Behavior of various complexity functions

Theoretical Computer Science 420 (2012), pp.36-47
(doi:10.1016/j.tcs.2011.11.1012)

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Abstract

For a nonempty closed set $\Omega \subset \mathbb{A}^{\mathbb{N}}$ with $2 \le \#\mathbb{A} < \infty$, we consider 3 complexity functions of $k = 0, 1, 2, \cdots$:
(1) (block complexity) $p_\Omega^{BL}(k) := \#\Omega|_{\{0,1,\cdots,k-1\}}$,
(2) (maximal pattern complexity) $p_\Omega^*(k) := \sup_{S \subset \mathbb{N},\ \#S=k} \#\Omega|_S$,
(3) (minimal pattern complexity) $p_{*\Omega}(k) := \inf_{S \subset \mathbb{N},\ \#S=k} \#\Omega|_S$,
where $\#$ denotes the number of elements in a set, and $\Omega|_S$ is the restriction of Ω to $S \subset \mathbb{N}$. If $p_\Omega^*(k) = p_{*\Omega}(k)$ $(k = 0, 1, 2, \cdots)$ holds, then the above 3 complexities coincide and are called uniform complexity, denoted by $p_\Omega(k)$.
Behaviors of these 4 complexity functions are discussed.

Key Words: block complexity; maximal pattern complexity; minimal pattern complexity; uniform complexity; maximal pattern entropy; uniform set; super-stationary set

1 Introduction

For a nonempty closed set $\Omega \subset \mathbb{A}^{\mathbb{N}}$, where $\mathbb{A}$ can be any alphabet (i.e. nonempty finite set of *letters*) and $\mathbb{N} = \{0, 1, 2, \cdots\}$, we consider 3 functions of $k = 0, 1, 2, \cdots$:
(1) $p_\Omega^{BL}(k) := \#\Omega|_{\{0,1,\cdots,k-1\}}$,
(2) $p_\Omega^*(k) := \sup_{S \subset \mathbb{N},\ \#S=k} \#\Omega|_S$,
(3) $p_{*\Omega}(k) := \inf_{S \subset \mathbb{N},\ \#S=k} \#\Omega|_S$,
where $\#$ denotes the number of elements in a set, and $\Omega|_S$ is the restriction of Ω to $S \subset \mathbb{N}$. They are called *block complexity*, *maximal pattern complexity* and *minimal pattern complexity* of Ω, respectively. If $p_\Omega^*(k) = p_{*\Omega}(k)$ $(k = 0, 1, 2, \cdots)$ holds, then we call Ω a *uniform set*. In this case, the above 3 complexities coincide and are called *uniform complexity*, denoted by $p_\Omega(k)$.

Behaviors of these 4 complexity functions are discussed. We always assume that $\Omega \subset \mathbb{A}^{\mathbb{N}}$ is a nonempty closed set and $\#\mathbb{A} \ge 2$ throughout this paper.

The notion of complexity was introduced for the first time as the block complexity of an infinite words $\omega = \omega(0)\omega(1)\omega(2)\cdots \in \mathbb{A}^{\mathbb{N}}$. That is,

$$p_\omega^{BL}(k) := \#\{\omega(n)\omega(n+1)\cdots\omega(n+k-1) \in \mathbb{A}^k;\ n \in \mathbb{N}\} \quad (k = 0, 1, 2, \cdots).$$

Let $T : \mathbb{A}^{\mathbb{N}} \to \mathbb{A}^{\mathbb{N}}$ be the *shift*, that is, $(T\omega)(n) = \omega(n+1)$ for any $\omega \in \mathbb{A}^{\mathbb{N}}$ and $n \in \mathbb{N}$. Let $\Omega \subset \mathbb{A}^{\mathbb{N}}$ be the closure of the orbit $\{T^n\omega;\ n \in \mathbb{N}\}$ of $\omega \in \mathbb{A}^{\mathbb{N}}$ with respect to the shift T. Then we have

$$p_\omega^{BL}(k) = p_\Omega^{BL}(k) \ (k = 0, 1, 2, \cdots).$$

The maximal pattern complexity was also introduced for an infinite word for the first time:

$$p_\omega^*(k) := \sup_{\tau \subset \mathbb{N},\ \#\tau = k} \#\{\omega[n+\tau];\ n \in \mathbb{N}\} \ (k = 0, 1, 2, \cdots),$$

where $\omega[n+\tau] = \omega(n+\tau_0)\omega(n+\tau_1)\cdots\omega(n+\tau_{k-1}) \in \mathbb{A}^k$ with $\tau = \{\tau_0 < \tau_1 < \cdots < \tau_{k-1}\} \subset \mathbb{N}$. Let Ω be the closure of $\{T^n\omega;\ n \in \mathbb{N}\}$. Then we have

$$p_\omega^*(k) = p_\Omega^*(k) \ (k = 0, 1, 2, \cdots).$$

The set Ω which is the orbit closure of a recurrent word in $\mathbb{A}^{\mathbb{N}}$ is *stationary*, that is, satisfies $T\Omega = \Omega$, and is *transitive*, that is, there exists $\omega \in \Omega$ such that the orbit of ω is dense in Ω. Therefore, statements on recurrent words are automatically translated into statements on stationary and transitive Ω. They are also closely related to topological dynamics. In this paper, we consider not only stationary and transitive Ω, but also general nonempty closed sets $\Omega \subset \mathbb{A}^{\mathbb{N}}$, where closedness is also irrelevant for most of the paper. The complexities for a non-closed Ω are same as those of its closure.

The uniform complexity functions behave regularly, so that the entropy $\lim_{k\to\infty} \frac{\log p_\Omega(k)}{k}$ exists and takes value $\log r$ with a positive integer r for any uniform set Ω. This property is shared by the maximal pattern complexity (Theorem 1) but not by the minimal pattern complexity (Example 1). This property for the maximal pattern complexity was proved by W. Huang and X. Ye [4] in the topological dynamics setting. Here, we generalize and simplify the proof for general Ω using the same idea. We discuss them in Section 2.

Let us give an example why the maximal pattern complexity is important. Consider a set of pictures of typical human faces as computer graphics. They are represented as configurations of digital data (colors, etc) at points in $\{t_0, t_1, t_2, \cdots\}$ which is a dense subset of a 2-dimensional domain. The set of digital data at a point is a finite set, say $\mathbb{A}$, so that a picture of a human face is an element in $\mathbb{A}^{\{t_0, t_1, t_2, \cdots\}}$, which we identify with $\mathbb{A}^{\mathbb{N}}$. Thus, the set of human faces can be identified with a set $\Omega \subset \mathbb{A}^{\mathbb{N}}$. We choose a subset S (*sampling set*) of $\mathbb{N}$ of a fixed size k to identify a human face $\omega \in \Omega$ by scanning and checking whether $\omega|_S$ coincides with the registered one or not. The best choice for the sampling set S is those which distinguish the faces in Ω as many as possible. In other words, the best S is that satisfying $\#\Omega|_S = p^*_\Omega(k)$.

In the above, if there exists an infinite set $\Sigma \subset \mathbb{N}$ such that for any $k = 1, 2, \cdots$ and $S \subset \Sigma$ with $\#S = k$, $\#\Omega|_S = p^*_\Omega(k)$ holds, then we call Σ an *optimal position*. In this case, we get the maximal information about the faces in Ω by taking sampling sets from Σ, so that Σ is considered as the best distinguishable combination of points for the faces in Ω. This also means that $\Omega|_\Sigma \subset \mathbb{A}^\Sigma$ is a uniform set with $p_{\Omega|_\Sigma}(k) = p^*_\Omega(k)$ $(k = 0, 1, 2, \cdots)$.

For a uniform set Ω, not only the entropy exists and takes value $\log r$ with positive integer r, $\lim_{k\to\infty} \dfrac{\log p_\Omega(k) - k \log r}{\log k}$ exists and takes nonnegative integer value (Theorem 12). Furthermore, if $p_\Omega(k)$ increases in a linear order, then $\lim_{k\to\infty} p_\Omega(k)/k$ exists and is a nonnegative integer. We don't know whether the former property is shared by the maximal pattern complexity or not, but as for the latter property, there exists Ω which is the orbit closure of a Toeplitz word, and hence, is stationary and transitive such that $\lim_{k\to\infty} p^*_\Omega(k)/k = 10/3$ (Example 5).

One of the aims of this paper is to compare the regularity of the complexity functions. The uniform complexity behaves most regularly, and the maximal pattern complexity is the next. The third is the minimal pattern complexity, and the last is the block complexity. Actually, we prove a necessary and sufficient condition for a function $\mathbb{N} \to \mathbb{N}$ to be a block complexity (Theorem 2), which is always satisfied by the other complexities.

In Section 3, we discuss the smallest unbounded increasing order of the complexity functions. If $\#\Omega = \infty$, then $p^*_\Omega(k) \geq k + 1$ $(k = 0, 1, 2, \cdots)$ holds (Theorem 3). Moreover, there exists Ω such that $p^*_\Omega(k) = k + 1$ $(k = 0, 1, 2, \cdots)$. On the other hand, $p_{*\Omega}(k)$ can be bounded in k even if $\#\Omega = \infty$ (Example 2). If $p_{*\Omega}(k)$ is unbounded, then $p_{*\Omega}(k) \geq C \log k$ $(k = 1, 2, \cdots)$ for some constant $C > 0$. Moreover, for any integer $d \geq 2$, there exists Ω such that $p_{*\Omega}(k) = \lceil \log k / \log d \rceil + 1$ (Theorem 4).

