Uniform Sets and Complexity

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Abstract

For a nonempty closed subset Ω of $\{0,1\}^\Sigma$, where Σ is a countably infinite set, let $p_\Omega(S) := \#\pi_S\Omega$ be the complexity function depending on the nonempty finite sets $S \subset \Sigma$, where $\#$ denotes the number of elements in a set and $\pi_S : \{0,1\}^\Sigma \to \{0,1\}^S$ is the projection. Define the maximal pattern complexity function $p^*_\Omega(k) := \sup_{S;\#S=k} p_\Omega(S)$ as a function of $k = 1, 2, \cdots$.

We call Ω a uniform set if $p_\Omega(S)$ depends only on $\#S = k$ and the complexity function $p_\Omega(k) := p_\Omega(S)$ as a function of $k = 1, 2, \cdots$ is called the uniform complexity function of Ω. Of course, we have $p_\Omega(k) = p^*_\Omega(k)$ in this case.

Such uniform sets appear, for example, as the partitions generated by congruent sets in a space with optimal positionings, or they appear as the restrictions of a symbolic system to optimal windows.

Let Ω' be the derived set (i.e. the set of accumulating points) of Ω and $\deg \Omega := \inf\{d;\ \Omega^{(d+1)} = \emptyset\}$ with $\Omega^{(1)} = \Omega', \Omega^{(2)} = (\Omega')', \cdots$.

We prove that for any nonempty closed subset Ω of $\{0,1\}^{\mathbb{N}}$, where $\mathbb{N} = \{0, 1, 2, \cdots\}$, such that $\deg(\Omega \circ \rho) < \infty$ for some injection $\rho : \mathbb{N} \to \mathbb{N}$, there exists an increasing injection $\phi : \mathbb{N} \to \mathbb{N}$ such that $\Omega \circ \phi \circ \psi = \Omega \circ \phi$ for any increasing injection $\psi : \mathbb{N} \to \mathbb{N}$. Such a set $\Omega \circ \phi$ is called a super-stationary set. Moreover, if $\deg(\Omega \circ \rho) = \infty$ for any injection $\rho : \mathbb{N} \to \mathbb{N}$, then $p^*_\Omega(k) = 2^k$ $(k = 1, 2, \cdots)$ holds.

A uniform set $\Omega \subset \{0,1\}^\Sigma$ is said to have a primitive factor $[\Omega \circ \phi]$ if there exists an injection $\phi : \mathbb{N} \to \Sigma$ such that $\Omega \circ \phi$ is a super-stationary set, where $[\Omega \circ \phi]$ is the isomorphic class containing $\Omega \circ \phi$. Then, any uniform set has at least one primitive factor, and hence, any uniform complexity function is realized by the uniform complexity function of a super-stationary set. It follows that the uniform complexity function $p_\Omega(k)$ is either 2^k for any k or a polynomial function of k for large k.

1 Introduction

An element $\omega \in \{0,1\}^{\mathbb{N}}$ is called an *infinite* 0-1-*word* which is a mapping from $\mathbb{N}$ to $\{0,1\}$, while it is also considered as an infinite sequence $\omega(0)\omega(1)\omega(2)\cdots$ of 0 and 1. On the other hand, an element u in $\{0,1\}^* := \cup_{k=0}^{\infty}\{0,1\}^k$ is called a *finite* 0-1-*word* and represented as a finite sequence $u_1u_2\cdots u_k$ of 0 and 1, where k is such that $u \in \{0,1\}^k$, which is called the *length* of u and is denoted by $|u|$. We also denote $\{0,1\}^+ = \cup_{k=1}^{\infty}\{0,1\}^k$. The *concatenation* $u\omega$ of $u \in \{0,1\}^*$ and $\omega \in \{0,1\}^* \cup \{0,1\}^{\mathbb{N}}$ is defined as the finite or infinite word $u_1u_2\cdots u_k\omega(0)\omega(1)\omega(2)\cdots$. In this case, u is called a *prefix* of $u\omega$ or equivalently, $u\omega$ is called an *extension* of u.

For $u \in \{0,1\}^*$, the *cylinder set* $[u]$ determined by u is defined by

$$[u] = \{\omega \in \{0,1\}^{\mathbb{N}};\ u \text{ is a prefix of } \omega\}.$$

The *prefix tree* $G(\Omega) = (V,E)$ of a nonempty closed set $\Omega \subset \{0,1\}^{\mathbb{N}}$ is defined to be a directed graph such that the set V of vertices is the set of cylinder sets $[u]$ which meet Ω, and the set E of edges is the set of the ordered pairs $([u],[v]) \in V \times V$ such that v is an immediate extension of u, that is, u is the prefix of v such that $|v| = |u| + 1$.

Two nonempty closed sets $\Omega,\ \Lambda \subset \{0,1\}^{\mathbb{N}}$ are said to be *isomorphic* to each other if their prefix trees are isomorphic to each other. The class of all closed subsets of $\{0,1\}^{\mathbb{N}}$ isomorphic to Ω is denoted by $[\Omega]$ and is called the *language structure* of (or determined by) Ω.

Define

$$\Theta_0 := \{0^\infty\}\ ,\quad \Theta_1 := \{1^\infty\},$$

$$\Theta_\delta := \{\omega \in \{0,1\}^{\mathbb{N}};\ \sum_{n\in\mathbb{N}} \omega(n) \le 1\},$$

$$\Theta_{1-\delta} := \{\omega \in \{0,1\}^{\mathbb{N}};\ \sum_{n\in\mathbb{N}} (1-\omega(n)) \le 1\},$$

$$\Theta_+ := \{\omega \in \{0,1\}^{\mathbb{N}};\ \omega \text{ is increasing}\},$$

$$\Theta_- := \{\omega \in \{0,1\}^{\mathbb{N}};\ \omega \text{ is decreasing}\},$$

where $a^\infty = aaa\cdots$ for $a \in \{0,1\}$ and $\omega \in \{0,1\}^{\mathbb{N}}$ is called *increasing* (*decreasing*) if $\omega(n) \le \omega(m)$ ($\omega(n) \ge \omega(m)$, respectively) for any $n < m$.

All of Θ_δ, $\Theta_{1-\delta}$, Θ_+, Θ_- are isomorphic to each other since for example, $G(\Theta_\delta)$ and $G(\Theta_+)$ are isomorphic (Figure 1). It also holds that $\Theta_\delta \cup \Theta_-$ and $\Theta_+ \cup \Theta_-$ are isomorphic, while $\Theta_\delta \cup \Theta_+$ is not isomorphic to $\Theta_\delta \cup \Theta_-$ (Figure 2).

