigcup_{a\in\mathbb{A}} \Omega_a = \bigcup_{a\in\mathbb{A}} \{a\}\Omega_a.$$

Proof Let $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\}$ be any infinite subset of $\mathbb{N}$. Take any $\omega \in \Omega_a$. Let $\mathcal{N}' = \{0\} \cup (\mathcal{N}+1) := \{0, N_0+1, N_1+1, \cdots\}$.

Since $a\omega \in \Omega$ and Ω is super-stationary, $(a\omega)[\mathcal{N}'] = a\omega[\mathcal{N}] \in \Omega$. Hence, $\omega[\mathcal{N}] \in \Omega_a$ and $\Omega_a[\mathcal{N}] \subset \Omega_a$.

Conversely, there exists $\theta \in \Omega$ such that $\theta[\mathcal{N}'] = a\omega$ since $a\omega \in \Omega$ and Ω is super-stationary. Let $\theta' := \theta(1)\theta(2)\cdots$. Then, $\theta' \in \Omega_a$ since $a\theta' = \theta \in \Omega$. Moreover, $\theta'[\mathcal{N}] = \omega$. Hence, $\Omega_a[\mathcal{N}] \supset \Omega_a$.

Thus, $\Omega_a[\mathcal{N}] = \Omega_a$ for any $\mathcal{N}$, and Ω_a is super-stationary if it is not empty.

It is clear by the definition that $\Omega = \bigcup_{a\in\mathbb{A}}\{a\}\Omega_a$. Since Ω is super-stationary and $\{a\}\Omega_a \subset \Omega$, we have $\Omega_a \subset \Omega$. Hence, $\Omega \supset \bigcup_{a\in\mathbb{A}} \Omega_a$ holds.

Conversely, for any $\omega \in \Omega$, since Ω is super-stationary, there exists $a \in \mathbb{A}$ such that $a\omega \in \Omega$. Therefore, $\omega \in \bigcup_{a\in\mathbb{A}} \Omega_a$. Hence, $\Omega \subset \bigcup_{a\in\mathbb{A}} \Omega_a$

Thus, $\Omega = \bigcup_{a\in\mathbb{A}} \Omega_a$, which completes the proof. □

Lemma 6. *Let $\Omega \subset \mathbb{A}^{\mathbb{N}}$ be a super-stationary set such that $\Omega = \mathcal{P}(\Xi)$ for some $\Xi \subset \mathbb{A}^+$. Then for any $a \in \mathbb{A}$, $\mathcal{P}(a^{-1}\Xi) = \Omega_a$ holds, and hence, $\mathcal{P}(a^{-1}\Xi)$ is super-stationary if it is not empty.*

Proof Take any $\omega \in \Omega_a$. Since both of ω and $a\omega$ belong to Ω by Lemma 5, $\omega \in \mathcal{P}(\Xi)$ and $a\omega \in \mathcal{P}(\{a\}\Xi')$ hold, where $\Xi' = \{\xi \in \mathbb{A}^*;\ a\xi \in \Xi\}$. The latter condition implies that $\omega \in \mathcal{P}(\Xi')$. Therefore, we have $\omega \in \mathcal{P}(a^{-1}\Xi)$, and hence, $\Omega_a \subset \mathcal{P}(a^{-1}\Xi)$.

Conversely, let $\omega \in \mathcal{P}(a^{-1}\Xi)$. Then, it is clear that $a\omega \in \mathcal{P}(\Xi)$ by the definitions. Hence, $a\omega \in \Omega$ and $\omega \in \Omega_a$. Therefore, $\Omega_a \supset \mathcal{P}(a^{-1}\Xi)$.

Thus, $\Omega_a = \mathcal{P}(a^{-1}\Xi)$ holds. □

5 Proof of Theorem 5

For $\Xi \subset \mathbb{A}^*$ and $n \in \mathbb{N}$, we denote

$$\mathbb{P}_n(\Xi) := \{\eta \in \mathbb{A}^n;\ \xi \ll \eta \text{ does not hold for any } \xi \in \Xi\}.$$

For $\xi = \xi_1\xi_2\cdots\xi_k \in \mathbb{A}^*$, we define its *reverse* $\widetilde{\xi} = \xi_k\cdots\xi_2\xi_1$. Also, for $\Xi \subset \mathbb{A}^*$, denote $\widetilde{\Xi} = \{\widetilde{\xi};\ \xi \in \Xi\}$.