If we restrict to stationary and transitive Ω, we know some more. That is, if $\#\Omega = \infty$, then $p^{BL}_\Omega(k) \geq k + 1$ $(k = 0, 1, 2, \cdots)$ and $p^*_\Omega(k) \geq 2k$ $(k = 1, 2, \cdots)$ (see [2], [5]). On the other hand, a stationary and transitive set Ω satisfies that $p^{BL}_\Omega(k) = k + 1$ $(k = 0, 1, 2, \cdots)$ if and only if it is the orbit closure of a Sturmian word. In this case, we call Ω a *Sturmian* set. In the same way, a stationary and transitive Ω satisfies that $p^*_\Omega(k) = 2k$ $(k = 1, 2, \cdots)$ if and only if it is the orbit closure of a recurrent pattern Sturmian word. In this case, we call Ω a *pattern Sturmian* set. It is known [5] that a Sturmian set is a pattern Sturmian set, but the converse is not true. We discuss them in Section 4.

The minimal pattern complexity for stationary and transitive sets Ω is studied by S. Ferenczi and P. Hubert [1]. They proved that if $\#\Omega = \infty$, then $p_{*\Omega}(k) \geq k+1$ ($k = 0, 1, 2, \cdots$) (Theorem 5), giving an example of stationary and transitive sets Ω with $p_{*\Omega}(k) = k+1$ ($k = 0, 1, 2, \cdots$) but not Sturmian (Example 4). They also proved that $p_{*\Omega}(k)$ increases exponentially if Ω is a strongly mixing subshift of finite type.

In Section 5, we study the uniform complexity. The uniform complexity has been studied by Rao Hui, Tan Bo, Xue Yumei and the author over the binary alphabet ([7], [10], [11]). Recently, it was generalized by the author over the general alphabet ([13]). The main fact is that it is realized by a *super-stationary set* which has 2 different characterizations (intersection form and union form) (Theorems 9 and 13). Using these facts, a uniform complexity function is proved to be *equivalent* (i.e. coincides except for finite places) to a function $f : \mathbb{N} \to \mathbb{Z}$ such that f is of the form $f(k) = \sum_{i=1}^{d} \mathcal{R}_i(k) i^k$, where $d = 1, 2, \cdots$ and $\mathcal{R}_i(k)$ ($i = 1, 2, \cdots, d$) are polynomials of k with rational coefficient (Theorem 11). Moreover, the set of functions of this form equivalent to some uniform complexity functions is a semi-ring. We discuss the basis of this semi-ring in Section 6.

The notion of uniform sets can be defined for any $\Omega \subset \mathbb{A}^{\Sigma}$ with an arbitrary infinite index set Σ. It is a class of sets with full symmetry in the sizes of the restrictions to finite index sets. One of the motivations to study it is this symmetry and the naturalness. In fact, we have many beautiful properties of the uniform complexity. Another motivation is that uniform sets come out as optimal positions of the problem to maximize informations of sampling sets. In [14], the problem to maximize the partition generated by k number of unit balls in n-dimensional Euclidean space is discussed. An optimal position exists in this problem, and a uniform set corresponds to it. Moreover, we can specify the super-stationary set contained in it. Actually, the problem of maximizing partitions and the problem of maximizing informations of sampling sets are dual. This duality is studied in [15] with an application to the problem of pattern recognition.

2 Exponentially increasing case and entropy

Let Ω be a nonempty closed subset of $\mathbb{A}^{\mathbb{N}}$, where $\mathbb{A}$ is an alphabet.

Definition 1. If the following limits exist, we call them the *block entropy*, the *maximal pattern entropy* and the *minimal pattern entropy* of Ω, respectively:
(1) $h^{BL}(\Omega) := \lim_{k\to\infty}(1/k)\log p_{\Omega}^{BL}(k)$,
(2) $h^{*}(\Omega) := \lim_{k\to\infty}(1/k)\log p_{\Omega}^{*}(k)$,
(3) $h_{*}(\Omega) := \lim_{k\to\infty}(1/k)\log p_{*\Omega}(k)$.
If Ω is a uniform set, the following limit is called the *uniform entropy* of Ω:
(4) $h(\Omega) := \lim_{k\to\infty}(1/k)\log p_{\Omega}(k)$.

Definition 2. For any positive integers r, h and $\Xi \subset \mathbb{A}^U$ with $U \subset \mathbb{N}$ (possibly, $U = \mathbb{N}$), a pair $((t_1, t_2, \cdots, t_h), \Theta)$ is called a *r-tree of size h* contained in Ξ, if $(t_1, t_2, \cdots, t_h)$ is a sequence of distinct elements in U and Θ is a subset of $\Xi|_{\{t_1,t_2,\cdots,t_h\}}$ such that for any $\eta \in \Theta|_{\{t_1,\cdots,t_i\}}$ with $i = 0, 1, \cdots, h-1$, we have

$$\#\{\eta' \in \Theta|_{\{t_1,\cdots,t_i,t_{i+1}\}};\ \eta'|_{\{t_1,\cdots,t_i\}} = \eta\} = r.$$

In other words, the following (V, E) is a r-tree:

$$\begin{aligned} V &= \{\eta(t_1)\cdots\eta(t_i);\ \eta \in \Theta,\ i = 0, 1, \cdots, h\}, \\ E &= \{(\eta(t_1)\cdots\eta(t_i),\ \eta(t_1)\cdots\eta(t_i)\eta(t_{i+1}));\ \eta \in \Theta,\ i = 0, 1, \cdots, h-1\}. \end{aligned}$$

In this case, we have $\#\Theta|_{\{t_1,\cdots,t_i\}} = r^i$ $(i = 0, 1, \cdots, h)$.

Theorem 1. (Huang-Kamae-Ye) *The maximal pattern entropy $h^*(\Omega)$ exists for any Ω, and takes value $\log r$ with a positive integer r. In this case, Ω contains a r-tree of an arbitrary large size.*

We need the following lemma to prove Theorem 1. We fix integers d and r such that $1 \le r < d = \#\mathbb{A}$.

Lemma 1. *For any $h = 1, 2, \cdots$ and δ with $0 < \delta \le 1$, there exists k_0 such that for any $S \subset \mathbb{N}$ with $k_0 \le \#S < \infty$ and $\Xi \subset \mathbb{A}^S$ with $\#\Xi \ge (r+\delta)^{\#S}$, Ξ contains a $(r+1)$-tree of size h.*

Proof We use the induction on h. Let $h = 1$. For any $\delta > 0$, let $k_0 = 1$. If $\Xi \subset \mathbb{A}^S$ with $1 \le \#S < \infty$ satisfies that $\#\Xi \ge (r+\delta)^{\#S}$, then there exists $i \in S$ such that $\Xi|_{\{i\}}$ contains more than r elements, since otherwise, we have a contradiction that $\#\Xi \le r^{\#S}$. Thus, our statement holds for $h = 1$.

Let $h \ge 1$ and assume that our statement holds for h. Let $0 < \delta \le 1$. Take any $S \subset \mathbb{N}$ with $K \le \#S < \infty$ and $\Xi \subset \mathbb{A}^S$ with $\#\Xi \ge (r+\delta)^{\#S}$, where K is a sufficiently large integer determined later. Let $S = \{s_1, s_2, \cdots, s_k\}$ with $k = \#S \ge K$. Take the maximum $i \le k$ such that

$$\frac{\#\Xi|_{\{s_1,\cdots,s_{i-1},s_i\}}}{\#\Xi|_{\{s_1,\cdots,s_{i-1}\}}} > r + \frac{\delta}{2}\ .$$

Since

$$(r+\delta)^k \le \#\Xi \le d^i (r + \frac{\delta}{2})^{k-i},$$

we have $i \ge Ck$ with

$$0 < C := \frac{\log(r+\delta) - \log(r+\frac{\delta}{2})}{\log d - \log(r+\frac{\delta}{2})} \le 1.$$

Let

$$\begin{aligned} A &= \{\xi \in \Xi|_{\{s_1,\cdots,s_{i-1}\}};\ \#\{\eta \in \Xi|_{\{s_1,\cdots,s_{i-1},s_i\}};\ \eta|_{\{s_1,\cdots,s_{i-1}\}} = \xi\} \le r\} \\ B &= \{\xi \in \Xi|_{\{s_1,\cdots,s_{i-1}\}};\ \#\{\eta \in \Xi|_{\{s_1,\cdots,s_{i-1},s_i\}};\ \eta|_{\{s_1,\cdots,s_{i-1}\}} = \xi\} \ge r+1\}. \end{aligned}$$

Then, since $\Xi|_{\{s_1,\cdots,s_{i-1}\}} = A \cup B$ (*disjoint*), we have

$$\begin{aligned} r\#A + d\#B &\geq \#\Xi|_{\{s_1,\cdots,s_{i-1},s_i\}} \\ &\geq (r + \frac{\delta}{2})\#\Xi|_{\{s_1,\cdots,s_{i-1}\}} = (r + \frac{\delta}{2})(\#A + \#B). \end{aligned}$$

Since $\#A \leq \dfrac{2(d-r)-\delta}{\delta}\#B$ follows from this, we have

$$\#\Xi|_{\{s_1,\cdots,s_{i-1},s_i\}} \leq r\#A + d\#B \leq \left(\frac{2(d-r)-\delta}{\delta}r + d\right)\#B.$$