Definition 1.1. For a nonempty closed set $\Omega \subset \{0,1\}^\Sigma$, define the *complexity function* $p_\Omega(S) := \#\pi_S\Omega$, which is a function of finite sets $S \subset \Sigma$, where # denotes the number of elements in a set and $\pi_S : \{0,1\}^\Sigma \to \{0,1\}^S$

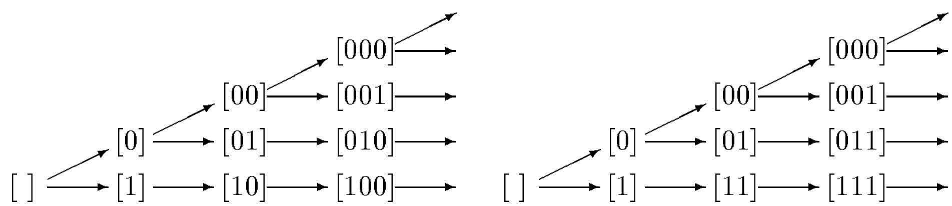

Figure 1: $G(\Theta_\delta)$ (left) and $G(\Theta_+)$ (right)

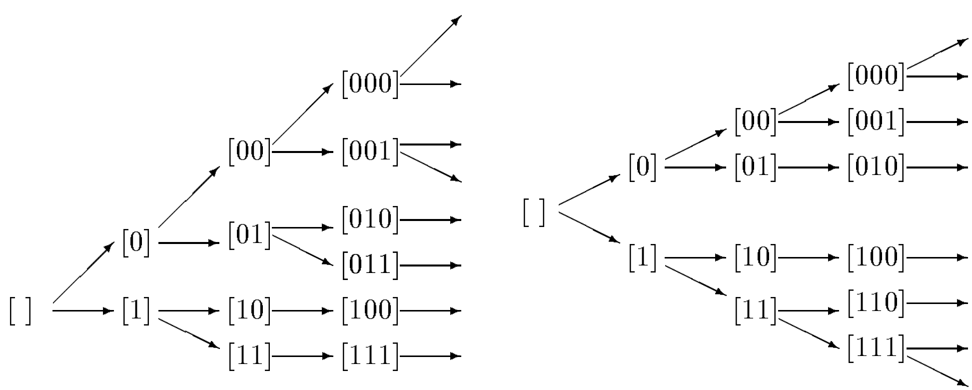

Figure 2: $G(\Theta_\delta \cup \Theta_+)$ (left) and $G(\Theta_\delta \cup \Theta_-)$ (right)

is the projection. We call Ω a *uniform set* if $p_\Omega(S)$ depends only on $\#S$. In this case, the function $p_\Omega(k) := p_\Omega(S)$ of $k = 1, 2, \cdots$, where $\#S = k$, is called the *uniform complexity function* of Ω. We also define the *maximal pattern complexity function* of Ω as $p^*_\Omega(k) := \sup_{S;\#S=k} p_\Omega(S)$ $(k = 1, 2, \cdots)$. Note that $p_\Omega(k) = p^*_\Omega(k)$ $(k = 1, 2, \cdots)$ if Ω is a uniform set.

Let $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\}$ be an infinite subset of $\mathbb{N}$. For $\omega \in \{0,1\}^{\mathbb{N}}$ and $\Omega \subset \{0,1\}^{\mathbb{N}}$, define $\omega[\mathcal{N}] \in \{0,1\}^{\mathbb{N}}$ and $\Omega[\mathcal{N}] \subset \{0,1\}^{\mathbb{N}}$ by

$$\begin{aligned} \omega[\mathcal{N}](n) &:= \omega(N_n) \quad (n \in \mathbb{N}) \\ \Omega[\mathcal{N}] &:= \{\omega[\mathcal{N}] \in \{0,1\}^{\mathbb{N}};\ \omega \in \Omega\}. \end{aligned}$$

We use the same notation for a finite set $S \subset \mathbb{N}$ in place of $\mathcal{N}$ to denote a set of finite words. Similarly, for $\Omega \subset \{0,1\}^{\Sigma}$, where Σ is a countably infinite set, and an injection $\psi : \mathbb{N} \to \Sigma$, denote

$$\Omega \circ \psi := \{\omega \circ \psi \in \{0,1\}^{\mathbb{N}};\ \omega \in \Omega\}.$$

Definition 1.2. A nonempty closed set $\Omega \subset \{0,1\}^{\mathbb{N}}$ is called a *super-stationary* set if $\Omega[\mathcal{N}] = \Omega$ holds for any infinite subset $\mathcal{N}$ of $\mathbb{N}$.

Note that a super-stationary set is a uniform set and all of Θ_0, Θ_1, Θ_δ, $\Theta_{1-\delta}$, Θ_+, Θ_- together with their unions are super-stationary sets.

Definition 1.3. Let $\Omega \subset \{0,1\}^{\mathbb{N}}$ be a nonempty closed set. For $\omega \in \Omega$ and $k \in \mathbb{N}$, we denote $\omega|_k = \omega(0)\omega(1)\cdots\omega(k-1) \in \{0,1\}^k$. Let Ω' be the set of accumulating points of Ω, that is,

$$\Omega' = \{\omega \in \Omega;\ \#([\omega|_k] \cap \Omega) = \infty \text{ for any } k \in \mathbb{N}\}.$$

We call Ω' the *derived set* of Ω. Clearly, Ω' is a closed set (possibly, the empty set). We denote $\Omega^{(0)} = \Omega$ and $\Omega^{(i)} = (\Omega^{(i-1)})'$ for $i = 1, 2, \cdots$. The *degree* of Ω is defined to be $d = 0, 1, 2, \cdots$ such that $\Omega^{(d)} \neq \emptyset$ and $\Omega^{(d+1)} = \emptyset$, if such d exists, otherwise, ∞. The degree of Ω is denoted by $\deg \Omega$. For completeness, we define $\emptyset' = \emptyset$ and $\deg \emptyset = -1$.

Definition 1.4. A nonempty closed set $\Omega \subset \{0,1\}^{\Sigma}$ is said to have a *primitive factor* $[\Omega \circ \phi]$ if $\Omega \circ \phi$ is a super-stationary set, where $\phi : \mathbb{N} \to \Sigma$ is an injection and $[\Omega \circ \phi]$ is the language structure determined by $\Omega \circ \phi$.

We will prove the following theorem.

Theorem 1.5. (Main Theorem) *Let Ω be a nonempty closed subset of $\{0,1\}^{\Sigma}$, where Σ is a countably infinite set.*
(1) *If there exists an injection $\rho : \mathbb{N} \to \Sigma$ such that $\deg(\Omega \circ \rho) < \infty$, then there exists an increasing injection $\phi : \mathbb{N} \to \mathbb{N}$ such that $\Omega \circ \rho \circ \phi$ is a super-stationary set.*

(2) *If* $\deg(\Omega \circ \rho) = \infty$ *for any injection* $\rho : \mathbb{N} \to \Sigma$*, then* $p^*_\Omega(k) = 2^k$ $(k = 1, 2, \cdots)$.
Hence, any uniform set has a primitive factor and any uniform complexity function is realized by a super-stationary set.

Remark 1.6. (1) of the Main Theorem can be generalized easily to the case of general finite alphabet.

The super-stationary set is characterized as a finite union of sets with a prohibited word by Kamae, Rao, Tan and Xue [1].