Lemma 7. *Let $\Xi \subset \mathbb{A}^+$, $\xi \in \mathbb{A}^n$, $\eta \in \mathbb{A}^m$ with $n, m \in \mathbb{N}$ and $a \in \mathbb{A}$.*
(1) $\xi \in \mathbb{P}_n(\Xi)$ if and only if $\epsilon \notin \xi^{-1}\Xi$.
(2) $\xi\eta \in \mathbb{P}_{n+m}(\Xi)$ if and only if $\epsilon \notin \xi^{-1}\Xi\eta^{-1}$.
(3) $a \in (\xi^{-1}\Xi\eta^{-1})_{min}$ if and only if $\xi\eta \in \mathbb{P}_{n+m}(\Xi)$ and $\xi a\eta \notin \mathbb{P}_{n+m+1}(\Xi)$.

Proof (1) Let $\zeta = \zeta_1\zeta_2\cdots\zeta_l \in \mathbb{A}^l$. Then, by (4.1), $\xi \in \mathbb{P}_n(\zeta)$ if and only if

$$\xi = \xi^1\zeta_1\xi^2\zeta_2\cdots\xi^{i-1}\zeta_{i-1}\xi^i$$

with $i \le l$ and $\xi^j \in (\mathbb{A} \setminus \{\zeta_j\})^*$ $(j = 1, 2, \cdots, i)$. Hence, $\xi \in \mathbb{P}_n(\zeta)$ if and only if $\xi^{-1}\zeta = \zeta_i\cdots\zeta_l \neq \epsilon$. Thus, $\xi \in \mathbb{P}_n(\Xi)$ if and only if $\epsilon \notin \xi^{-1}\Xi$.

(2) The statement that $\epsilon \notin \xi^{-1}\Xi\eta^{-1}$ is equivalent to $\xi \in \mathbb{P}_n(\Xi\eta^{-1})$ by (1). Hence, it is equivalent to $\widetilde{\xi} \in \mathbb{P}_n(\widetilde{\eta}^{-1}\widetilde{\Xi})$. By (1), this is equivalent to $\epsilon \notin \widetilde{\xi}^{-1}\widetilde{\eta}^{-1}\widetilde{\Xi} = (\widetilde{\eta}\widetilde{\xi})^{-1}\widetilde{\Xi}$, which is equivalent to $\widetilde{\eta}\widetilde{\xi} \in \mathbb{P}_{n+m}(\widetilde{\Xi})$ again by (1). Thus, $\epsilon \notin \xi^{-1}\Xi\eta^{-1}$ if and only if $\xi\eta \in \mathbb{P}_{n+m}(\Xi)$.

(3) Assume that $a \in (\xi^{-1}\Xi\eta^{-1})_{min}$. Since $\epsilon \notin \xi^{-1}\Xi\eta^{-1}$, $\xi\eta \in \mathbb{P}_{n+m}(\Xi)$ by (2). Since $a \in \xi^{-1}\Xi\eta^{-1}$, $a \notin \mathbb{P}_1(\xi^{-1}\Xi\eta^{-1})$. Hence by (1), $\epsilon \in a^{-1}\xi^{-1}\Xi\eta^{-1} = (\xi a)^{-1}\Xi\eta^{-1}$. Therefore by (2), $\xi a\eta \notin \mathbb{P}_{n+m+1}(\Xi)$.

Conversely, assume that $a \notin (\xi^{-1}\Xi\eta^{-1})_{min}$. Then, either $\epsilon \in \xi^{-1}\Xi\eta^{-1}$ or both a and ϵ are not in $\xi^{-1}\Xi\eta^{-1}$. In the former case, we have $\xi\eta \notin \mathbb{P}_{n+m}(\Xi)$ by (2). In the latter case, we have $a \in \mathbb{P}_1(\xi^{-1}\Xi\eta^{-1})$. Hence, $\epsilon \notin a^{-1}\xi^{-1}\Xi\eta^{-1} = (\xi a)^{-1}\Xi\eta^{-1}$ by (1). Therefore, $\xi a\eta \in \mathbb{P}_{n+m+1}(\Xi)$ by (2), which completes the proof. □

Lemma 8. *Let $\Xi \subset \mathbb{A}^+$ satisfies the condition (#). Then, for any $\eta \in \mathbb{P}_n(\Xi)$ with an arbitrary $n \in \mathbb{N}$, there exists $\omega \in \mathcal{P}(\Xi)$ such that η is a prefix of ω. In particular, $\mathcal{P}(\Xi) \neq \emptyset$.*