Therefore, $\#B \geq D\#\Xi|_{\{s_1,\cdots,s_{i-1},s_i\}} \geq D(r+\delta)^i$ holds with

$$D := \left(\frac{2(d-r)-\delta}{\delta}r + d\right)^{-1} > 0.$$

Let k_1 be such that $D(r+\delta)^i \geq (r + \frac{\delta}{2})^i$ for any $i \geq k_1$. Then, $B \subset \mathbb{A}^{\{s_1,\cdots,s_{i-1}\}}$ and $\#B \geq (r + \frac{\delta}{2})^{i-1}$ holds if $i \geq k_1$. Moreover, if $i - 1 \geq k_0$, where k_0 is the value in the statement of Lemma 1 for h and $\delta/2$, then by the induction hypothesis, B contains a $(r+1)$-tree of size h. Let this be $((t_1,\cdots,t_h),\Theta)$. Since $\Theta \subset B|_{\{t_1,\cdots,t_h\}}$ and each element in B has at least $r+1$ extensions to the coordinate s_i in the set Ξ, we can find an extension of Θ to $(t_1,\cdots,t_h,s_i)$, say Θ', such that $((t_1,\cdots,t_h,s_i),\Theta')$ is a $(r+1)$-tree of size $h+1$ contained in Ξ. To complete the proof, we remark that if $K \geq ((k_0+1)\vee k_1)/C$, then the requirements that $i \geq k_0+1$ and $i \geq k_1$ are satisfied since $i \geq Ck \geq CK$. □

Proof of Theorem 1:

Let $\log H = \limsup_{k\to\infty} \log p^*_\Omega(k)/k$. If $H = 1$, then "lim inf" has the same value 0, and hence, "lim" exists and takes value $0 = \log 1$. Assume that $H > 1$. Let $\epsilon > 0$ be smaller than the fractional part of H if H is not an integer, otherwise, let $0 < \epsilon < 1$. Then for any K, there exists $S \subset \mathbb{N}$ such that $K \leq \#S < \infty$ and $\#\Omega|_S \geq (H-\epsilon)^{\#S}$. By Lemma 1, $\Omega|_S$ contains an $(r+1)$-tree of size h with $r = \lfloor H - \epsilon \rfloor$, where h can be arbitrary large corresponding to K. Hence, Ω contains a $(r+1)$-tree of an arbitrary large size. This implies that $\log(r+1) \leq \liminf_{k\to\infty} \log p^*_\Omega(k)/k$. Since

$$\log(r+1) \leq \liminf_{k\to\infty} \log p^*_\Omega(k)/k \leq \limsup_{k\to\infty} \log p^*_\Omega(k)/k = \log H,$$

we must have $r + 1 \leq H$. This is possible only if H is an integer and $H = r+1$. Hence, the equality holds in the above formula. Thus, $h^*(\Omega)$ exists and takes value $\log(r+1)$ with a positive integer r. □

Corollary 1. *For a uniform set Ω, $h(\Omega)$ exists and coincides with $h^*(\Omega)$.*

Example 1. For $\omega \in \mathbb{A}^{\mathbb{N}}$ with $\#\mathbb{A} = 2$, let $\tilde{\omega} \in \mathbb{A}^{\mathbb{N}}$ be such that

$$\tilde{\omega}(n) = \omega(\lfloor n/2 \rfloor) \quad (n \in \mathbb{N}).$$

Let $\Omega = \{\tilde{\omega};\ \omega \in \mathbb{A}^{\mathbb{N}}\}$. Then, it is easy to see that $p_{*\Omega}(k) = 2^{\lceil k/2 \rceil}$ for $k = 0, 1, 2, \cdots$. Therefore, $h_*(\Omega) = \log \sqrt{2}$. If we replace Ω by $\Omega \cup T\Omega$, the minimal pattern entropy remains unchanged, so that we get a stationary and transitive set with $\log \sqrt{2}$ as the minimal pattern entropy.

3 Smallest increasing case

Let Ω be a nonempty closed subset of $\mathbb{A}^{\mathbb{N}}$, where $\mathbb{A}$ is an alphabet.

Theorem 2. *A necessary and sufficient condition for an increasing function $f : \mathbb{N} \to \mathbb{N}$ with $f(0) = 1$ to be a block complexity of some $\Omega \subset \mathbb{A}^{\mathbb{N}}$ over some alphabet $\mathbb{A}$ is that $\sup_{n \in \mathbb{N}} f(n+1)/f(n) < \infty$.*

Proof Let $f(k) = p_{\Omega}^{BL}(k)$ $(k = 0, 1, 2, \cdots)$ for $\Omega \subset \mathbb{A}^{\mathbb{N}}$ with $\#\mathbb{A} = d$. Then, $f : \mathbb{N} \to \mathbb{N}$ is an increasing function with $f(0) = 1$. Since

$$\Omega|_{\{0,1,\cdots,k\}} \subset \Omega|_{\{0,1,\cdots,k-1\}} \times \mathbb{A}^{\{k\}},$$

where $\mathbb{A}^{\{k\}}$ is the set of words over $\mathbb{A}$ defined on the one-point set $\{k\}$, we have $p_{\Omega}^{BL}(k+1) \leq d p_{\Omega}^{BL}(k)$ $(k = 0, 1, 2, \cdots)$. Thus,

$$\sup_{n \in \mathbb{N}} f(n+1)/f(n) \leq d < \infty.$$

Conversely, let $f : \mathbb{N} \to \mathbb{N}$ be an increasing function with $f(0) = 1$ such that $\sup_{n \in \mathbb{N}} f(n+1)/f(n) < \infty$. Let d be an integer such that $d \geq \sup_{n \in \mathbb{N}} f(n+1)/f(n)$. Let $\mathbb{A} = \{0, 1, \cdots, d-1\}$. We can construct $\Omega_k \subset \mathbb{A}^{\{0,1,\cdots,k-1\}}$ with $\#\Omega_k = f(k)$ inductively for $k = 0, 1, 2, \cdots$ so that $\Omega_{k+1} \subset \Omega_k \times \mathbb{A}^{\{k\}}$ and $\Omega_{k+1}|_{\{0,1,\cdots,k-1\}} = \Omega_k$. Let Ω be the projective limit of Ω_k with $k = 0, 1, 2, \cdots$. Then, we have $p_{\Omega}^{BL}(k) = f(k)$ $(k = 0, 1, 2, \cdots)$. $\square$

Theorem 3. *If $\Omega \subset \mathbb{A}^{\mathbb{N}}$ is an infinite set, then $p_{\Omega}^*(k) \geq k+1$ for any $k = 0, 1, 2, \cdots$. Moreover, there exists $\Omega \subset \mathbb{A}^{\mathbb{N}}$ such that $p_{\Omega}^*(k) = k+1$ for any $k = 0, 1, 2, \cdots$.*

Proof Assume that there exists k such that $p_{\Omega}^*(k) \leq k$. Let k_0 be the smallest k as this. Since $p_{\Omega}^*(0) = 1$, $k_0 \geq 1$. Since $p_{\Omega}^*(k_0 - 1) \geq k_0 \geq p_{\Omega}^*(k_0)$, we have $p_{\Omega}^*(k_0 - 1) = p_{\Omega}^*(k_0) = k_0$. Take $S \subset \mathbb{N}$ with $\#S = k_0 - 1$ such that $\#\Omega|_S = k_0$. Since $p_{\Omega}^*(k_0) = k_0$, $\omega(n)$ for any $n \in \mathbb{N}$ and $\omega \in \Omega$ is determined by $\omega|_S$. This implies that $\#\Omega = \#\Omega|_S = k_0$. Thus, $\#\Omega < \infty$, which proves the first claim.

Let $\Omega = \{\omega \in \mathbb{A}^{\mathbb{N}};\ \sum_{n \in \mathbb{N}} \omega(n) \leq 1\}$ with $\mathbb{A} = \{0, 1\}$. Then, it is clear that $p_{\Omega}^*(k) = k+1$ for any $k = 0, 1, 2, \cdots$. $\square$

Example 2. Let $\mathbb{N} = N_1 \cup N_2$ be such that $N_1 \cap N_2 = \emptyset$ and $\#N_1 = \#N_2 = \infty$. Let $\Omega = \{0,1\}^{N_1} \times \{0^{\mathbb{N}_2}\} \subset \mathbb{A}^{\mathbb{N}}$ with $\mathbb{A} = \{0,1\}$, where $0^{\mathbb{N}_2}$ is the 0-valued word defined on $\mathbb{N}_2$. Then, it is clear that $\#\Omega = \infty$ and $p_{*\Omega}(k) = 1$ $(k = 0,1,2,\cdots)$ since $\#\Omega|_S = 1$ if $S \subset \mathbb{N}_2$.

Theorem 4. *If $p_{*\Omega}(k)$ is unbounded, then there exists $C > 0$ such that $p_{*\Omega}(k) \geq C \log k$ $(k = 1,2,\cdots)$. Moreover, for any integer $d \geq 2$, there exists Ω such that $p_{*\Omega}(k) = \lceil \log k / \log d \rceil + 1$ $(k = 1,2,\cdots)$.*

Proof Assume that $\liminf_{k\to\infty} p_{*\Omega}(k)/\log k = 0$ for some $\Omega \subset \mathbb{A}^{\mathbb{N}}$ with $\#\mathbb{A} = d$. Then, $p_{*\Omega}(kd^k) \leq k$ holds for infinitely many k. Hence, there exist an arbitrarily large k and $S \subset \mathbb{N}$ satisfying that $\#S = kd^k$ and $\#\Omega|_S \leq k$. Let $V := \{(\omega(i);\ \omega \in \Omega|_S) \in \mathbb{A}^{\Omega|_S};\ i \in S\}$. Since $\#\Omega|_S \leq k$, there are at most d^k different elements in V, while $\#S = kd^k$. Hence, there is $S_0 \subset S$ with $\#S_0 \geq k$ such that all of $(\omega(i);\ \omega \in \Omega|_S)$ with $i \in S_0$ coincide. This implies that $\Omega|_{S_0}$ consists of constant elements a^{S_0} with $a \in \mathbb{A}$. Therefore, $\#\Omega|_{S_0} \leq d$. Since k can be arbitrarily large, $p_{*\Omega}(k) \leq d$ for any k, which proves the first claim.