For $\xi = \xi_1\xi_2 \cdots \xi_k \in \{0,1\}^k$ and $\eta = \eta_1\eta_2 \cdots \eta_l \in \{0,1\}^l$ with $k \le l$, we say that ξ is a *super-subword* of η, if $\xi = \eta_{s_1}\eta_{s_2} \cdots \eta_{s_k}$ holds for some $1 \le s_1 < s_2 < \cdots < s_k \le l$. For this ξ and $\omega \in \{0,1\}^{\mathbb{N}}$, we say that ξ is a *super-subword* of ω, if $\xi = \omega(s_1)\omega(s_2)\cdots\omega(s_k)$ holds for some $0 \le s_1 < s_2 < \cdots < s_k < \infty$. In this case, we denote $\xi \ll \eta$ or $\xi \ll \omega$. For $\xi \in \{0,1\}^*$, denote

$$\mathcal{P}(\xi) := \{\omega \in \{0,1\}^{\mathbb{N}};\ \xi \ll \omega \text{ does not hold}\},$$

that is, $\mathcal{P}(\xi)$ is the set of infinite 0-1-words with the prohibited word ξ as its super-subword. Denote for $\Xi \subset \{0,1\}^*$,

$$\mathcal{Q}(\Xi) := \bigcup_{\xi\in\Xi} \mathcal{P}(\xi) \text{ and } \mathcal{P}(\Xi) := \bigcap_{\xi\in\Xi} \mathcal{P}(\xi).$$

We call $\eta \in \{0,1\}^*$ a *cover* of Ξ if $\xi \ll \eta$ holds for any $\xi \in \Xi$. It is called a *minimal cover* if in addition, any $\zeta \underset{\neq}{\ll} \eta$ is not a cover of Ξ. Let $\mathrm{L}(\Xi)$ be the set of minimal covers of Ξ.

Theorem 1.7. [1] *The class of super-stationary sets other than* $\{0,1\}^{\mathbb{N}}$ *coincides with the class of sets* $\mathcal{Q}(\Xi)$ *with nonempty finite sets* $\Xi \subset \{0,1\}^+$*. It also coincides with the class of sets* $\mathcal{P}(L(\Xi))$ *with nonempty finite sets* $\Xi \subset \{0,1\}^+$.

Theorem 1.8. [1] *The complexity function* $p_\Omega(k)$ *of a super-stationary set* Ω *other than* $\{0,1\}^{\mathbb{N}}$ *is a polynomial function of* k *for large* k.

The following corollary follows from Theorems 1.5 and 1.8.

Corollary 1.9. *The complexity function* $p_\Omega(k)$ *of a uniform set* Ω *is either* 2^k $(k = 1, 2, \cdots)$ *or a polynomial function of* k *for large* k.

Uniform sets are introduced by Kamae, Rao, Tan and Xue [2] to study recurrent pattern Sturmian words. By a k-*window* τ, we mean a subset of $\{0, 1, 2, \cdots\}$ with cardinality k. For a word $\alpha \in \{0,1\}^{\mathbb{N}}$ and a k-window $\tau = \{\tau_0 < \tau_1 < \cdots < \tau_{k-1}\}$, we define

$$p_\alpha(\tau) := \#\{\alpha(n+\tau_0)\alpha(n+\tau_1)\cdots\alpha(n+\tau_{k-1}) \in \{0,1\}^k;\ n = 0, 1, 2, \cdots\}$$

and the *maximal pattern complexity* function p_α^* of α by

$$p_\alpha^*(k) = \sup_\tau p_\alpha(\tau) \quad (k = 1, 2, 3, \cdots),$$

where the supremum is taken over all k-windows τ, while the *block complexity* B_α is defined by

$$B_\alpha(k) = p_\alpha(\{0, 1, \cdots, k-1\}).$$

It is well known (Morse and Hedrund [3]) that a word $\alpha \in \{0,1\}^{\mathbb{N}}$ is eventually periodic if and only if $B_\alpha(k) < k+1$ for some $k = 1, 2, \cdots$. A word α with $B_\alpha(k) = k+1$ $(k = 1, 2, \cdots)$ is known as a *Sturmian word*.

In a similar way, Kamae and Zamboni [4] characterized the eventual periodicity in terms of maximal pattern complexity. A word α is eventually periodic if and only if $p_\alpha^*(k) < 2k$ for some $k = 1, 2, \cdots$. Accordingly, a word α with $p_\alpha^*(k) = 2k$ $(k = 1, 2, \cdots)$ is called a *pattern Sturmian word*.

It is shown that Sturmian words are pattern Sturmian. Indeed, the class of pattern Sturmian words is larger than that of Sturmian words. Till now, three classes of pattern Sturmian words are known: rotation words, simple Toeplitz words and a class of words with rare 1, where the first two of them are recurrent, while the last ones are not ([4, 5]).

For a recurrent pattern Sturmian word α, there exists an *optimal window* $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$, that is, for any $k = 1, 2, \cdots$ and any subset τ of $\mathcal{N}$ with size k, $p_\alpha(\tau) = p_\alpha^*(k)$ holds. This implies that $\Omega := \overline{O}(\alpha)[\mathcal{N}]$ is a uniform set with $p_\Omega(k) = 2k$ $(k = 1, 2, \cdots)$, where $\overline{O}(\alpha)$ is the closure of the orbit $\{T^n\alpha;\ n = 0, 1, 2, \cdots\}$ of α with respect to the *shift* T, that is, $(T\alpha)(n) = \alpha(n+1)$ $(n \in \mathbb{N},\ \alpha \in \{0,1\}^{\mathbb{N}})$.

Theorem 1.10. [1] *For any rotation word $\alpha \in \{0,1\}^{\mathbb{N}}$ and any optimal window $\mathcal{N} \subset \mathbb{N}$ of α, the uniform set $\overline{O}(\alpha)[\mathcal{N}]$ has a unique primitive factor $[\Theta_\delta \cup \Theta_-]$. On the other hand, for any simple Toeplitz word $\alpha \in \{0,1\}^{\mathbb{N}}$ and any optimal window $\mathcal{N} \subset \mathbb{N}$ of α, the uniform set $\overline{O}(\alpha)[\mathcal{N}]$ has a unique primitive factor $[\Theta_\delta \cup \Theta_+]$.*

Example 1.11. Let

$$\Omega := \{\omega \in \{0,1\}^{\mathbb{Z}};\ \omega \text{ is increasing or } \sum_{n \in \mathbb{Z}} \omega(n) \leq 1\}.$$

Then, Ω is a uniform set having 2 primitive factors $[\Theta_\delta \cup \Theta_+]$ and $[\Theta_\delta \cup \Theta_-]$ given by the injections $n \mapsto n$ and $n \mapsto -n$ from $\mathbb{N}$ to $\mathbb{Z}$, respectively.

We give some references related to the subject. For general notions and basic properties of dynamical system and complexity, refer [10] and [11]. Block complexity of Toeplitz words is discussed in [9] and [14]. Complexity with respect to arithmetic windows are discussed in [8] and [12]. A combinatorial lemma which implies that the maximal pattern complexity less than exponential order implies polynomial order is proved in [17]. Pattern Sturmian words for general alphabet or $\Sigma = \mathbb{Z}^2$ are discussed in [15] or [16].