Proof Take any $\eta \in \mathbb{P}_n(\Xi)$. Then, $\epsilon \notin \eta^{-1}\Xi$ holds by Lemma 7. Since Ξ satisfies (#), there exists $a \in \mathbb{A}$ such that $a \notin \eta^{-1}\Xi$. Together with $\epsilon \notin \eta^{-1}\Xi$, this implies that $a \in \mathbb{P}_1(\eta^{-1}\Xi)$. Then, by Lemma 7, $\epsilon \notin a^{-1}\eta^{-1}\Xi$ follows, and again by Lemma 7, $\eta^1 := \eta a \in \mathbb{P}_{n+1}(\Xi)$ follows. In this way, we can define $\eta^i \in \mathbb{P}_{n+i}(\Xi)$ $(i = 1, 2, \cdots)$ so that η^{i-1} is a prefix of η^i, where we put $\eta^0 = \eta$. Then, $\omega \in \mathbb{A}^{\mathbb{N}}$ is determined so that each η^i is a prefix of ω $(i = 0, 1, 2, \cdots)$. Thus, $\omega \in \mathcal{P}(\Xi)$ and η is a prefix of ω.

Let $\eta = \epsilon$ in the above. Since $\epsilon \notin \Xi$, $\epsilon \in \mathbb{P}_0(\Xi)$ holds. Hence, there exists $\omega \in \mathcal{P}(\Xi)$ and $\mathcal{P}(\Xi) \neq \emptyset$. □

Lemma 9. *Let $\Xi \subset \mathbb{A}^+$ satisfies the condition $(\#)$. Then, $\mathcal{P}(\Xi)$ is a super-stationary set.*

Proof By Lemma 8, $\mathcal{P}(\Xi)$ is not empty. It is clearly closed. It is also clear that for any infinite subset $\mathcal{N}$ of $\mathbb{N}$, $\mathcal{P}(\Xi)[\mathcal{N}] \subset \mathcal{P}(\Xi)$. Therefore, it is sufficient to prove that $\mathcal{P}(\Xi)[\mathcal{N}] \supset \mathcal{P}(\Xi)$, that is, for any $\omega \in \mathcal{P}(\Xi)$, there exists $\omega' \in \mathcal{P}(\Xi)$ such that $\omega'[\mathcal{N}] = \omega$. In another word, any $\omega \in \mathcal{P}(\Xi)$ situated at the places of $\mathcal{N}$ can be extended to an element in $\mathcal{P}(\Xi)$. We extend ω step by step. In fact, we'll prove that for any $\omega \in \mathcal{P}(\Xi)$ and $n \in \mathbb{N}$, there exists $a \in \mathbb{A}$ such that

$$\omega(0)\omega(1)\cdots\omega(n-1)a\omega(n)\omega(n+1)\cdots \in \mathcal{P}(\Xi). \tag{5.1}$$

If we prove this, then we can repeat this process to get $\omega' \in \mathcal{P}(\Xi)$ such that

$$\omega'[\{N_0, N_1, \cdots, N_k, N_k+1, N_k+2, \cdots\}] = \omega.$$

The required result follows from the compactness of $\mathcal{P}(\Xi)$ by letting $k \to \infty$.

Let us prove (5.1). Take large $N > n$. Let $\xi = \omega(0)\omega(1)\cdots\omega(n-1)$ and $\eta = \omega(n)\omega(n+1)\cdots\omega(N-1)$. Since Ξ satisfies the condition $(\#)$, there exists $a \in \mathbb{A}$ such that $a \notin (\xi^{-1}\Xi\eta^{-1})_{min}$. Then by Lemma 7, $\xi a\eta \in \mathbb{P}_{N+1}(\Xi)$ since $\xi\eta \in \mathbb{P}_N(\Xi)$. Thus, we have

$$\xi a\eta = \omega(0)\omega(1)\cdots\omega(n-1)a\omega(n)\omega(n+1)\cdots\omega(N-1) \in \mathbb{P}_{N+1}(\Xi)$$

Letting $N \to \infty$, we have (5.1) for some $a \in \mathbb{A}$. □

To complete the proof of Theorem 5, by Lemma 4 and 9, it is sufficient to prove that if $\mathcal{P}(\Xi)$ is a super-stationary set for some noncomparable set $\Xi \subset \mathbb{A}^+$, then there exists a noncomparable set $\Xi_\infty \subset \mathbb{A}^+$ satisfying the condition $(\#)$ such that $\mathcal{P}(\Xi_\infty) = \mathcal{P}(\Xi)$.