The second claim follows from the following Example 3. $\square$

Example 3. Let $d \geq 2$ be an integer. For $i \in \mathbb{N}$, define $\eta^i \in \mathbb{A}^{\mathbb{N}}$, where $\mathbb{A} = \{0,1,\cdots,d-1\}$, by $n = \sum_{i=0}^{\infty} \eta^i(n) d^i$ $(\forall n \in \mathbb{N})$. We also define $\eta^\infty := 0^{\mathbb{N}}$. Let $\Omega = \{\eta^i;\ i \in \mathbb{N} \cup \{\infty\}\}$. Then, Ω is a closed subset of $\mathbb{A}^{\mathbb{N}}$. Let us prove that $p^*_\Omega(k) = d^k$ and $p^{BL}_\Omega(k) = p_{*\Omega}(k) = \lceil \log k/\log d \rceil + 1$.

For any $k = 1,2,\cdots$ and $j = 0,1,\cdots,k-1$, let $s_j = \sum_{n=0}^{d^k-1} \eta^j(n) d^n$. Then, we have $s_0 < s_1 < \cdots < s_{k-1}$. Let $S = \{s_0, s_1, \cdots, s_{k-1}\} \subset \mathbb{N}$. Since $\eta^i(s_j) = \eta^j(i)$ for any $i = 0,1,\cdots,d^k-1$ and $j = 0,1,\cdots,k-1$, we have

$$\sum_{j=0}^{k-1} \eta^i(s_j) d^j = \sum_{j=0}^{k-1} \eta^j(i) d^j = i$$

for any $i = 0,1,\cdots,d^k-1$. This implies that all of $\eta^i|_S$ for $i = 0,1,\cdots,d^k-1$ are distinct, and hence, $\#\Omega|_S = d^k$. Thus, $p^*_\Omega(k) = d^k$ for any $k = 1,2,\cdots$.

Note that $p^{BL}_\Omega(k) = h+1$ $(k = 1,2,\cdots)$ if $d^{h-1} < k \leq d^h$, since all of $\eta^i|_{\{0,1,\cdots,k-1\}}$ for $i = 0,1,\cdots,h$ are distinct and $\eta^i|_{\{0,1,\cdots,k-1\}} = 0^{\{0,1,\cdots,k-1\}}$ for any $i = h, h+1, \cdots$. Thus, $p^{BL}_\Omega(k) = h+1 = \lceil \log k/\log d \rceil + 1$.

Moreover, take any $S \subset \mathbb{N}$ with $\#S = k$. Let $r := \#\Omega|_S$. Since for any pair $i,j \in S$ with $i \neq j$, there exists $\omega \in \Omega$ with $\omega|_S \neq 0^S$ such that $\omega(i) \neq \omega(j)$, all the functions $\omega \mapsto \omega(i)$ from $\Omega'|_S$ to $\mathbb{A}$, where $\Omega' := \{\omega \in \Omega;\ \omega|_S \neq 0^S\}$, for $i \in S$ are distinct. Since $\#\Omega'|_S = r-1$, this implies $k \leq d^{r-1}$, and hence, $h \leq r-1$. Therefore, $\#\Omega|_S \geq \lceil \log k/\log d \rceil + 1$ for any $S \subset \mathbb{N}$ with $\#S = k$ and $k = 1,2,\cdots$. Hence, $p_{*\Omega}(k) \geq \lceil \log k/\log d \rceil + 1$. Together with $p^{BL}_\Omega(k) = \lceil \log k/\log d \rceil + 1$, we have

$$p^{BL}_\Omega(k) = p_{*\Omega}(k) = \lceil \log k/\log d \rceil + 1 \ (k = 1,2,\cdots).$$

4 Stationary and transitive sets

In this section, we always assume that $\Omega \subset \mathbb{A}^{\mathbb{N}}$ is stationary and transitive. The following theorem except for the statement on $p_{*\Omega}$ is just a copy of well known results (see [2], [5]). The statement on $p_{*\Omega}$ is proved in [1]. Here, we reproduce the proof for the sake of self-containedness.

Theorem 5. *If $\Omega \subset \mathbb{A}^{\mathbb{N}}$ is stationary and transitive with $\#\Omega = \infty$, then $p_{\Omega}^{BL}(k) \geq p_{*\Omega}(k) \geq k+1$ and $p_{\Omega}^{*}(k) \geq 2k$ $(k = 1, 2, \cdots)$. On the other hand, $p_{\Omega}^{BL}(k) = k+1$ $(k = 1, 2, \cdots)$ holds if and only if Ω is a Sturmian set and $p_{\Omega}^{*}(k) = 2k$ $(k = 1, 2, \cdots)$ holds if and only if Ω is a pattern Sturmian set. Moreover, a Sturmian set is always a pattern Sturmian set.*

Proof Assume that $\#\Omega = \infty$. Then, clearly $p_{*\Omega}(1) \geq 2$ since Ω is stationary. Assume that there exists $k = 1, 2, \cdots$ such that $p_{*\Omega}(k) \leq k$. Let k_0 be the minimum k as this. Since $p_{*\Omega}(1) \geq 2$, $k_0 \geq 2$. Since $p_{*\Omega}(k_0 - 1) \geq k_0$ and $p_{*\Omega}(k)$ is increasing in k, we have $p_{*\Omega}(k_0 - 1) = p_{*\Omega}(k_0) = k_0$. Let $S \subset \mathbb{N}$ satisfy $\#S = k_0$ and $\#\Omega|_S = k_0$. Let $S = \{s_1 < s_2 < \cdots < s_{k_0}\}$. Then, $\#\Omega|_{\{s_1, \cdots, s_{k_0 - 1}\}} = k_0$ since

$$k_0 = \#\Omega|_S \geq \#\Omega|_{\{s_1, \cdots, s_{k_0 - 1}\}} \geq p_{*\Omega}(k_0 - 1) = k_0.$$

This implies that $\omega(s_{k_0})$ is determined by $\omega|_{\{s_1, \cdots, s_{k_0 - 1}\}}$ in Ω. Hence, $\omega(s_{k_0})$ is determined by $\omega|_{\{0, 1, \cdots, s_{k_0} - 1\}}$ in Ω. Since Ω is stationary, there exists a function $f : \mathbb{A}^{s_{k_0}} \to \mathbb{A}$ such that

$$\omega(n + s_{k_0}) = f(\omega(n), \omega(n+1), \cdots, \omega(n + s_{k_0} - 1))$$

for any $n \in \mathbb{N}$ and $\omega \in \Omega$. This implies that ω is ultimately periodic with period at most $(\#\mathbb{A})^{s_{k_0}}$ and the period start before $n = (\#\mathbb{A})^{s_{k_0}}$ for any $\omega \in \Omega$. Hence, we have a contradiction that $\#\Omega < \infty$, which proves that $p_{*\Omega}(k) \geq k + 1$ $(k = 1, 2, \cdots)$. $\square$

Example 4. (S. Ferenczi and P. Hubert [1])
Let $\Omega \subset \mathbb{A}^{\mathbb{N}}$ be a Sturmian set with $\mathbb{A} = \{0, 1\}$. For $\omega \in \mathbb{A}^{\mathbb{N}}$, define $\tilde{\omega} \in \mathbb{A}^{\mathbb{N}}$ by $\tilde{\omega}(n) = \omega(\lfloor n/2 \rfloor)$ $(\forall n \in \mathbb{N})$. Let

$$\tilde{\Omega} := \{\tilde{\omega};\ \omega \in \Omega\} \cup \{T\tilde{\omega};\ \omega \in \Omega\}.$$

Then, $\tilde{\Omega} \subset \mathbb{A}^{\mathbb{N}}$ is stationary and transitive. It is clear that $p_{*\tilde{\Omega}}(k) = k + 1$ by Theorem 5 since $\#\tilde{\Omega}|_{\{0, 2, \cdots, 2(k-1)\}} = \#\Omega|_{\{0, 1, \cdots, k-1\}} = k + 1$.

On the other hand, for any large $k \in \mathbb{N}$, we have $p_{\tilde{\Omega}}^{BL}(k) = k + 3$. This is because there exists K such that $\omega|_{\{0, 1, \cdots, K-1\}}$ contains both 0 and 1 for any $\omega \in \Omega$, since Ω is a Sturmian set and uniformly recurrent with respect to the shift T. Then,

$$\tilde{\eta}|_{\{0, 1, \cdots, 2(K-1)\}} \neq (T\tilde{\zeta})|_{\{0, 1, \cdots, 2(K-1)\}}$$

holds for any $\eta,\ \zeta \in \Omega$ since $\tilde{\eta}(i) \neq \tilde{\eta}(i+1)$ implies i is odd while$(\widetilde{T\zeta})(i) \neq (\widetilde{T\zeta})(i+1)$ implies i is even. It follows that if $k \geq K$, then

$$\#\tilde{\Omega}|_{\{0,1,\cdots,2(k-1)\}} = \#\Omega|_{\{0,1,\cdots,k-1\}} + \#\Omega|_{\{0,1,\cdots,k-1\}} = (k+1) + (k+1)$$

and

$$\#\tilde{\Omega}|_{\{0,1,\cdots,2k-1\}} = \#\Omega|_{\{0,1,\cdots,k-1\}} + \#\Omega|_{\{0,1,\cdots,k\}} = (k+1) + (k+2),$$

which proves the requirement.