2 Optimal positions and uniform sets

Let X be a metrizable space with a continuous group or semi-group action G. For a family of subsets $A_1, A_2, \cdots, A_k$ of X, let $\mathbb{P}(\{A_i; i = 1, 2, \cdots, k\})$ denote the *partition* of X generated by these subsets, that is, the family of nonempty sets of the form

$$A_1^{i_1} \cap A_2^{i_2} \cap \cdots \cap A_k^{i_k} \quad (i_1, i_2, \cdots, i_k \in \{0, 1\}),$$

where for a set $A \subset X$, we denote $A^1 = A$ and $A^0 = X \setminus A$.

Let D be a nonempty subset of X. Define the *maximal pattern complexity* function $p^*_{X,G,D}$ of the triple (X, G, D), where if X is obvious, we drop X in the notation, by

$$p^*_{G,D}(k) = \sup_{\tau \subset G,\ \#\tau = k} \#\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \tau\}) \quad (k = 1, 2, \cdots). \tag{1}$$

Definition 2.1. For a set U and $k \in \mathbb{N}$, $\mathcal{F}_k(U)$ denotes the family of sets $S \subset U$ with $\#S = k$. A countably infinite subset Σ of G is called an *optimal position* of the triple (X, G, D) (or the pair (G, D) if X is obvious) if

$$\#\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \tau\}) = p^*_{G,D}(k), \tag{2}$$

holds for any $k = 1, 2, \cdots$ and $\tau \in \mathcal{F}_k(\Sigma)$. We say that the pair (G, D) admits *finitely determined optimal positioning* if there exists k_0 such that any countably infinite subset Σ of G satisfying that $\#\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \tau\}) = p^*_{G,D}(k)$ for any $\tau \in \mathcal{F}_k(\Sigma)$ with $k \le k_0$ is an optimal position of the pair (G, D) and such Σ does exist.

Let $\Sigma \subset G$ be a countably infinite set. We call $\omega \in \{0, 1\}^\Sigma$ a *name* of the partition $\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \Sigma\})$ if there exists $x \in X$ such that

$$\omega(\sigma) = \begin{cases} 1 & x \in \sigma^{-1}D \\ 0 & x \notin \sigma^{-1}D. \end{cases} \tag{3}$$

The closure of the set of names of the partition $\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \Sigma\})$ is called the *name set* of Σ with respect to the pair (G, D).

The following theorem is clear from the definitions.

Theorem 2.2. *The name set of any optimal position Σ of a pair (G, D) is a uniform set with the complexity function $p^*_{G,D}$.*

Example 2.3. Let $X = G = \mathbb{R}/\mathbb{Z}$. The action of $g \in G$ maps $x \in X$ to $x + g \in X$. Let D be an interval $[a, b)$ in X such that $a < b < a + 1$. Then, we have $p^*_{D,G}(k) = 2k$ $(k = 1, 2, \cdots)$. In this case, a countably infinite subset Σ of G is an optimal position of (G, D) if and only if for any $\sigma, \sigma' \in \Sigma$ with $\sigma \neq \sigma'$, $D - \sigma$ and $D - \sigma'$ intersect as well as their complements.

Let Ω be the name set of an optimal position Σ. Then, Ω is known to have the unique primitive factor $[\Theta_\delta \cup \Theta_-] = [\mathcal{Q}(11, 01)]$ ([2]).

Example 2.4. Let $X = \mathbb{R}^2$ and $G = (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}^2$. The action of $(\theta, (u, v))$ in G maps $(x, y) \in X$ to the following $(x', y') \in X$:

$$\begin{cases} x' = x\cos\theta - y\sin\theta + u \\ y' = x\sin\theta + y\cos\theta + v. \end{cases}$$

Let D be a line in X. Then, $g^{-1}D$ is also a line for any $g \in G$ and we have $p^*_{G,D}(k) = (1/2)k^2 + (1/2)k + 1$ $(k = 1, 2, \cdots)$. In this case, Σ is an optimal position if and only if Σ is a countably infinite subset of G such that
(1) for any $\sigma, \sigma' \in \Sigma$ with $\sigma \neq \sigma'$, $\sigma^{-1}D \cap \sigma'^{-1}D \neq \emptyset$, and
(2) for any $\sigma, \sigma', \sigma'' \in \Sigma$ which are different each other,

$$\sigma^{-1}D \cap \sigma'^{-1}D \cap \sigma''^{-1}D = \emptyset.$$

Let Ω be the name set of an optimal position Σ. Then,

$$\Omega = \{\omega \in \{0,1\}^{\Sigma};\ \sum_{\sigma\in\Sigma} \omega(\sigma) \leq 2\}.$$

Hence, Ω has the unique primitive factor $[\mathcal{Q}(111)]$.

Example 2.5. (Xue [6]) Let $X = G = \mathbb{R}^2$. The action of $g = (g_1, g_2) \in G$ maps $(x, y) \in X$ to $(x+g_1, y+g_2) \in X$. Let $D := \{(x, y) \in \mathbb{R}^2;\ x^2+y^2 \leq 1\}$ be the unit disk. Then, we have $p^*_{G,D}(k) = k^2 - k + 2$ $(k = 1, 2, \cdots)$. In this case, a countably infinite subset Σ of G is an optimal position if and only if $\#\mathbb{P}(\{\sigma^{-1}D;\ \sigma \in \tau\}) = p^*_{G,D}(3) = 8$ for any $\tau \in \mathcal{F}_3(\Sigma)$. Moreover, Σ satisfies this condition if $\Sigma \subset \{g \in G;\ g_1^2 + g_2^2 = r^2\}$ with $0 < r < 1$. The name set Ω for any optimal position Σ has a unique primitive factor $[\mathcal{Q}(101, 010)]$.

Example 2.6. Let X be the 2-adic group. That is, $X = \{0,1\}^{\mathbb{N}}$ with the addition so that $\gamma = \alpha + \beta$ for $\alpha, \beta, \gamma \in \{0,1\}^{\mathbb{N}}$ implies that

$$\sum_{0\leq i<n} \alpha(i)2^i + \sum_{0\leq i<n} \beta(i)2^i \equiv \sum_{0\leq i<n} \gamma(i)2^i \pmod{2^n}$$

for any $n \in \mathbb{N}$. Let $G := \mathbb{Z}$ which is considered as a subgroup of X in the sense that $m \in \mathbb{Z}$ with $m \geq 0$ is identified with $\omega \in \{0,1\}^{\mathbb{N}}$ having finitely many 1's such that $m = \sum_{n\in\mathbb{N}} \omega(n)2^n$ and $m < 0$ is identified with $\omega \in \{0,1\}^{\mathbb{N}}$ having finitely many 0's such that $-1-m = \sum_{n\in\mathbb{N}}(1-\omega(n))2^n$. Let

$$D := \{\omega \in X;\ \inf\{n \in \mathbb{N};\ \omega(n) = 1\} \text{ is finite and even}\}.$$

Then, we have $p^*_{G,D}(k) = 2k$ $(k = 1, 2, \cdots)$ and Σ is an optimal position if and only if Σ is a countably infinite subset of G such that
(1) for any $\sigma, \sigma' \in \Sigma$, $e(\sigma - \sigma') \geq 1$, and

(2) for any $\sigma, \sigma', \sigma'' \in \Sigma$ which are different each other, we have

$$\max\{e(\sigma - \sigma'), e(\sigma' - \sigma''), e(\sigma'' - \sigma)\} \geq \\ \min\{e(\sigma - \sigma'), e(\sigma' - \sigma''), e(\sigma'' - \sigma)\} + 2,$$

where for $n \in \mathbb{Z} \setminus \{0\}$, $e(n)$ denotes the maximum k such that $2^k | n$ ([2]).