Starting from $\Xi_0 := \Xi$, we define Ξ_i $(i = 0, 1, 2, \cdots)$ inductively as follows:

$$\Xi_{i+1} = \left(\Xi_i \cup \{\xi\eta;\ (\xi^{-1}\Xi_i\eta^{-1})_{min} = \mathbb{A}\}\right)_{min}.$$

Then, the following lemma is clear that

Lemma 10. *For any $i < j$ and $\xi \in \Xi_i$, there exists $\eta \in \Xi_j$ such that $\eta \ll \xi$.*

Lemma 11. *If $\Xi_k = \Xi_{k+1}$, then Ξ_k satisfies the condition $(\#)$.*

Proof Assume that there exists a pair (ξ, η) such that $(\xi^{-1}\Xi_k\eta^{-1})_{min} = \mathbb{A}$. Then, by Lemma 7, $\xi\eta \in \mathbb{P}_{|\xi|+|\eta|}(\Xi_k)$. This implies that $\zeta \ll \xi\eta$ does not hold for any $\zeta \in \Xi_k$. On the other hand, there exists $\zeta \in \Xi_{k+1}$ such that $\zeta \ll \xi\eta$ by the above definition. Hence, $\Xi_k \neq \Xi_{k+1}$. □

Lemma 12. $\mathcal{P}(\Xi_0) = \mathcal{P}(\Xi_1) = \mathcal{P}(\Xi_2) = \cdots$

Proof Suppose that there exist (ξ, η) with $(\xi^{-1}\Xi_0\eta^{-1})_{min} = \mathbb{A}$ and $\omega \in \mathcal{P}(\Xi_0)$ such that $\xi\eta \ll \omega$. Since $\mathcal{P}(\Xi_0)$ is a super-stationary set, there exists $\omega' \in \mathcal{P}(\Xi_0)$ such that $\xi a\eta \ll \omega'$ for some $a \in \mathbb{A}$. This is a contradiction since $\xi a\eta \notin \mathbb{P}_{|\xi|+|\eta|+1}(\Xi_0)$ by Lemma 7. Thus, $\xi\eta \ll \omega$ does not hold for any ξ, η with $(\xi^{-1}\Xi_0\eta^{-1})_{min} = \mathbb{A}$ and $\omega \in \mathcal{P}(\Xi_0)$. This implies that

$$\mathcal{P}(\Xi_0) = \mathcal{P}(\Xi_0 \cup \{\xi\eta;\ (\xi^{-1}\Xi_0\eta^{-1})_{min} = \mathbb{A}\}) = \mathcal{P}(\Xi_1).$$

In the same way, we have $\mathcal{P}(\Xi_0) = \mathcal{P}(\Xi_1) = \mathcal{P}(\Xi_2) = \cdots$. □

Lemma 13. $\cup_{i=0}^{\infty}\Xi_i$ *is a finite set.*

Proof Suppose that $\cup_{i=0}^{\infty}\Xi_i$ is an infinite set. Since each Ξ_i is a finite set, there exist $\xi \in \Xi_i$ and $\eta \in \Xi_j$ with $i < j$ such that $\xi \underset{\neq}{\ll} \eta$ by Lemma 3. Since there is $\zeta \in \Xi_j$ such that $\zeta \ll \xi$, we have $\zeta \underset{\neq}{\ll} \eta$, which contradicts the fact that Ξ_j is noncomparable. □

Let $\Xi'_i = \{\xi \in \cup_{j=0}^{\infty}\Xi_j;\ \eta \ll \xi \text{ for some } \eta \in \Xi_i\}$ $(i = 0, 1, 2, \cdots)$. Then, we have $\mathcal{P}(\Xi_i) = \mathcal{P}(\Xi'_i)$ and $\Xi'_0 \subset \Xi'_1 \subset \Xi'_2 \subset \cdots$. Since $\cup_{j=0}^{\infty}\Xi_j$ is a finite set, there exists k such that $\Xi'_k = \Xi'_{k+1}$. Therefore, $\Xi_k = \Xi_{k+1}$. Hence, Ξ_k satisfies the condition (#) by Lemma 11. Together with $\mathcal{P}(\Xi_k) = \mathcal{P}(\Xi_0)$, $\Xi_\infty := \Xi_k$ has the desired property.