Theorem 6. *If $\Omega \subset \mathbb{A}^{\mathbb{N}}$ is stationary and transitive, then all of the entropies $h^{BL}(\Omega),\ h^*(\Omega),\ h_*(\Omega)$ exist and $h_*(\Omega) \leq h^{BL}(\Omega) \leq h^*(\Omega)$ holds.*

Proof The existence of $h^{BL}(\Omega)$ is just a classical result. The existence of $h^*(\Omega)$ is proved in Theorem 1 in more general setting. To prove the existence of $h_*(\Omega)$, it is sufficient to prove that

$$p_{*\Omega}(k_1 + k_2) \leq p_{*\Omega}(k_1) p_{*\Omega}(k_2)$$

for any $k_1, k_2 \in \mathbb{N}$. Let $S_i \subset \mathbb{N}$ satisfy that $\#S_i = k_i$ and $\#\Omega|_{S_i} = p_{*\Omega}(k_i)$ for $i = 1, 2$. Since Ω is stationary, $\#\Omega|_{S_i+n} = \#\Omega|_{S_i}$ holds for any $n \in \mathbb{N}$, where $S_i + n = \{s + n;\ s \in S_i\}$. Therefore taking $S_2 + n$ instead of S_2 if necessary, we may assume that $S_1 \cap S_2 = \emptyset$. Then, we have

$$p_{*\Omega}(k_1 + k_2) \leq \#\Omega|_{S_1 \cup S_2} \leq \#\Omega|_{S_1} \#\Omega|_{S_2} = p_{*\Omega}(k_1) p_{*\Omega}(k_2).$$

The existence of $h_*(\Omega)$ follows from this by the subadditivity of $\log p_{*\Omega}(k)$.
It is clear that $h_*(\Omega) \leq h^{BL}(\Omega) \leq h^*(\Omega)$. □

Example 5. Let $\varphi : \{0,1\} \to \{0,1\}^r$ with $r \geq 2$ be a primitive substitution such that $\varphi(0)$ begins by 0. Let $\alpha \in \{0,1\}^{\mathbb{N}}$ with $\alpha(0) = 0$ be the fixed point of φ, that is $\varphi(\alpha) = \alpha$. Let Ω^φ be the closure of $\{T^n \alpha;\ n \in \mathbb{N}\}$. Then, it is known [12] that either $p^*_{\Omega^\varphi}(k) = 2^k$ $(k = 0, 1, 2, \cdots)$ or $p^*_{\Omega^\varphi}(k)$ increases in a linear order of k.

Let

$$\varphi(0) = 010100 \quad , \quad \varphi(1) = 011100,$$

Then, the fix point α of φ is a Toeplitz word (see [3]):

$$\alpha = (01?100)^\infty \lhd (01?100)^\infty \lhd \cdots .$$

By Theorem 4 in [3], we have

$$\lim_{k\to\infty} \frac{p^*_{\Omega^\varphi}(k)}{k} = \lim_{k\to\infty} \frac{p^*_\alpha(k)}{k} = 2 + \max_{\substack{L \subset \{0,1,\cdots,5\} \\ \#L \geq 2}} \frac{E(\xi, L)}{\#L - 1},$$

where $\xi = (01?100)^\infty$ and $E(\xi, L) = \#(\pi_{\mathbb{A}} F_\xi(L) \cup \{0^{\#L}, 1^{\#L}\}) - 2\#L$. Here, $F_\xi(L) = \{\xi[n + L];\ n \in \mathbb{N}\}$ and $\pi_{\mathbb{A}} F_\xi(L)$ is the set of finite words over $\{0, 1\}$

obtained by substituting the letter ? by 0 or 1 arbitrary for each element in $F_\xi(L)$. For example, if $L = \{0, 1, 2, 3\}$, then

$$
\begin{aligned}
&F_\xi(L) = \{01?1, 1?10, ?100, 1000, 0001, 001?\},\\
&\pi_{\mathbb{A}} F_\xi(L) = \{0101, 0111, 1010, 1110, 0100, 1100, 1000, 0001, 0010, 0011\}\\
&\#(\pi_{\mathbb{A}} F_\xi(L) \cup \{0000, 1111\}) = 12
\end{aligned}
$$

Hence, we have $E(\xi, L) = 12 - 8 = 4$ and $E(\xi, L)/(\#L - 1) = 4/3$. It is not difficult to check that the maximum of $E(\xi, L)/(\#L - 1)$ is attained by this L. Thus, $\lim_{k\to\infty} p^*_{\Omega^\varphi}(k)/k = 10/3$, which is not an integer.

5 Uniform complexity

Theorem 7. *Let $p(k)$ and $q(k)$ be uniform complexity functions of $k \in \mathbb{N}$. Then, $p(k)+q(k)-\mathbf{1}_{k=0}$ and $p(k)q(k)$ are also uniform complexity functions.*

Proof Let $U \subset \mathbb{A}^{\mathbb{N}}$ and $V \subset \mathbb{B}^{\mathbb{N}}$ be uniform sets such that $p(k) = p_U(k)$ and $q(k) = p_V(k)$ for any $k = 0, 1, 2, \cdots$. We may assume that $\mathbb{A} \cap \mathbb{B} = \emptyset$. Then, $U \cup V \subset (\mathbb{A} \cup \mathbb{B})^{\mathbb{N}}$ and $U \times V \subset (\mathbb{A} \times \mathbb{B})^{\mathbb{N}}$ are uniform sets with $p_{U\cup V}(k) = p(k) + q(k) - \mathbf{1}_{k=0}$ and $p_{U\times V}(k) = p(k)q(k)$. □

Uniform complexity has been studied well in [13]. We summarize some known results.

For an infinite set $\mathcal{N} = \{N_0 < N_1 < N_2 < \cdots\} \subset \mathbb{N}$, $\omega \in \mathbb{A}^{\mathbb{N}}$ and $\Omega \subset \mathbb{A}^{\mathbb{N}}$, define $\omega[\mathcal{N}] \in \mathbb{A}^{\mathbb{N}}$ and $\Omega[\mathcal{N}] \subset \mathbb{A}^{\mathbb{N}}$ by

$$\omega[\mathcal{N}](n) := \omega(N_n)\ (n \in \mathbb{N})$$

and

$$\Omega[\mathcal{N}] := \{\omega[\mathcal{N}] \in \mathbb{A}^{\mathbb{N}};\ \omega \in \Omega\}.$$

We call Ω a *super-stationary set* if $\Omega[\mathcal{N}] = \Omega$ holds for any infinite subset $\mathcal{N}$ of $\mathbb{N}$.

Theorem 8. [13] *Let $\Omega \subset \mathbb{A}^{\mathbb{N}}$ be a uniform set. Then, there exists an infinite subset $\mathcal{N} \subset \mathbb{N}$ such that $\Omega[\mathcal{N}]$ is a super-stationary set. Hence, all the uniform complexity functions are realized by super-stationary sets.*

The set of finite words over $\mathbb{A}$ is denoted by $\mathbb{A}^*$, that is $\mathbb{A}^* = \cup_{k=0}^{\infty} \mathbb{A}^k$. We also denote $\mathbb{A}^+ = \cup_{k=1}^{\infty} \mathbb{A}^k = \mathbb{A}^* \setminus \{\epsilon\}$, where ϵ is the empty word. For $\xi \in \mathbb{A}^*$, k such that $\xi \in \mathbb{A}^k$ is called the *length* of ξ and is denoted by $|\xi|$. In this case, we denote $\xi = \xi_1\xi_2\cdots\xi_k$ with $\xi_i \in \mathbb{A}$ $(i = 1, 2, \cdots, k)$. For $\xi = \xi_1\xi_2\cdots\xi_k$, $\eta = \eta_1\eta_2\cdots\eta_l$ in $\mathbb{A}^*$ with $0 \le k = |\xi| \le l = |\eta|$ and $\omega \in \mathbb{A}^{\mathbb{N}}$, ξ is called a *super-subword* of η or ω if there exists $S = \{s_1 < s_2 < \cdots < s_k\}$ which is a subset of $\{1, 2, \cdots, l\}$ or $\mathbb{N}$, respectively, such that $\xi = \eta[S] := \eta_{s_1}\eta_{s_2}\cdots\eta_{s_k}$ or $\xi = \omega[S] := \omega(s_1)\omega(s_2)\cdots\omega(s_k)$. We denote

$$\xi \ll \eta \ \text{ or } \ \xi \ll \omega$$

if ξ is a super-subword of η or ω, respectively.

We denote by Ξ_{min} the set of all minimal words in $\Xi \subset \mathbb{A}^*$ with respect to $\ll$, that is, the set of $\xi \in \Xi$ such that $\eta \underset{\neq}{\ll} \xi$ does not hold for any $\eta \in \Xi$. It is known [13] that Ξ_{min} is a finite set for any $\Xi \subset \mathbb{A}^*$.

For $\xi \in \mathbb{A}^*$, denote

$$\mathcal{P}(\xi) := \{\omega \in \mathbb{A}^{\mathbb{N}};\ \xi \ll \omega \text{ does not hold}\},$$

and for $\Xi \subset \mathbb{A}^*$, denote

$$\mathcal{P}(\Xi) := \bigcap_{\xi \in \Xi} \mathcal{P}(\xi).$$

Note that $\mathcal{P}(\Xi) = \emptyset$ if $\epsilon \in \Xi$ and $\mathcal{P}(\emptyset) = \mathbb{A}^{\mathbb{N}}$. Also, $\mathcal{P}(\Xi) = \mathcal{P}(\Xi_{min})$.