Take an optimal position Σ and let Ω be the name set. Then it is known that Ω has a unique primitive factor $[\Theta_\delta \cup \Theta_+] = [\mathcal{Q}(11, 10)]$ ([2]).

All the examples so far admit finitely determined optimal positioning whenever an optimal position exists. We do not know whether this is true in general or not. The following example does not admit an optimal position.

Example 2.7. Let $X = \mathbb{T}_1 \cup \mathbb{T}_2$ and $G = \mathbb{T}_1 \times \mathbb{T}_2$, where $\mathbb{T}_i \cong \mathbb{R}/\mathbb{Z}\,(i = 1, 2)$ and $\mathbb{T}_1, \mathbb{T}_2$ are disjoints of each other. The action of $g = (g_1, g_2) \in G$ maps $x \in \mathbb{T}_i$ to $x + g_i \in \mathbb{T}_i$ for $i = 1, 2$. Let $D = [a_1, b_1) \cup [a_2, b_2)$, where $[a_i, b_i) \subset \mathbb{T}_i$ and $a_i < b_i < a_i + 1$ for $i = 1, 2$.

Then, we have $p^*_{G,D}(k) = 4k - 4$ $(k = 2, 3, \cdots)$. In this case, there is no optimal position since for any infinite subset Σ of G, there exists a sequence $g_n = (g_{n,1}, g_{n,2}) \in \Sigma$ for $n = 1, 2, \cdots$ such that $g_{n,i}$ converges monotonously to, say $c_i \in \mathbb{T}_i$, for $i = 1, 2$. Then, for any sufficiently large n_0, $\#\mathbb{P}(\{g_n^{-1}D;\ n = n_0 + 1, n_0 + 2, n_0 + 3\}) = 6$ but not 8.

Definition 2.8. A nonempty closed set $\Omega \subset \{0, 1\}^{\mathbb{N}}$ is called a *stationary* set if $T\Omega = \Omega$, where $T : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ is the shift. Note that a super-stationary set is always stationary since $T\Omega = \Omega[\{1, 2, \cdots\}]$. We call $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$ an *optimal window* of Ω if $p_\Omega(S) = p^*_\Omega(k)$ for any $k = 1, 2, \cdots$ and $S \subset \mathcal{N}$ with $\#S = k$.

Take a stationary set $\Omega \subset \{0, 1\}^{\mathbb{N}}$ as X and the additive semi-group $\mathbb{N}$ as G. Let the action of $n \in \mathbb{N}$ to $\omega \in \Omega$ be $T^n\omega$. Let $D = \{\omega \in \Omega;\ \omega(0) = 1\}$. In this case, it is easy to see that

Theorem 2.9. *For an infinite subset $\mathcal{N}$ of $\mathbb{N}$, $\mathcal{N}$ is an optimal position of $(\Omega, \mathbb{N}, D)$ if and only if $\mathcal{N}$ is an optimal window of Ω.*

Hence, the following theorem follows from Theorem 4.1 of [2].

Theorem 2.10. *Let $\alpha \in \{0, 1\}^{\mathbb{N}}$ be a recurrent pattern Sturmian word. Let $X = \overline{O}(\alpha)$, $G = \mathbb{N}$, $D = \{\omega \in \Omega;\ \omega(0) = 1\}$ and the action of $n \in \mathbb{N}$ to $\omega \in \Omega$ be $T^n\omega$. Then, an optimal position exists.*

Example 2.11. Let $\Omega = \overline{O}(\alpha)$ with the non-simple Toeplitz word $\alpha \in \{0, 1\}^{\mathbb{N}}$ defined in Example 3 in [13]. Then, $p^*_\Omega(k) = 2^k$ $(k = 1, 2, \cdots)$ holds. In this case, an optimal window does not exist. Take an arbitrary $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$. For any $k \in \mathbb{N}$, there exists $K \in \mathbb{N}$ with $K \geq k$ and $\xi \in \{0, 1\}^K$ such that $\alpha = (\xi a_0)(\xi a_1)(\xi a_2) \cdots$ holds with

$a_0, a_1, a_2 \cdots \in \{0,1\}$. There exists such a K together with the property that there exist 3 elements in $\mathcal{N}$, say $N_u < N_v < N_w$ with $N_u \not\equiv N_v \equiv N_w$ modulo $K+1$. Then, either 001 or 101 is not in $\Omega[\{N_u, N_v, N_w\}]$. Hence, $\mathcal{N}$ is not an optimal window.

3 Derived sets

Let Ω be a nonempty closed subset of $\{0,1\}^\Sigma$. where Σ is a countably infinite set. To prove the Main Theorem, we may assume without loss of generality that $\Sigma = \mathbb{N}$, so that from now on, we take $\mathbb{N}$ as Σ unless mentioned otherwise.

Lemma 3.1. *For any injection $\psi : \mathbb{N} \to \mathbb{N}$ and $k = 1, 2, \cdots$, we have $(\Omega \circ \psi)^{(k)} \subset \Omega^{(k)} \circ \psi$. Hence, $\deg(\Omega \circ \psi) \leq \deg \Omega$.*

Proof Since $\omega \mapsto \omega \circ \psi$ is a continuous mapping from Ω onto $\Omega \circ \psi$ and Ω is compact, it is clear that $\Omega' \circ \psi \supset (\Omega \circ \psi)'$. Hence,

$$\Omega'' \circ \psi \supset (\Omega' \circ \psi)' \supset (\Omega \circ \psi)''.$$

In this way, we can prove that

$$\Omega^{(k)} \circ \psi \supset (\Omega \circ \psi)^{(k)} \quad (k = 1, 2, \cdots).$$

Thus, $\deg \Omega \geq \deg(\Omega \circ \psi)$. □

Lemma 3.2. *If $\deg \Omega = 0$, then there exists an infinite subset $\mathcal{N}$ of $\mathbb{N}$ such that $\Omega[\mathcal{N}] \subset \{0^\infty, 1^\infty\}$.*

Proof Since $\deg \Omega = 0$ and Ω is compact, Ω is a finite set. Let $\Omega = \{\omega_1, \omega_2, \cdots, \omega_K\}$ with $K < \infty$. Then, there exists an infinite subset $\mathcal{N}$ of $\mathbb{N}$ and $\eta \in \{0,1\}^K$ such that $\omega_1(n)\omega_2(n)\cdots\omega_K(n) = \eta$ for any $n \in \mathcal{N}$. This implies that $\omega_i[\mathcal{N}] \in \{0^\infty, 1^\infty\}$ for any $i = 1, 2, \cdots, K$. Thus, we have $\Omega[\mathcal{N}] \subset \{0^\infty, 1^\infty\}$. □