6 Proof of Theorem 6

Let $\lambda = \lambda_1\lambda_2\cdots\lambda_k \in \Lambda(\mathbb{A})$ be considered as a subset of $\mathbb{A}^{\mathbb{N}}$ in the sense of the concatenations (Definition 1) among the sets λ_i. Since it is obtained starting from super-stationary sets $B^{\mathbb{N}}$ with $\emptyset \neq B \subset \mathbb{A}$ by applying the operations of succession or a-succession ($a \in \mathbb{A}$) finite number of times, it is super-stationary by Lemma 1. The finite union of them is also super-stationary by Lemma 1.

Now, we prove the opposite implication. Let $\Omega \subset \mathbb{A}^{\mathbb{N}}$ be a super-stationary set. By Lemma 4, there exists a noncomparable set $\Xi \subset \mathbb{A}^+$ such that $\Omega = \mathcal{P}(\Xi)$. Moreover, Ξ is a finite set by Lemma 3.

We use the induction on $\sum_{\xi\in\Xi}|\xi|$ to prove that

Claim: *If $\mathcal{P}(\Xi)$ is super-stationary, then there exist $j \geq 1$ and $\lambda^i \in \Lambda(\mathbb{A})$ $(i = 1, 2, \cdots, j)$ such that $\mathcal{P}(\Xi) = \bigcup_{i=1}^{j}\lambda^i$.*

If $\sum_{\xi\in\Xi}|\xi| = 0$, then $\Xi = \emptyset$ and $\mathcal{P}(\Xi) = \mathbb{A}^{\mathbb{N}}$. Therefore, our Claim holds since $\mathcal{P}(\Xi) = I_{\mathbb{A}} \in \Lambda(\mathbb{A})$. Assume that $\sum_{\xi\in\Xi}|\xi| \geq 1$ and our Claim holds for the case of smaller $\sum_{\xi\in\Xi}|\xi|$.

If $a^{-1}\Xi \neq \Xi$ for some $a \in \mathbb{A}$, then we have either $\epsilon \in a^{-1}\Xi$ or $a^{-1}\Xi \subset \mathbb{A}^+$ with $\sum_{\xi\in a^{-1}\Xi}|\xi| < \sum_{\xi\in\Xi}|\xi|$. Hence, either $\mathcal{P}(a^{-1}\Xi) = \emptyset$ or $\mathcal{P}(a^{-1}\Xi)$ can be written as a finite union of sets of the form $\lambda \in \Lambda(\mathbb{A})$ by the induction hypothesis and Lemma 6.

Case 1: $a^{-1}\Xi \neq \Xi$ for all $a \in \mathbb{A}$.

In this case, we have

$$\Omega = \bigcup_{a\in\mathbb{A}} \Omega_a = \bigcup_{a\in\mathbb{A}} \mathcal{P}(a^{-1}\Xi)$$

by Lemma 5 and Lemma 6. Moreover, each $\mathcal{P}(a^{-1}\Xi)$ is empty or a finite union of sets in $\Lambda(\mathbb{A})$ by the induction hypothesis. Hence, Ω is a finite union of sets in $\Lambda(\mathbb{A})$.

Case 2: $B := \{a \in \mathbb{A};\ a^{-1}\Xi = \Xi\} \neq \emptyset$.

In this case, a is not a prefix of ξ for any $a \in B$ and $\xi \in \Xi$. Hence, $\omega \in \mathbb{A}^{\mathbb{N}}$ belongs to $\mathcal{P}(\Xi)$ if and only if $\omega \in B^{\mathbb{N}}$ or $\omega = \eta a \omega'$ with $\eta \in B^*$ and $\omega' \in \mathcal{P}(a^{-1}\Xi)$ for some $a \in \mathbb{A} \setminus B$. Hence, we have

$$\Omega = B^{\mathbb{N}} \cup \left(B^* \bigcup_{a\in\mathbb{A}\setminus B} \{a\}\mathcal{P}(a^{-1}\Xi) \right).$$

By the induction hypothesis, each $\mathcal{P}(a^{-1}\Xi)$ in the above is empty or a finite union of sets in $\Lambda(\mathbb{A})$. Let $\lambda \in \Lambda(\mathbb{A})$ be one component in the union representing $\mathcal{P}(a^{-1}\Xi)$. Then, $B^*\{a\}\lambda$ becomes one component in the union representing Ω. Since Ω is super-stationary, $B^*\{a,\epsilon\}\lambda$ is also contained in Ω. Thus, by replacing $B^*\{a\}\lambda$ by $I_B\delta_a\lambda$, we can represent Ω as a finite union of sets in $\Lambda(\mathbb{A})$.