For $\zeta = \zeta_1\zeta_2 \cdots \zeta_{l-1}\zeta_l \in \mathbb{A}^*$ and $a \in \mathbb{A}$, denote

$$a^{-1}\zeta = \begin{cases} \zeta_2 \cdots \zeta_l & (\text{if } \zeta_1 = a) \\ \zeta_1\zeta_2 \cdots \zeta_l & (\text{if } \zeta_1 \neq a) \end{cases},\ \zeta a^{-1} = \begin{cases} \zeta_1 \cdots \zeta_{l-1} & (\text{if } \zeta_l = a) \\ \zeta_1 \cdots \zeta_{l-1}\zeta_l & (\text{if } \zeta_l \neq a) \end{cases}.$$

For $\zeta \in \mathbb{A}^*$, $\xi = \xi_1 \cdots \xi_k \in \mathbb{A}^*$ and $\Xi \subset \mathbb{A}^*$, denote

$$\xi^{-1}\zeta = \xi_k^{-1} \cdots \xi_1^{-1}\zeta\ ,\ \zeta\xi^{-1} = \zeta\xi_k^{-1} \cdots \xi_1^{-1}$$
$$\xi^{-1}\Xi = \{\xi^{-1}\zeta;\ \zeta \in \Xi\}\ ,\ \Xi\xi^{-1} = \{\zeta\xi^{-1};\ \zeta \in \Xi\}.$$

We define the condition (#) for $\Xi \subset \mathbb{A}^*$ as follows.

(#) *There are no words $\xi, \eta \in \mathbb{A}^*$ such that $(\xi^{-1}\Xi\eta^{-1})_{min} = \mathbb{A}$, where each letter in $\mathbb{A}$ here is considered as a word with length 1.*

Theorem 9. [13] *The class of super-stationary sets over $\mathbb{A}$ coincides with the class of sets $\mathcal{P}(\Xi)$ with $\Xi \subset \mathbb{A}^+$ satisfying (#). Moreover, since $\mathcal{P}(\Xi) = \mathcal{P}(\Xi_{min})$, we may assume that Ξ is a finite set.*

For $\Xi \subset \mathbb{A}^*$ satisfying the condition (#), we denote $\mathbf{p}(\Xi)$ the function $\mathbb{N} \to \mathbb{N}$ such that

$$\mathbf{p}(\Xi)(k) = \#\{\eta \in \mathbb{A}^k;\ \xi \ll \eta \text{ does not hold for any } \xi \in \Xi\}.$$

Theorem 10. [13] *Let $\Omega \subset \mathbb{A}^{\mathbb{N}}$ be such that $\Omega = \mathcal{P}(\Xi)$ with $\Xi \subset \mathbb{A}^+$ satisfying (#). Then, we have*
(1) $p_\Omega(k) = \mathbf{p}(\Xi)(k)$ $(k = 0, 1, 2, \cdots)$,
(2) $\mathbf{p}(\Xi) = \mathbf{p}(\Xi_{min})$,
(3) $\mathbf{p}(\Xi)(0) = 1$ *if* $\epsilon \notin \Xi$ *and* $\mathbf{p}(\Xi)(k) = 0$ $(k = 0, 1, 2, \cdots)$ *if* $\epsilon \in \Xi$,
(4) $\mathbf{p}(B)(k) = (\#\mathbb{A} - \#B)^k$ $(k = 0, 1, 2, \cdots)$ *if* $B \underset{\neq}{\subset} \mathbb{A}$.

For $r = 1, 2, \cdots$, we denote $\tau(r)$ the function $\mathbb{N} \to \mathbb{N}$ such that

$$\tau(r)(k) = r^k.$$

For a function $u : \mathbb{N} \to \mathbb{N}$, we define a function $Su : \mathbb{N} \to \mathbb{N}$ by

$$(Su)(k) = \begin{cases} u(k-1) & (k \geq 1) \\ 1 & (k = 0) \end{cases} .$$

The *convolution* $u \otimes v$ between functions $u, v : \mathbb{N} \to \mathbb{N}$ is defined as

$$(u \otimes v)(k) = \sum_{l=0}^{k} u(l)v(k-l) \ (k = 0, 1, 2, \cdots).$$

Theorem 11. [13] (1) *For $\Xi \subset \mathbb{A}^+$ satisfying the condition (#), we have*

$$\mathbf{p}(\Xi) = \tau(\#\mathbb{A} - \#\Xi_{pre}) \otimes S\left(\sum_{a \in \Xi_{pre}} \mathbf{p}(a^{-1}\Xi)\right),$$

where $\Xi_{pre} := \{a \in \mathbb{A};\ a$ is a prefix of some $\xi \in \Xi\}$ and $\tau(0)(k) = \mathbf{1}_{k=0}$.
(2) *The class of uniform complexity functions over $\mathbb{A}$ is included in the minimal class of functions containing all $\tau(r)$ with $r = 1, 2, \cdots, \#\mathbb{A}$, closed under the operations of S, convolution and summation.*
(3) *Any uniform complexity function $p_\Omega(k)$ over $\mathbb{A}$ with $\#\mathbb{A} = d$ satisfies either $p_\Omega(k) = d^k$ $(\forall k \in \mathbb{N})$ or there exist polynomials R_r $(r = 1, 2, \cdots, d-1)$ with rational coefficients such that $p_\Omega(k) = \sum_{r=1}^{d-1} R_r(k) r^k$ holds for any sufficiently large k.*

Example 6. Let $\mathbb{A} = \{0, 1, 2\}$ and $\Xi = \{001, 021, 10\}$. Then, Ξ satisfies the condition (#). Applying Theorems 10 and Theorem 11, we have

$$\begin{aligned} &\mathbf{p}(\Xi) = \mathbf{p}(001, 021, 10) \\ &= \tau(1) \otimes S(\mathbf{p}(01, 21, 10) + \mathbf{p}(0)) \\ &= \tau(1) \otimes S(S(\mathbf{p}(1) + \mathbf{p}(1) + \mathbf{p}(21, 0)) + \mathbf{p}(0)) \\ &= \tau(1) \otimes S(S(\mathbf{p}(1) + \mathbf{p}(1) + \tau(1) \otimes S\mathbf{p}(1, 0)) + \mathbf{p}(0)) \\ &= \tau(1) \otimes S(S(\tau(2) + \tau(2) + \tau(1) \otimes S\tau(1)) + \tau(2)). \end{aligned}$$

For $k \in \mathbb{N}$, we have

$$\begin{gathered} (\tau(1) \otimes S\tau(1))(k) = \textstyle\sum_{l=0}^{k} 1 \cdot 1 = k + 1 \\ (\tau(2) + \tau(2) + \tau(1) \otimes S\tau(1))(k) = 2 \cdot 2^k + k + 1 \\ S(\tau(2) + \tau(2) + \tau(1) \otimes S\tau(1))(k) = 2^k + k \\ (S(\tau(2) + \tau(2) + \tau(1) \otimes S\tau(1)) + \tau(2))(k) = 2 \cdot 2^k + k \\ S(S(\tau(2) + \tau(2) + \tau(1) \otimes S\tau(1)) + \tau(2))(k) = \begin{cases} 2^k + k - 1 & (k \geq 1) \\ 1 & (k = 0) \end{cases} \\ \mathbf{p}(\Xi)(k) = (\tau(1) \otimes S(S(\tau(2) + \tau(2) + \tau(1) \otimes S\tau(1)) + \tau(2)))(k) \\ = \textstyle\sum_{l=1}^{k} (2^l + l - 1) + 1 = 2 \cdot 2^k + (1/2)k^2 - (1/2)k - 1. \end{gathered}$$

Definition 3. (1) Two functions $f,\ g : \mathbb{N} \to \mathbb{R}$ are said to be *equivalent* if $f(k) = g(k)$ holds except for finitely many k.
(2) For functions $f,\ g : \mathbb{N} \to \mathbb{R}$, we denote $f \leq_\infty g$ if $f(k) \leq g(k)$ holds except for finitely many k.
(3) A function of k of the form $\sum_{r=1}^{d} R_r(k) r^k$ with a positive integer d and polynomials R_r $(r = 1, 2, \cdots, d)$ with rational coefficients is called an *expolynomial*. The set of expolynomials is a totally ordered ring with respect to the addition, the multiplication and the ordering $\leq_\infty$.
(4) An expolynomial $f : \mathbb{N} \to \mathbb{R}$ which is equivalent to some uniform complexity function is called a *uniform complexity expolynomial*.

The following corollary follows from Theorems 7, 10 and 11.

Corollary 2. (1) *For any uniform complexity function, there exists a unique uniform complexity expolynomial equivalent to it.*
(2) *The set of uniform complexity expolynomials is closed under summation and multiplication. Hence, it is a totally ordered semi-ring with respect to $\leq_\infty$, which is denoted by $\mathcal{U}$. The basis $\mathcal{U}_0$ of $\mathcal{U}$ as semi-ring consists of $f \in \mathcal{U}$ such that f is not written as a summation or a multiplication of elements in $\mathcal{U} \setminus \{f\}$.*

Example 7. The mapping from a uniform complexity function to the uniform complexity expolynomial equivalent to it is not one-to-one. In fact, let

$$\Omega_1 = \mathcal{P}(0011, 0101, 0110, 1001, 1010, 1100) \subset \{0,1\}^{\mathbb{N}}$$
$$\Omega_2 = \mathcal{P}(11, 12, 21, 022, 202, 220) \subset \{0,1,2\}^{\mathbb{N}}.$$

Then, $p_{\Omega_1}(k) = p_{\Omega_2}(k) = 2k + 2$ for $k = 3, 4, \cdots$, but

$$2 = p_{\Omega_1}(1) \neq p_{\Omega_2}(1) = 3\ ,\ 4 = p_{\Omega_1}(2) \neq p_{\Omega_2}(2) = 6.$$

Theorem 12. *Let a uniform complexity function $p_\Omega(k)$ of a uniform set $\Omega \subset \mathbb{A}^{\mathbb{N}}$ be equivalent to an expolynomial $\sum_{i=1}^{l} R_i(k) {r_i}^k$ such that $\{r_1 < r_2 < \cdots < r_l\} \subset \mathbb{Z}_+$ and R_i $(i = 1, 2, \cdots, l)$ are nonzero polynomials with rational coefficients. Then, we have $h(\Omega) = \log r_l$ and*

$$D(\Omega) := \lim_{k\to\infty} \frac{\log p_\Omega(k) - kh(\Omega)}{\log k} = \deg(R_l) \in \mathbb{N}.$$

Moreover, if $h(\Omega) = 0$ and $D(\Omega) \leq 1$, then $\lim_{k\to\infty} p_\Omega(k)/k$ exists and is a nonnegative integer.