Lemma 3.3. *If $\deg \Omega[\mathcal{N}] = \infty$ for any infinite subset $\mathcal{N}$ of $\mathbb{N}$, then we have $p^*_\Omega(k) = 2^k$ $(k = 1, 2, \cdots)$.*

Proof Suppose that $p_\Omega(k_0) < 2^{k_0}$ for some $k_0 = 1, 2, \cdots$. Then, for any $S \in \mathcal{F}_{k_0}(\mathbb{N})$ (see Definition 2.1) there exists $\xi \in \{0,1\}^{k_0}$ such that $\xi \notin \Omega[S]$. For each $S \in \mathcal{F}_{k_0}(\mathbb{N})$, choose one of ξ as this and call it the color of S. Thus, each element in $\mathcal{F}_{k_0}(\mathbb{N})$ is colored by an element in $\{0,1\}^{k_0}$. By the infinitary Ramsey Theorem [7], there exists an infinite subset $\mathcal{N}$ of $\mathbb{N}$ such that $\mathcal{F}_{k_0}(\mathcal{N})$ is monochromatic. That is, there exists $\xi \in \{0,1\}^{k_0}$ such that $\xi \notin \Omega[S]$ for any $S \in \mathcal{F}_{k_0}(\mathcal{N})$. Hence, $\Omega[\mathcal{N}] \subset \mathcal{P}(\xi)$.

Since $\mathcal{P}(u_1 \cdots u_{k-1}u_k)' = \mathcal{P}(u_1 \cdots u_{k-1})$ for any $u_1 \cdots u_{k-1}u_k \in \{0,1\}^+$, it holds that $\mathcal{P}(\xi)^{(k_0-1)} = \mathcal{P}(\xi_1) \neq \emptyset$ and $\mathcal{P}(\xi)^{(k_0)} = \emptyset$. Hence, $\deg \mathcal{P}(\xi) = k_0 - 1$. Thus, $\deg \Omega[\mathcal{N}] \leq k_0 - 1 < \infty$, which completes the proof. □

Corollary 3.4. *Let* $\Omega \subset \{0,1\}^\Sigma$ *be a uniform set other than* $\{0,1\}^\Sigma$*. Then, there exists an injection* $\rho : \mathbb{N} \to \Sigma$ *such that* $\deg(\Omega \circ \rho) < \infty$*.*

It is well known that

Lemma 3.5. *For a finite family of closed subsets* Ω_i $(i = 1, 2, \cdots, k)$*, it holds that*

$$\deg(\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_k) = \sup_{i=1,2,\cdots,k} \deg \Omega_i.$$

4 Proof of the Main Theorem

(2) of the Main Theorem was already proved in Lemma 3.3. Also, the last statement follows from the Corollary 3.4.

We prove (1). Let $\Omega \subset \{0,1\}^{\mathbb{N}}$ be a nonempty closed set satisfying that $\deg(\Omega \circ \rho) < \infty$ for some injection $\rho : \mathbb{N} \to \mathbb{N}$. Then, there exists an increasing injection $\psi : \mathbb{N} \to \mathbb{N}$ such that $\rho(\psi(0)) < \rho(\psi(1)) < \rho(\psi(2)) < \cdots$. Let $\mathcal{N} := \{\rho(\psi(0)) < \rho(\psi(1)) < \rho(\psi(2)) < \cdots\}$. Since $\Omega[\mathcal{N}] = \Omega \circ \rho \circ \psi$, we have $\deg \Omega[\mathcal{N}] \leq \deg(\Omega \circ \rho) < \infty$ by Lemma 3.1. We denote this $\Omega[\mathcal{N}]$ by Ω and assume that $d := \deg \Omega < \infty$.

If $d = 0$, then by Lemma 3.2, there exists an infinite subset $\mathcal{N}$ of $\mathbb{N}$ such that $\emptyset \neq \Omega[\mathcal{N}] \subset \{0^\infty, 1^\infty\}$ holds. Hence, $\Omega[\mathcal{N}]$ is super-stationary.

Let $d \geq 1$ and assume that our theorem holds for degrees $0, 1, \cdots, d-1$. Since $\Omega^{(d)}$ is a finite set, there exists an infinite subset $\mathcal{N}$ of $\mathbb{N}$ such that

$$\Omega[\mathcal{N}]^{(d)} \subset \Omega^{(d)}[\mathcal{N}] \subset \{0^\infty, 1^\infty\}$$

by Lemmas 3.1 and 3.2. If $\Omega[\mathcal{N}]^{(d)} = \emptyset$, then by the induction hypothesis, there exists an infinite subset $\mathcal{M}$ of $\mathbb{N}$ such that $\Omega[\mathcal{N} \circ \mathcal{M}] := \Omega[\mathcal{N}][\mathcal{M}]$ is super-stationary.

Thus, the Main Theorem holds with $\mathcal{N} \circ \mathcal{M}$.

Hence, we may assume that $\emptyset \neq \Omega[\mathcal{N}]^{(d)} \subset \{0^\infty, 1^\infty\}$. We write this $\Omega[\mathcal{N}]$ as Ω. Then, we have 3 cases:

$$\Omega^{(d)} = \{0^\infty\},\ \Omega^{(d)} = \{1^\infty\} \text{ or } \Omega^{(d)} = \{0^\infty, 1^\infty\}.$$

The 2nd case is just parallel to the 1st case. So we consider only 2 cases, namely, the 1st case and the 3rd case. For $\Lambda \subset \{0,1\}^{\mathbb{N}}$ and $\xi \in \{0,1\}^*$, denote

$$\begin{aligned} \xi\Lambda &= \{\xi\omega;\ \omega \in \Lambda\} \\ \xi^{-1}\Omega &= \{\omega;\ \xi\omega \in \Lambda\}. \end{aligned}$$

Case 1: $\Omega^{(d)} = \{0^\infty\}$. In this case, we have

$$\Omega = \bigcup_{n=0}^{\infty} 0^n 1 \Omega_n^0 \cup \{0^\infty\}$$

and Ω_n^0 $(n = 0, 1, 2, \cdots)$ are closed sets (possibly, empty) with degrees $\leq d-1$. Moreover, there are infinitely many n's with $\Omega_n^0 \neq \emptyset$, since $\deg \Omega = d$. Denoting the set of these n's by $\mathcal{N}$ and taking $\Omega[\mathcal{N}]$ for Ω, we may assume that $\Omega_n^0 \neq \emptyset$ for all n's. We always assume this in the following similar settings.