Thus, we complete the proof of Theorem 6.

7 Formula for the uniform complexity

Theorem 7. *Let $\Omega \subset \mathbb{A}^{\mathbb{N}}$ be a super-stationary set such that $\Omega = \mathcal{P}(\Xi)$ with $\Xi \subset \mathbb{A}^+$ satisfying the condition (#). Then, $p_\Omega(k) = \#\mathbb{P}_k(\Xi)$ $(k = 1, 2, \cdots)$.*

Proof By Lemma 8, the set of $\xi \in \mathbb{A}^k$ such that ξ is a prefix of some word in $\mathcal{P}(\Xi)$ coincides with $\mathbb{P}_k(\Xi)$, which implies our theorem. □

For $\Xi \subset \mathbb{A}^*$ satisfying the condition (#), we denote $\mathbf{p}(\Xi)$ the function $\mathbb{N} \to \mathbb{N}$ such that

$$\mathbf{p}(\Xi)(k) = \#\mathbb{P}_k(\Xi),$$

where note that $\mathbf{p}(\Xi)(0) = 1$ if $\epsilon \notin \Xi$ and $\mathbf{p}(\Xi)(k) = 0$ $(k = 0, 1, 2, \cdots)$ if $\epsilon \in \Xi$. For $r = 1, 2, \cdots$, we denote $\tau(r)$ the function $\mathbb{N} \to \mathbb{N}$ such that

$$\tau(r)(k) = r^k.$$

For a function $u : \mathbb{N} \to \mathbb{N}$, we define a function $Su : \mathbb{N} \to \mathbb{N}$ by

$$(Su)(k) = \begin{cases} u(k-1) & (k \geq 1) \\ 1 & (k = 0) \end{cases} .$$

The *convolution* $u \otimes v$ between functions $u, v : \mathbb{N} \to \mathbb{N}$ is defined as

$$(u \otimes v)(k) = \sum_{l=0}^{k} u(l)v(k-l) \ (k = 0, 1, 2, \cdots).$$

Theorem 8. (1) *For $\Xi \subset \mathbb{A}^+$ satisfying the condition (#), we have*

$$\mathbf{p}(\Xi) = \tau(\#\mathbb{A} - \#\Xi_{pre}) \otimes S\left(\sum_{a \in \Xi_{pre}} \mathbf{p}(a^{-1}\Xi)\right),$$

where $\Xi_{pre} := \{a \in \mathbb{A};\ a$ is a prefix of some $\xi \in \Xi\}$ and $\tau(0)(k) = \mathbf{1}_{k=0}$.
(2) *The class of uniform complexity functions over $\mathbb{A}$ is included in the minimal class of functions containing all $\tau(r)$ with $r = 1, 2, \cdots, \#\mathbb{A}$, closed under the operations of S, convolution and summation.*
(3) *Any uniform complexity function $p_\Omega(k)$ over $\mathbb{A}$ with $\#\mathbb{A} = d$ satisfies either $p_\Omega(k) = d^k$ ($\forall k \in \mathbb{N}$) or there exist polynomials R_r ($r = 1, 2, \cdots, d-1$) with rational coefficient such that $p_\Omega(k) = \sum_{r=1}^{d-1} R_r(k) r^k$ holds for any sufficiently large k.*

Proof (1) Since

$$\mathbb{P}_{k+1}(\Xi) = \bigcup_{a \in \mathbb{A}} \{a\}\mathbb{P}_k(a^{-1}\Xi) = \bigcup_{a \in \Xi_{pre}} \{a\}\mathbb{P}_k(a^{-1}\Xi) \cup \bigcup_{a \in \mathbb{A} \setminus \Xi_{pre}} \{a\}\mathbb{P}_k(\Xi)$$

holds for any $k \in \mathbb{N}$ with disjoint unions, we have

$$\mathbf{p}(\Xi)(k+1) = \sum_{a \in \Xi_{pre}} \mathbf{p}(a^{-1}\Xi)(k) + r\mathbf{p}(\Xi)(k),$$

where we put $r = \#\mathbb{A} - \#\Xi_{pre}$ Hence, we have

$$\mathbf{p}(\Xi)(k+1) = \sum_{l=0}^{k} r^l \sum_{a \in \Xi_{pre}} \mathbf{p}(a^{-1}\Xi)(k-l) + r^{k+1}\mathbf{p}(\Xi)(0).$$