Proof The first two claims are clear. Let us prove the last claim. Assume that $h(\Omega) = 0$ and $D(\Omega) \leq 1$. Then, $l = 1,\ r_1 = 1$ and $\deg(R_l) \leq 1$. Thus, $p_\Omega(k)$ is equivalent to $R_l(k)$ with $\deg(R_l) \leq 1$, and hence, there exist rational numbers a and b such that $p_\Omega(k) = ak + b$ holds for any sufficiently large k. Since $p_\Omega(k)$ is $\mathbb{N}$-valued and increasing in k, $a = p_\Omega(k+1) - p_\Omega(k) \in \mathbb{N}$ holds for any sufficiently large k. Thus, $a \in \mathbb{N}$, which proves that $a = \lim_{k\to\infty} p_\Omega(k)/k$ exists and is a nonnegative integer. $\square$

6 Basis of uniform complexity expolynomials

As in the Corollary 1, $\mathcal{U}$ denotes the set of uniform complexity expolynomials and $\mathcal{U}_0$ denotes its basis as semi-ring. Our final end is to characterize the set $\mathcal{U}_0$ which is still a long way off. However we step towards it. For $f \in \mathcal{U}$, we denote by $h(f)$ and $D(f)$, the $h(\Omega)$ and $D(\Omega)$, respectively, such that f is equivalent to p_Ω.

We have another characterization of the class of super-stationary sets than Theorem 9 ([13]). We summarize it here.

Definition 4. The *concatenation* UV of subsets U and V of $\mathbb{A}^* \cup \mathbb{A}^{\mathbb{N}}$ is defined as

$$UV = (U \cap \mathbb{A}^{\mathbb{N}}) \cup \{uv;\ u \in U \cap \mathbb{A}^*,\ v \in V\},$$

which is a subset of $\mathbb{A}^* \cup \mathbb{A}^{\mathbb{N}}$.

Definition 5. For $\emptyset \neq B \subset \mathbb{A}$, we denote $I_B = B^* \cup B^{\mathbb{N}}$. For $a \in \mathbb{A}$, we denote $\delta_a = \{a, \epsilon\}$. Denote

$$I(\mathbb{A}) = \{I_B;\ \emptyset \neq B \subset \mathbb{A}\} \text{ and } \delta(\mathbb{A}) = \{\delta_a;\ a \in \mathbb{A}\},$$

which are considered as alphabets (i.e. sets of just letters) as well as the families of sets of words. Denote by $\Lambda(\mathbb{A})$ the set of nonempty finite words λ over the alphabet $I(\mathbb{A}) \cup \delta(\mathbb{A})$ satisfying that

(1) the first and the last letters of λ belong to $I(\mathbb{A})$,

(2) there are no neighboring letters in λ both of which belong to $\delta(\mathbb{A})$,

(3) if I_B and $I_{B'}$ are neighboring in λ, then neither $B \subset B'$ nor $B' \subset B$ hold, and

(4) if I_B and δ_a are neighboring in λ, then $a \notin B$.

The above $\lambda = \lambda_1\lambda_2 \cdots \lambda_k \in \Lambda(\mathbb{A})$ can be considered as a subset of $\mathbb{A}^{\mathbb{N}}$ defined by the concatenations among the sets $\lambda_1, \lambda_2, \cdots, \lambda_k$ of words in the sense of Definition 4 and collecting all the infinite words.

Theorem 13. [13] *The class of super-stationary sets over $\mathbb{A}$ coincides with the class of sets which are nonempty finite unions of sets belonging to $\Lambda(\mathbb{A})$.*

Definition 6. Let $\lambda = \lambda_1\lambda_2 \cdots \lambda_k \in \Lambda(\mathbb{A})$ and $\eta = \eta_1\eta_2 \cdots \eta_l \in \Lambda(\mathbb{A})$. We call λ a *super-subword in the wide sense* of η if there exists a sequence $\{s_1 \leq s_2 \leq \cdots \leq s_k\} \subset \{1, 2, \cdots, l\}$ such that
(1) if $\lambda_i = I_B$, then $\eta_{s_i} = I_C$ with $B \subset C$ for any $i = 1, 2, \cdots, k$, and
(2) if $\lambda_i = \delta_a$, then either $\eta_{s_i} = \delta_a$ or $\eta_{s_i} = I_C$ with $a \in C$ for any $i = 1, 2, \cdots, k$.
In this case, we denote $\lambda \ll_w \eta$.

Theorem 14. *It holds that $\lambda \subset \eta$ as sets of words in $\mathbb{A}^{\mathbb{N}}$ if and only if $\lambda \ll_w \eta$.*

Proof The proof is not simple, but straightforward from the definitions, so we omit it. □

Definition 7. For $\lambda = \lambda_1\lambda_2\cdots\lambda_k \in \Lambda(\mathbb{A})$, the maximal number $\#B$ such that $I_B = \lambda_i$ for some $i = 1, 2, \cdots, k$ is denoted by $c(\lambda)$. We denote by $d(\lambda)$ the number of i such that $\lambda_i = I_B$ for some B with $\#B = c(\lambda)$.

Theorem 15. *For a super-stationary set Ω with $\Omega = \cup_{\lambda\in\mathcal{L}}\lambda$, where $\mathcal{L}$ is a finite subset of $\Lambda(\mathbb{A})$, it holds that $h(\Omega) = \log\max_{\lambda\in\mathcal{L}} c(\lambda)$. Moreover, $D(\Omega)$ coincides with the maximum $d(\lambda) - 1$ among $\lambda \in \mathcal{L}$ such that $c(\lambda) = \max_{\eta\in\mathcal{L}} c(\eta)$.*

Proof Let $\Omega = \cup_{\lambda\in\mathcal{L}}\lambda$ with $c = \max_{\lambda\in\mathcal{L}} c(\lambda)$ and d being the maximum $d(\lambda)$ among $\lambda \in \mathcal{L}$ such that $c(\lambda) = c$. Let $\lambda \in \mathcal{L}$ be as this, that is, $c(\lambda) = c$ and $d(\lambda) = d$. Let $|\lambda| = l$.

Then, $I_B \subset \lambda$ holds for some $B \subset \mathbb{A}$ with $c = \#B$ by Theorem 14. Therefore, $p_\Omega(k) \geq p_{I_B}(k) = c^k$ $(k = 0, 1, 2, \cdots)$ and $h(\Omega) \geq \log c$.

Assume that $c \geq 2$. Then, there exists $B_1, B_2, \cdots, B_l \subset \mathbb{A}$ such that $\#B_i \leq c$ $(i = 1, 2, \cdots, l)$, $\#B_i = c$ holds for d numbers of i, and that $\lambda \ll_w I_{B_1}I_{B_2}\cdots I_{B_l} \in \Lambda(\mathbb{A})$. In fact, B_i is chosen so that

$$I_{B_i} = \begin{cases} \lambda_i & \text{if } \lambda_i \in I(\mathbb{A}) \\ I_{\{a\}} & \text{if } \lambda_i = \delta_a \end{cases},$$

where $\lambda = \lambda_1\lambda_2\cdots\lambda_l$. Then, $\lambda \subset I_{B_1}I_{B_2}\cdots I_{B_l}$ holds by Theorem 14. Hence, there exists a constant $\varepsilon > 0$ such that for any large $k \in \mathbb{N}$, we have

$$\begin{aligned}
&\varepsilon p_\Omega(k) \leq p_\lambda(k) \leq p_{I_{B_1}I_{B_2}\cdots I_{B_l}}(k) \\
&\leq \binom{k+d-1}{d-1}c^k + \sum_{i=0}^{k-1}\binom{i+d-1}{d-1}c^i\binom{k-i+l-d-1}{l-d-1}(c-1)^{k-i} \\
&\leq (k+1)^{d-1}c^k + \sum_{i=0}^{k-1}(k+1)^{d-1}c^i\binom{k-i+l-d-1}{l-d-1}(c-1)^{k-i} \\
&\leq (k+1)^{d-1}c^k + (k+1)^{d-1}c^k\sum_{i=1}^{k}\binom{d'+i}{d'}((c-1)/c)^i \\
&\leq (k+1)^{d-1}c^k + (k+1)^{d-1}c^k\sum_{i=1}^{\infty}\binom{d'+i}{d'}((c-1)/c)^i \\
&\leq (1+C)(k+1)^{d-1}c^k
\end{aligned}$$

with a constant $C \geq 0$ independent of k, where $d' := l - d - 1$ and we always assume that $\binom{i}{j} = 0$ if either $i < j$ or $j < 0$. Hence, $h(\Omega) \leq \log c$ and $D(\Omega) \leq d - 1$.

Assume next that $c = 1$. Then, there exists

$$\eta = I_{\{b_1\}}\delta_{a_1}I_{\{b_2\}}\delta_{a_2}\cdots\delta_{a_{d-1}}I_{\{b_d\}} \in \Lambda(\mathbb{A})$$

such that $\lambda \ll_w \eta$. Hence, there exists a constant $\varepsilon > 0$ such that for any large $k \in \mathbb{N}$,

$$\varepsilon p_\Omega(k) \le p_\lambda(k) \le p_\eta(k) \le 2^{d-1}\binom{k+d-1}{d-1},$$

which also implies that $h(\Omega) \le 0 = \log c$ and $D(\Omega) \le d-1$.