By the induction hypothesis, there exists an infinite subset $\mathcal{K} = \{K_0 < K_1 < K_2 < \cdots\}$ of $\mathbb{N}$ such that $\Omega_0^0[\mathcal{K}]$ is a super-stationary set. Then, we have

$$\Omega[\{0\} \cup (\mathcal{K}+1)] = \bigcup_{n=0}^{\infty} 0^n 1 \Omega_n^1 \cup \{0^\infty\},$$

where $\Omega_0^1 = \Omega_0^0[\mathcal{K}]$ is a super-stationary set and

$$\begin{aligned} &\Omega_{n+1}^1 = \Omega_{K_n+1}^0[\mathcal{K} - (K_n+1)] \\ &\cup \bigcup_{m=0}^{n} (0^{n-m}1)^{-1} \Big(\bigcup_{K_{m-1}+1 < i \leq K_m} \Omega_i^0[\mathcal{K} - i] \Big) \\ &(n = 0, 1, 2, \cdots), \end{aligned}$$

where we put $K_{-1} = -1$. Let $\mathcal{M}^1 = \{0\} \cup (\mathcal{K}+1)$. Since

$$\Omega[\mathcal{M}^1] = \bigcup_{n=0}^{\infty} 0^n 1 \Omega_n^1 \cup \{0^\infty\}$$

and Ω_n^1's are nonempty closed sets with degrees $\leq d-1$, we can apply the induction hypothesis. Then, there exists an infinite subset $\mathcal{L}$ of $\mathbb{N}$ such that $\Omega_1^1[\mathcal{L}]$ is a super-stationary set. Let $\mathcal{M}^2 = \mathcal{M}^1 \circ (\{0,1\} \cup (\mathcal{L}+2))$. Then, we have

$$\Omega[\mathcal{M}^2] = \bigcup_{n=0}^{\infty} 0^n 1 \Omega_n^2 \cup \{0^\infty\},$$

where Ω_0^2 and Ω_1^2 are super-stationary sets with $\Omega_0^2 = \Omega_0^1$.

In this way, we can continue so that for any $k = 1, 2, \cdots$, there exists an infinite subset $\mathcal{M}^k$ of $\mathbb{N}$ such that

$$\Omega[\mathcal{M}^k] = \bigcup_{n=0}^{\infty} 0^n 1 \Omega_n^k \cup \{0^\infty\},$$

where Ω_n^k's for $n = 0, 1, \cdots, k-1$ are super-stationary sets with

$$\Omega_n^k = \Omega_n^{n+1} \quad (n = 0, 1, \cdots, k-1).$$

Moreover, since $\mathcal{M}^{n+1} = \mathcal{M}^n \circ (\{0, 1, \cdots, n\} \cup (\mathcal{H}+n+1))$ holds for the infinite subset $\mathcal{H}$ of $\mathbb{N}$ such that $\Omega_n^{n+1} = \Omega_n^n[\mathcal{H}]$ is super-stationary, it holds that $\mathcal{M}^0 \supset \mathcal{M}^1 \supset \mathcal{M}^2 \supset \cdots$ and

$$\mathcal{M} := \bigcap_{k=0}^{\infty} \mathcal{M}^k = \{M_0^0 < M_1^1 < M_2^2 < \cdots\},$$

where $\mathcal{M}^0 = \mathbb{N}$ and $\mathcal{M}^k = \{M_0^k < M_1^k < M_2^k < \cdots\}$ $(k = 0, 1, 2, \cdots)$. Putting $\Omega_n = \Omega_n^{n+1}$, we have

$$\Omega[\mathcal{M}] = \bigcup_{n=0}^{\infty} 0^n 1 \Omega_n \cup \{0^\infty\},$$

where Ω_n's are super-stationary sets with degrees $\leq d - 1$.

Now we prove that there exists an infinite subset $\mathcal{N}$ of $\mathbb{N}$ such that $\Omega[\mathcal{M}][\mathcal{N}]$ is a super-stationary set. We may assume that Ω in place of $\Omega[\mathcal{M}]$ satisfies the above decomposition. That is,

$$\Omega = \bigcup_{n=0}^{\infty} 0^n 1 \Omega_n \cup \{0^\infty\}$$

holds with super-stationary sets Ω_n's with degrees $\leq d - 1$. Since there are only finitely many super-stationary sets with degrees $\leq d-1$, there are finitely many different Ω_n's. Let Θ_0 be their union. There are also finitely many different Ω_n's that appear in the sequence $\Omega_0, \Omega_1, \Omega_2, \cdots$ infinitely many times. Let their union be Θ_1. Choose one of sets, say Θ_2, that appears in the sequence $\Omega_0, \Omega_1, \Omega_2, \cdots$ infinitely many times. Take $\mathcal{N} = \{1 \leq N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$ satisfying the following conditions:

$$\begin{gathered} \textstyle\bigcup_{0 \leq i < N_0} \Omega_i = \Theta_0 \\ \textstyle\bigcup_{N_n < i < N_{n+1}} \Omega_i = \Theta_1 \quad (n = 0, 1, 2, \cdots) \\ \Omega_{N_n} = \Theta_2 \quad (n = 0, 1, 2, \cdots). \end{gathered}$$

Then, we have

$$\Omega[\mathcal{N}] = \Theta_0 \cup \bigcup_{n=1}^{\infty} 0^n \Theta_1 \cup \bigcup_{n=0}^{\infty} 0^n 1 \Theta_2 \cup \{0^\infty\}.$$

Since $\Theta_0 \supset \Theta_1 \supset \Theta_2$ and they are super-stationary sets, it is easy to see that $\Omega[\mathcal{N}]$ is a super-stationary set.

Case 2: $\Omega^{(d)} = \{0^\infty, 1^\infty\}$. In this case, we have

$$\begin{aligned} \Omega &= \Theta \cup \Lambda \\ \Theta &= \bigcup_{n=1}^{\infty} 0^n 1 \Theta_n^0 \cup \{0^\infty\} \\ \Lambda &= \bigcup_{n=1}^{\infty} 1^n 0 \Lambda_n^0 \cup \{1^\infty\} \end{aligned}$$

and Θ_n^0, Λ_n^0 $(n = 0, 1, 2, \cdots)$ are nonempty closed sets with degrees $\leq d-1$. Then, as in Case 1, there exists an infinite subset $\mathcal{K}$ of $\mathbb{N}$ such that $\Omega_0 := (0^{-1}\Theta)[\mathcal{K}]$ is a super-stationary set. Hence

$$(1^{-1}\Lambda)[\mathcal{K}] = \bigcup_{n=0}^{\infty} 1^n 0 \Lambda_n^1 \cup \{1^\infty\}$$

holds with nonempty closed sets Λ_n^1's having degrees $\leq d-1$. Applying the same argument as in Case 1, there exists an infinite subset $\mathcal{M}$ of $\mathbb{N}$ such that $(1^{-1}\Lambda)[\mathcal{K}][\mathcal{M}]$ has a decomposition

$$(1^{-1}\Lambda)[\mathcal{K}][\mathcal{M}] = \bigcup_{n=0}^{\infty} 1^n 0 \Lambda_n \cup \{1^\infty\},$$

where Λ_n's are super-stationary sets with degrees $\leq d-1$. Therefore, $\Omega[\{0\} \cup (\mathcal{K} \circ \mathcal{M} + 1)]$ has a decomposition that

$$\Omega[\{0\} \cup (\mathcal{K} \circ \mathcal{M} + 1)] = 0\Omega_0 \cup (\bigcup_{n=1}^{\infty} 1^n 0 \Lambda_n) \cup \{1^\infty\},$$

where Ω_0 is a super-stationary set with degree $\leq d$ and Λ_n's are super-stationary sets with degrees $\leq d-1$.