Since $\mathbf{p}(\Xi)(0) = 1$, we have

$$\mathbf{p}(\Xi)(k+1) = \left(\tau(r) \otimes S\left(\sum_{a \in \Xi_{pre}} \mathbf{p}(a^{-1}\Xi)\right)\right)(k+1)$$

for $k \in \mathbb{N}$. Moreover, $\mathbf{p}(\Xi)(0) = 1 = \tau(r)(0) \cdot 1$, we have the formula.

(2) By Theorem 4 and 5, any uniform complexity function is written as $\mathbf{p}(\Xi)$ with a finite set $\Xi \subset \mathbb{A}^+$ satisfying the condition (#). By applying the formula in (1) repeatedly, we arrive at either $\mathbf{p}(a^{-1}\Xi) = \tau(j)$ where

$a^{-1}\Xi \subset \mathbb{A}$ with $\#a^{-1}\Xi = d-j \leq d-1$ by the condition (#) or $\mathbf{p}(a^{-1}\Xi) \equiv 0$ where $\epsilon \in a^{-1}\Xi$. Thus, we have (2).

(3) If $p_\Omega(k) = d^k$ ($\forall k \in \mathbb{N}$) does not hold, then $\Omega = \mathbf{p}(\Xi)$ holds with $\emptyset \neq \Xi \subset \mathbb{A}^+$. Therefore, $r = \#\mathbb{A} - \#\Xi_{pre} \leq d-1$. Moreover, in each step applying the formula in (1), we have the same situation, hence in (2), we can represent p_Ω without using the function $\tau(d)$. Therefore, p_Ω is in the minimal class of functions $\mathbb{N} \to \mathbb{N}$ containing all $\tau(r)$ with $r = 1, 2, \cdots, d-1$, closed under the operations of S, convolution and summation. On the other hand, the class of functions p such that $p(k) = \sum_{r=1}^{d-1} R_r(k) r^k$ holds for any sufficiently large k, where R_r are polynomials with rational coefficient, contains all $\tau(r)$ with $r = 1, 2, \cdots, d-1$ and satisfies the same closedness property as above. Thus, p_Ω is in this class. □

Example 4. Let $\mathbb{A} = \{0, 1, 2\}$ and $\Xi = \{12, 100\}$, which satisfies the condition (#). Then by Theorem 8, we have

$$\begin{gathered}
\mathbf{p}(\Xi) = \tau(2) \otimes S(\mathbf{p}(2, 00)) \\
\mathbf{p}(2, 00) = \tau(1) \otimes S(\mathbf{p}(\epsilon, 00) + \mathbf{p}(2, 0)) \\
\mathbf{p}(2, 0) = \tau(1) \otimes S(\mathbf{p}(\epsilon, 0) + \mathbf{p}(2, \epsilon)) \\
\mathbf{p}(\epsilon, 00) = \mathbf{p}(\epsilon, 0) = \mathbf{p}(2, \epsilon) = 0^\infty.
\end{gathered}$$

Hence, $\mathbf{p}(2, 0)(k) = \tau(1)(k) = 1$ and $\mathbf{p}(2, 00)(k) = \sum_{i=0}^{k} 1 = k + 1$ for $k = 0, 1, 2, \cdots$. Hence,

$$\mathbf{p}(\Xi)(k) = \sum_{l=0}^{k-1} 2^l (k - l) + 2^k = 3 \cdot 2^k - k - 2 \quad (k = 0, 1, 2, \cdots)$$