To complete the proof, it is sufficient to prove that $D(\Omega) \ge d-1$. Let $B_1, B_2, \cdots, B_d$ be the sequence of subsets of $\mathbb{A}$ such that $\#B_i = c$ ($i = 1, 2, \cdots, d$) and $I_{B_1}, I_{B_2}, \cdots, I_{B_d}$ appears in λ in this order. Let $a_i \in \mathbb{A}$ ($i = 1, 2, \cdots, d-1$) be such that $a_i \notin B_i$ and that either the next letter in λ after I_{B_i} is δ_{a_i} or I_B with $a_i \in B$. Then, $\xi^1 a_1 \xi^2 a_2 \cdots a_{d-1}\xi^d \in \lambda$ holds for any $\xi^i \in B_i^*$ ($i = 1, 2, \cdots, d-1$) and $\xi_d \in B_d^{\mathbb{N}}$. Therefore,

$$p_\Omega(k) \ge p_\lambda(k) \ge \binom{k}{d-1}c^{k-d+1} \ge Ck^{d-1}c^k \quad (\forall k \in \mathbb{N})$$

with some constant $C > 0$. Thus, $D(\Omega) \ge d-1$. □

Lemma 2. *The minimum $f \in \mathcal{U}$ with respect to $\le_\infty$ among $f \in \mathcal{U}$ with $h(f) = \log c$ and $D(f) = d-1$ is $f = Q_{c,d}\tau(c)$, that is, $f(k) = Q_{c,d}(k)c^k$ with*

$$Q_{c,d}(k) = \sum_{i=0}^{d-1} c^{-i}\binom{k}{i} \quad (\forall k \in \mathbb{N}).$$

Proof Let $\eta = (I_B\delta_a)^{d-1}I_B \in \Lambda(\mathbb{A})$ with $a \in \mathbb{A}$, $B \subset \mathbb{A}$ such that $\#B = c$ and $a \notin B$. Then, for any large $k \in \mathbb{N}$, we have

$$p_\eta(k) = \sum_{i=0}^{d-1}\binom{k}{i}c^{k-i} = Q_{c,d}(k)c^k.$$

Hence, it is sufficient to prove that for any $f \in \mathcal{U}$ with $h(f) = \log c$ and $D(f) = d-1$, we have $p_\eta \le_\infty f$. Let f as this be equivalent to p_Ω with $\Omega = \cup_{\lambda\in\mathcal{L}}\lambda$ such that $c = \max_{\lambda\in\mathcal{L}} c(\lambda)$ and d being the maximum $d(\lambda)$ among $\lambda \in \mathcal{L}$ such that $c(\lambda) = c$. Let $\lambda \in \mathcal{L}$ satisfy that $c(\lambda) = c$ and $d(\lambda) = d$. Let $|\lambda| = l$. As in the proof of Theorem 15, there exist a sequence of subsets $B_1, B_2, \cdots, B_d$ of $\mathbb{A}$ such that $\#B_i = c$ ($i = 1, 2, \cdots, d$) and $I_{B_1}, I_{B_2}, \cdots, I_{B_d}$ appears in λ in this order, and a sequence $a_1, a_2, \cdots, a_{d-1}$ with $a_i \in \mathbb{A}$ ($i = 1, 2, \cdots, d-1$) such that $a_i \notin B_i$ and that either the next letter in λ after I_{B_i} is δ_{a_i} or I_B with $a_i \in B$. Then, $\xi^1 a_1 \xi^2 a_2 \cdots a_i \xi^{i+1} \in \lambda$

holds for any $0 \leq i \leq d-1$, $\xi^j \in B_j^*$ $(j = 1, 2, \cdots, i)$ and $\xi^{i+1} \in B_{i+1}^{\mathbb{N}}$. Moreover, there is no overlapping between them. Therefore, we have

$$p_\Omega(k) \geq p_\lambda(k) \geq \sum_{i=0}^{d-1} \binom{k}{i} c^{k-i}$$

for any large $k \in \mathbb{N}$, which completes the proof. $\square$

Theorem 16. *The expolynomial $Q_{c,d}\tau(c) \in \mathcal{U}$ for any c, $d \in \mathbb{Z}_+$ is the minimum (w.r.t. $\leq_\infty$) among $f \in \mathcal{U}$ with $h(f) = \log c$ and $D(f) = d-1$. Hence, it is in the additive basis of $\mathcal{U}$. Moreover, it is in the basis $\mathcal{U}_0$ of $\mathcal{U}$ as semi-ring except for the case that c with $c > 1$ is not a prime number and $d = 1$, when we have the multiplicative decomposition $\tau(c) = \tau(c_1)\tau(c_2)$.*

Proof If $f = Q_{c,d}\tau(c)$ is not in the additive basis of $\mathcal{U}$, then there exists $u, v \in \mathcal{U} \setminus \{f\}$ such that $f = u + v$. Since $h(f) = c$ and $D(f) = d-1$, at least one of u or v, say u, satisfies that $h(u) = \log c$ and $D(u) = d-1$. Since $u \leq_\infty f$ and $u \neq f$, this contradicts the minimality of f. Thus, f belongs to the additive basis of $\mathcal{U}$.

If $f = f_1 f_2$ with $f_i \in \mathcal{U}$ and $f_i \neq 1$ $(i = 1, 2)$, then we have

$$c = c_1 c_2,\ d-1 = d_1 - 1 + d_2 - 1 \text{ and } (c_i, d_i) \neq (1, 1)\ (i = 1, 2),$$

where $h(f_i) = \log c_i$ and $D(f_i) = d_i - 1$ for $i = 1, 2$. Then by the minimality of $Q_{c_i,d_i}\tau(c_i)$ $(i = 1, 2)$, we have $Q_{c_i,d_i}\tau(c_i) \leq_\infty f_i$ $(i = 1, 2)$. On the other hand, the minimality of f implies that $f \leq_\infty Q_{c_1,d_1}Q_{c_2,d_2}\tau(c_1)\tau(c_2)$. Since

$$f \leq_\infty Q_{c_1,d_1}Q_{c_2,d_2}\tau(c_1)\tau(c_2) \leq_\infty f_1 f_2 = f,$$

we have $Q_{c,d}\tau(c) = Q_{c_1,d_1}Q_{c_2,d_2}\tau(c_i)\tau(c_2)$, and hence, $Q_{c,d} = Q_{c_1,d_1}Q_{c_2,d_2}$. Since the leading term of $Q_{c,d}(k)$ is $c^{-d+1}((d-1)!)^{-1}k^{d-1}$, it follows that

$$c^{d-1} \cdot (d-1)! = c_1^{d_1-1} c_2^{d_2-1} \cdot (d_1-1)!(d_2-1)!,$$

and hence,

$$c_1^{d_2-1} c_2^{d_1-1} \binom{d-1}{d_1-1} = 1,$$

which is possible only when $d = d_1 = d_2 = 1$. Together with the relation $c = c_1 c_2$, it holds that f is not in the multiplicative basis of $\mathcal{U}$ if and only if $d = 1$ and c with $c > 1$ is not a prime number. In this case, we have a multiplicative decomposition $f = \tau(c) = \tau(c_1)\tau(c_2)$. $\square$

Example 8. Let $f \in \mathcal{U}$ satisfy $h(f) = 0$ and $D(f) \leq 1$. Then by Theorem 12, $f(k) = ak + b$ with $a, b \in \mathbb{N}$ such that $(a, b) \neq (0, 0)$. Let $\Omega \subset \mathbb{A}^{\mathbb{N}}$ be a super-stationary set with $p_\Omega(k) = f(k)$ for any sufficiently large $k \in \mathbb{N}$.

Then by Theorem 15, there exists a finite set $\mathcal{L} \subset \Lambda(\mathbb{A})$ with $\Omega = \cup_{\lambda \in \mathcal{L}} \lambda$ such that any of $\lambda \in \mathcal{L}$ is one of the following sets:

$$I_{\{b\}},\ I_{\{b\}} I_{\{b'\}}\ (b \neq b'),\ I_{\{b\}} \delta_a I_{\{b\}}\ (b \neq a),\ I_{\{b\}} \delta_a I_{\{b'\}}\ (b, b', a \text{ are distinct}),$$

where $b, b', a \in \mathbb{A}$. By considering finite unions of them, we obtain $f \in \mathcal{U}_0$ with $h(f) = 0$ and $D(f) \leq 1$ up to some stage as follows:

$$\begin{gathered} 1,\ k+1,\ 2k,\ 3k-2,\ 4k-4,\ 5k-4,\ 6k-5,\ 7k-7,\ 8k-9, \\ 9k-10,\ 10k-12,\ 11k-14,\ 12k-16,\ 13k-17, \\ 14k-19,\ 15k-21,\ 16k-23, 17k-25,\ 18k-27,\ \cdots \end{gathered}$$

7 Open problems

1. Does $h_*(\Omega)$ exist in general?
2. Does $\lim_{k \to \infty} (\log p^*_\Omega(k) - k h^*(\Omega)) / \log k$ exist in general and take non-negative integer value?
3. How to characterize the basis of the semi-ring $\mathcal{U}$ of the uniform complexity expolynomials?

In particular,

4. Determine the function $\varphi : \mathbb{N} \to \mathbb{Z}$ such that $ak + \varphi(a)$ as the function of k is in the basis $\mathcal{U}_0$ of $\mathcal{U}$ for $a = 0, 1, 2, \cdots$.

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