To prove that there exists an infinite subset $\mathcal{N}$ of $\mathbb{N}$ such that $\Omega[\{0\} \cup (\mathcal{K} \circ \mathcal{M} + 1)][\mathcal{N}]$ is a super-stationary set, we denote $\Omega[\{0\} \cup (\mathcal{K} \circ \mathcal{M} + 1)]$ by Ω and assume that

$$\Omega = 0\Omega_0 \cup (\bigcup_{n=1}^{\infty} 1^n 0 \Lambda_n \cup \{1^\infty\}),$$

where Ω_0 is a super-stationary set with degree $\leq d$ and Λ_n's are super-stationary sets with degrees $\leq d-1$. Since there are only finitely many super-stationary sets with degrees $\leq d-1$, there are finitely many different Λ_n's. Let Ω_1 be their union. Moreover, there are finitely many different Λ_n's that appear in the sequence $\Lambda_0, \Lambda_1, \Lambda_2, \cdots$ infinitely many times. Let their union be Ω_2. Choose a set, say Ω_3 that appears in the sequence $\Lambda_0, \Lambda_1, \Lambda_2, \cdots$ infinitely many times. Take $\mathcal{N} = \{1 \leq N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$ satisfying the following conditions:

$$\begin{gathered} \textstyle\bigcup_{0 \leq i < N_0} \Lambda_i = \Omega_1 \\ \textstyle\bigcup_{N_n < i < N_{n+1}} \Lambda_i = \Omega_2 \quad (n = 0, 1, 2, \cdots) \\ \Lambda_{N_n} = \Omega_3 \quad (n = 0, 1, 2, \cdots) \end{gathered}$$

Then, we have

$$\Omega[\mathcal{N}] = \Omega_0 \cup \Omega_1 \cup \bigcup_{n=1}^{\infty} 1^n \Omega_2 \cup \bigcup_{n=0}^{\infty} 1^n 0 \Omega_3 \cup \{1^\infty\}.$$

Since $\Omega_1 \supset \Omega_2 \supset \Omega_3$ and they are super-stationary sets together with Ω_0, it is easy to see that $\Omega[\mathcal{N}]$ is a super-stationary set.

5 Uniform sets with low complexity

For a nonempty finite subset Ξ of $\{0,1\}^+$ and $k = 0, 1, 2, \cdots$, let

$$Q(\Xi)(k) = \#\{\eta \in \{0,1\}^k;\ \xi \ll \eta \text{ does not hold for some } \xi \in \Xi\}.$$

Then, it holds that

Lemma 5.1. [1] *If $\mathcal{Q}(\Xi)$ is a super-stationary set, then $p_{\mathcal{Q}(\Xi)}(k) = Q(\Xi)(k)$ $(k = 1, 2, \cdots)$ holds. Moreover, we have the following formula:*
For a nonempty finite set $\Xi \subset \{0,1\}^+$ with $\Xi = \Xi_0 0 \cup \Xi_1 1$,

$$Q(\Xi)(k) = Q(\Xi_0 \cup \Xi_1 1)(k-1) + Q(\Xi_0 0 \cup \Xi_1)(k-1) \quad (k = 1, 2, \cdots)$$

holds, and if $\Xi = \Xi_0 0$, then

$$Q(\Xi)(k) = 1 + \sum_{i=0}^{k-1} Q(\Xi_0)(i) \quad (k = 0, 1, 2, \cdots)$$

holds (the same formula holds for $\Xi = \Xi_1 1$).

Let us list up all uniform complexity functions with degree ≤ 1. If Ξ contains ξ with $|\xi| \geq 3$, then $p_{\mathcal{Q}(\Xi)}(k)$ is a polynomial of degree ≥ 2 since

$$p_{\mathcal{Q}(\Xi)}(k) \geq p_{\mathcal{P}(\xi)}(k) = \sum_{i=0}^{|\xi|-1} \binom{k}{i}.$$

Therefore, any uniform complexity function with degree ≤ 1 is realized by a union of $\mathcal{P}(0)$, $\mathcal{P}(1)$, $\mathcal{P}(00)$, $\mathcal{P}(01)$, $\mathcal{P}(10)$, $\mathcal{P}(11)$ by Theorem 1.7.

The following list contains all irreducible union of the above sets up to the symmetry of exchanging 0 and 1. The first column is the super-stationary set Ω represented by the notation introduced in Section 1. The second column is its representation as $\mathcal{Q}(\Xi)$. The 3rd column is its representation as $\mathcal{P}(\mathrm{L}(\Xi))$. The 4th column is the complexity function $p_\Omega(k)$ and the 5th column is the minimum k_0 such that the formula holds for $k \geq k_0$.

In the list, Θ_δ and Θ_+ are isomorphic, also, $\Theta_\delta \cup \Theta_-$ and $\Theta_+ \cup \Theta_-$ are isomorphic, but they are not isomorphic to $\Theta_\delta \cup \Theta_+$ (Figures 1, 2).

Ω	Ξ	L(Ξ)	$p_\Omega(k)$	k_0
Θ_0	1	1	1	1
$\Theta_0 \cup \Theta_1$	1,0	10,01	2	1
Θ_δ	11	11	$k+1$	1
Θ_+	10	10		
$\Theta_\delta \cup \Theta_1$	11,0	110,101,011	$k+2$	2
$\Theta_\delta \cup \Theta_+$	11,10	110,101	$2k$	1
$\Theta_\delta \cup \Theta_-$	11,01	101,011		
$\Theta_+ \cup \Theta_-$	10,01	101,010		
$\Theta_\delta \cup \Theta_{1-\delta}$	11,00	1100,1010,1001,0110,0101,0011	$2k+2$	3
$\Theta_\delta \cup \Theta_+ \cup \Theta_-$	11,10,01	101,0101,0110	$3k-2$	2
$\Theta_\delta \cup \Theta_+ \cup \Theta_{1-\delta}$	11,10,00	1100,1010,1001,0110,0101	$3k-1$	3
$\Theta_\delta \cup \Theta_+ \cup \Theta_{1-\delta} \cup \Theta_-$	11,10,01,00	1010,1001,0110,0101	$4k-4$	2

Example 5.2. By Lemma 5.1, we have

$$\begin{aligned}
& Q(11,10,00)(k) \\
= \; & Q(11,1,0)(k-1) + Q(1,10,00)(k-1) \\
= \; & Q(11,0)(k-1) + Q(10,00)(k-1) \\
= \; & Q(11)(k-2) + Q(1,0)(k-2) + 1 + \sum_{i=0}^{k-2} Q(1,0)(i) \\
= \; & 1 + \sum_{i=0}^{k-3} Q(1)(i) + 2 + 1 + 1 + \sum_{i=1}^{k-2} 2 \\
= \; & 5 + \sum_{i=0}^{k-3} 1 + 2(k-2) = 3k-1.
\end{aligned}$$

Hence, $p_{\Theta_\delta \cup \Theta_+ \cup \Theta_{1-\delta}}(k) = 3k-1$.

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(Kamae's papers are downloadable from http://www14.plala.or.jp/kamae.)

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