Example 5. Let $\mathbb{A} = \{0, 1, 2\}$ and $\Xi = \{10, 21\}$, which satisfies the condition (#). Since

$$\begin{aligned}
\mathbf{p}(1^{-1}\Xi)(k) &= \mathbf{p}(\{0, 21\})(k) \\
&= \sum_{l=0}^{k-1} 1^l \mathbf{p}(\{0, 1\})(k - l - 1) + 1^k \\
&= \sum_{l=0}^{k-1} 1 + 1 = k + 1
\end{aligned}$$

and

$$\mathbf{p}(2^{-1}\Xi)(k) = \mathbf{p}(\{10, 1\})(k) = \mathbf{p}(\{1\})(k) = 2^k,$$

we have

$$\begin{aligned}
\mathbf{p}(\Xi)(k) &= \sum_{l=0}^{k-1} 1^l (\mathbf{p}(1^{-1}\Xi)(k - l - 1) + \mathbf{p}(2^{-1}\Xi)(k - l - 1))) + 1^k \\
&= \sum_{l=0}^{k-1} (k - l + 2^{k-l-1}) + 1 = 2^k + \frac{k(k+1)}{2}
\end{aligned}$$

by Theorem 8 for $k = 0, 1, 2, \cdots$.

8 Philosophical remark

Let $\Omega \subset \mathbb{A}^{\mathbb{N}}$ be super-stationary. Then, it is stationary in the sense that $T\Omega = \Omega[\{1, 2, \cdots\}] = \Omega$, where T is the shift. Stationarity implies that we have the same observations whenever we start to observe. That is, they are invariant under time lag. Super-stationarity is much stronger. We have the same observations whenever we choose the observation points, provided that they keep the time order. Therefore, super-stationary sets represent phenomena which do not depend on time scaling but are sensitive only to the time order. They capture the essence once we lose quantative sense of time. In another word, they suggest what remain meaningful after losing quantitative sense of time. Our Theorem 6 shows that they consist of several stages of two kinds, either eternal eras like $B^{\mathbb{N}} \cup B^*$ that can continue forever or temporary epochs like δ_a that can appear but disappear at once. The temporary epochs do not occur at the begining nor are they consecutive. After a finite number of stages, they become stable and finish with some really eternal era $B^{\mathbb{N}}$ (without B^* part). Moreover, by Theorem 5, these phenomena are also characterized by events prohibited from occurring in time order. These sets of taboos are consistent in the sense that they satisfy our condition (#).

Acknowledgement: The author thanks Professor Rao Hui (HuaZhong Normal University), Professor Tan Bo (Huazhong University of Science and Technology) and Professor Xue Yu-Mei (Beijing University of Aeronautics and Astronautics) for their useful discussions with the author. The basic idea of Theorem 6 was suggested by Professor Rao Hui. The author also thanks the anonymous referee(s) for the usefull suggestions.

References

[1] R. McCutcheon, Elementary Methods in Ergodic Ramsey Theory, Lecture Note in Mathematics 1722, Springer, 1999 (in Chapter 2)

[2] T. Kamae and L. Zamboni, Sequence entropy and the maximal pattern complexity of infinite words, *Ergod. Th. & Dynam. Sys.* **22** (2002), pp. 1191-1199.

[3] T. Kamae and L. Zamboni, Maximal pattern complexity for discrete systems, *Ergod. Th. & Dynam. Sys.* **22** (2002), pp. 1201-1214.

[4] T. Kamae, H. Rao, B. Tan, Y.-M. Xue, Language structure of pattern sturmian word, *Discrete Math.* **306** (2006), pp. 1651-1668.

[5] N. Gjini, T. Kamae, B. Tan, Y.-M. Xue, Maximal pattern complexity for Toeplitz words, *Ergod. Th. & Dynam. Sys.* **26** (2006), pp. 1-14.

[6] T. Kamae, H. Rao, Y.-M. Xue, Maximal pattern complexity for 2-dimensional words, *Theoret. Comput. Sci.* **359** (2006), pp. 15-27.

[7] T. Kamae, H. Rao, Maximal pattern complexity over ℓ letters, *European J. Combin.* **27** (2006), pp. 125-137.

[8] T. Kamae, Uniform set and complexity, *Discrete Math.* **309** (2009), pp. 3738-3747.

[9] T. Kamae, H. Rao, B. Tan, Y.-M. Xue, Super-stationary set, subword problem and the complexity *Discrete Math.* **309** (2009), pp. 4417-4427.

[10] Y.-M. Xue, T. Kamae, Partitions by congruent sets and optimal positions, *Ergod. Th. & Dynam. Sys.* (DOI 10.1017/S0143385709001175), to appear

[11] Teturo Kamae, Pavel V. Salimov, On Maximal Pattern Complexity of Some of Automatic Words, *Ergod. Th. & Dynam. Sys.* (DOI 10.1017/S0143385710000453), to appear

All the above papers except for [1] can be downloaded from the site:

`http://www14.plala.or.jp/kamae`